INTERCITY SEMINAR ALGEBRAIC GEOMETRY, Autumn 2007

Derived categories and stability conditions

Proposal for the programme. (Version of 29 August 2007)

The proposed theme of the seminar is a study of derived categories and the role they play in algebraic geometry, representation theory, string theory, and non-commutative geometry. (The focus will be on algebraic geometry though.) In the past decade (with some essential ideas going further back) there has been a revival of derived categories, not so much for their (co)homological use but based on the insight that the derived category of a variety X is actually a manageable invariant that contains a lot of interesting information about X. Further, as derived categories also appear in other areas, such as representation theory, we get to possible links between various fields through equivalences of derived categories. A nice illustration of this principle is provided by Kontsevich's homological formulation of Mirror Symmetry. The seminar will mainly focus on work of Mukai, Bondal, Orlov, and Bridgeland.

What follows are suggestions for a number of talks in the seminar. The precise shape of the talks is of course to be determined by the lecturers! I'm happy to discuss further details and suggestions for other topics that could or should be discussed. —Ben Moonen

Lecture 1. Bas Edixhoven: *General introduction.*

1. Start with some basic explanation about derived categories and derived functors. This should really only be a quick summary, and it seems best to restrict to some concrete examples. E.g., describe $D(A)$ for A a field or a hereditary algebra. The main example for us will be the bounded derived category of a variety. A nice reference is [29], beginnings of sections 2 and 3. See also [25] or [19], or the book [23] by Gelfand and Manin, or one of the many more technical accounts.
2. The Grothendieck group of $D(X)$ and its connection with the Chow ring. See [22], Chap. 15. (State clearly that $K(X) \otimes \mathbb{Q} \cong \mathrm{CH}(X) \otimes \mathbb{Q}$, which is a bit hidden in [22].)
3. Serre functors. Point-like objects and a preview of results of Bondal and Orlov. (To be discussed in detail in Lecture 3.) Reference: [25], Chap. 4.
4. Gabriel's result that X is determined by $\mathrm{Coh}(X)$. Also mention Balmer's result in [1]. References: [25], Cor. 5.24, [44], Section 3.

Lecture 2. Gerard van der Geer: *Mukai's work on abelian varieties and K3 surfaces.*

1. Mukai's equivalence $\Phi_{\mathscr{P}}\colon D(X) \xrightarrow{\sim} D(X^t)$ for an abelian variety X. Also discuss the corresponding theory for the Chow ring, and Beauville's decomposition. Stress that $\Phi_{\mathscr{P}}$ does not preserve the codimension. References: [35]
2. As far as time permits, discuss some applications. (Some options: the theory of semihomogeneous bundles and the theory of M-regularity. The latter would probably take too much time.) References: [34], [35], [38], [39].
3. The second half of the lecture should be devoted to K3 surfaces. The main goal should be to discuss the Mukai equivalence $\Phi_{\mathscr{P}}\colon D(X) \xrightarrow{\sim} D(M)$, where X is a K3, M is a component of

the coarse moduli space of sheaves, and $\mathscr{P}$ is the universal sheaf over $X \times M$. This result has motivated many of the later developments, in particular the results to be discussed in Lectures 6 and 8. Stress that in the "classical" theory for abelian varieties, the dual X^t should also be seen as a moduli space. References: [25], Chap. 10, [36], [37] and [26].

Lecture 3. Ben Moonen: *Orlov's theorem and reconstruction results.*

1. Start by recalling the definition of the Fourier-Mukai functor $\Phi_P\colon D(X) \to D(Y)$ for an object P in $D(X\times Y)$. Give some simple examples. Discuss some basic properties: exactness, left and right adjoints, composition of two FM-functors is again a FM-functor. References: [25], Section 5.1.
2. State Orlov's Theorem in the following form: *Let X and Y be non-singular projective varieties over a field k. Let $F\colon D(X) \to D(Y)$ be an exact additive fully faithful functor. Then there exists an object $P \in D(X\times Y)$, unique up to isomorphism, such that F is isomorphic to Φ_P.* Try to give some ideas about the proof; though you probably cannot do any details. (Note: the assumptions imply, by results of Bondal-van den Bergh [10], that F has left and right adjoints.)
3. As application of Orlov's theorem we want to discuss some results about how much information we can recover from $D(X)$. Suppose X and Y are non-singular projective varieties over a field k such that $D(X)$ and $D(Y)$ are equivalent as triangulated categories. State the following results:
   (i) $\dim(X) = \dim(Y)$;
   (ii) the canonical rings of X and Y are isomorphic;
   (iii) the Kodaira dimensions of X and Y are equal;
   (iv) if ω_X is ample or anti-ample then $X \cong Y$;
   (v) the Hochschild (co)homologies of X and Y are isomorphic; in particular, for every n we have $\sum_{p-q=n} h^{p,q}(X) = \sum_{p-q=n} h^{p,q}(Y)$.

   These results should probably take up half of the lecture. Give some of the proofs; in particular try to explain the use of Orlov's theorem. Point-like objects have already been introduced in Lecture 1. References: [25], especially Cor. 5.21 and section 6.1; see also [43], Section 2.1. Remark that in (iv) the (anti-)ampleness is necessary; example: take an abelian variety that is not isomorphic to its dual.
5. State some results on the structure of $\mathrm{Aut}\big(D(X)\big)$ if ω_X is ample/anti-ample, and also for X an abelian variety. If time permits give the proof in case ω_X is ample. References: [25], especially Prop. 4.17 and section 9.5.

Lecture 4. Jozef Steenbrink: *An introduction to the minimal model program.*

The main goal of the lecture should be to give an overview of what the MMP is about. Notions that should be discussed include:

— flips and flops;
— the discrepancy of a birational morphism; crepant resolutions;
— a brief overview of what is known and what not, especially in dimension 3. Most relevant for us are the existence and termination of flips and flops, chains of flops connecting two minimal models, and the (non?)existence of crepant resolutions.

Some references: [18], [20], [21], [30], [31], [33], [40].

Lecture 5. Jochen Heinloth: *Semi-orthogonal decompositions and t-structures.*

1. Define what is a semi-orthogonal decomposition of a k-linear triangulated category. Also say what is an orthogonal decomposition. A good reference is [25], Chap. 1. See also [8], start of section 2, [43], [11], section 2.

2. Define complete exceptional sequences, and explain how they give rise to an equivalence between $D(X)$ and $D(A)$ for some algebra A. As a key example, discuss the theory for $X = \mathbb{P}^n$. If possible, mention some generalisations for flag varieties, Fanos, etc. (Helix theory, work of Kapranov, etc.) References: [4], [19], [24], [28].
3. Discuss some first connections with geometry. In particular, explain that for a smooth projective and connected X the category $D(X)$ has no non-trivial orthogonal decomposition, and that if $\omega_X = \mathscr{O}_X$ then there is no semi-orthogonal decomposition. Next explain that results as in 2 cannot hold if $K(X)$ is not free. Discuss [11], Prop. 2.3 and the related [43], Thm. 2.3, and the philosophy that the MMP should be related to finding "minimal" subcategories in $D(X)$. References: [11], [43].
4. Now we turn to t-structures. Give the definition of a t-structure and its heart. (Note: we shall usually call $\mathscr{D}^{\leqslant 0}$ the t-structure. It is a right-admissible subcategory, so this relates to the notions just discussed. However, we tend to think a bit differently about t-structures and admissible subcategories.) Example: the trivial t-structure on $D(\mathscr{A})$ for $\mathscr{A}$ an abelian category. The heart of a t-structure is abelian. Note: if $\mathscr{C}$ is the heart of a t-structure on $\mathscr{D}$ then this does not mean that $\mathscr{D}$ is equivalent with $D(\mathscr{C})$. Define bounded t-structures, and explain how in this case the t-structure can be recovered from the heart.
5. Try to say something reasonable about the theory of perverse sheaves. (I admit, this is too vague...) We shall not need perverse sheaves in the classical topological sense, but we shall need (notably in Lecture 8) a notion of "coherent perverse sheaves".

Lecture 6. NN: *The McKay correspondence.*

Here I leave the details to the lecturer. The goal would be to discuss the paper [16], for which we should by now be well-prepared. Depending on the lecturer's taste this could be preceded by a short, or long, general discussion of the McKay correspondence. References for this include [41] and [42]. Also state the results in the symplectic case due to Bezrukavnikov and Kaledin; see [6] and [27].

Lecture 7. Jan Stienstra: *Stability conditions.*

1. Start with a brief "rappel" about (semi-)stability conditions on vector bundles, or, more generally, coherent sheaves. Discuss the HN-filtration. (Perhaps only the case of curves?) As this only serves as motivation, it's probably best to be brief here. Reference: [26].
2. Give Bridgeland's definition of a stability condition as in [12] and [15]. Explain that to give a stability condition is the same as giving a bounded t-structure with a stability function on its heart that satisfies a HN-condition. (Refer to the notions discussed in Lecture 5; in passing mention again that a bounded t-structure is determined by its heart.) Mention that the categories $\mathscr{P}(\varphi)$ are abelian.
3. For a variety X, define the space $\mathrm{Stab}(X)$ as in [15], Def. 3.6. Give [15], Lemma 3.8 = [12], Lemma 8.2. Explain the main results, Thm. 1.2 and Cor. 1.3, from [12].
4. The remainder of the lecture should be spent on examples: elliptic curves, abelian surfaces, curves in general,... References: [15], [12], [13], [32]. In the course of the semester we should decide how the last lectures are to be filled in; perhaps the case of K3's should be done in a separate lecture.

Lecture 8. Eduard Looijenga: *The Bondal-Orlov conjecture, and Bridgeland's theorem for 3-folds.*

The goal of this lecture is to discuss the Bondal-Orlov conjecture (see [43], Conj. 1.1) and the known results in dimension $\leqslant 3$. After a brief review of the situation in dimensions 1 and 2 (use what has been discussed in earlier lectures, quote the main result of [17]), the largest part of the lecture should be devoted to Bridgeland's paper [11].

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