Cohomology of Varieties—selected exercises

(to be extended during the semester)

Note: It is recommended that you do lots and lots of exercises; only this way you will really learn the subject. The exercises given here are only intended to get you going; you should look in the literature for more.

CATEGORY THEORY.

Exercise 1. Answer the questions below for each of the following categories:

Sets	sets
Top	topological spaces with continuous maps
Top_*	pointed topological spaces with base point preserving continuous maps
Mod_R	modules over the commutative ring R
CAlg_R	commutative algebras over the commutative ring R

Questions:

(i) Is there an initial object? A terminal object? A null object?
(ii) Do kernels exist in this category?
(iii) Do fibre products exist in this category? Do pushouts exist?
(iv) Is this an additive category? An abelian category?

If a question has a negative answer then subsequent questions may become meaningless. E.g., if there is no null object then you may skip (ii) and (iv). Further, in (iii) you should not only give the answer but also describe the fibre products and pushouts, if they exist.

Exercise 2. Let k be a field. If V is a vector space over k then by a *descending filtration* of V we mean a chain

$$F^\bullet : \qquad \cdots \supset F^{-2} \supset F^{-1} \supset F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} \supset \cdots$$

of linear subspaces of V. The pair $(V, F^\bullet)$ is called a *filtered vector space*.

If $(V, F^\bullet)$ and $(W, G^\bullet)$ are filtered vector spaces, then a linear map $f\colon V \to W$ is said to be *compatible* with the given filtrations if $f(F^n) \subseteq G^n$ for all $n \in \mathbb{Z}$.

Let FVec_k be the category of filtered k-vector spaces, i.e.,

objects:	filtered k-vector spaces
morphisms:	linear maps that are compatible with the filtrations

Note that FVec_k is an additive (even k-linear) category.

(i) Let $f\colon (V, F^\bullet) \to (W, G^\bullet)$ be a morphism in FVec_k. Let K be the kernel of $f\colon V \to W$, in the usual sense of linear algebra. On K we have the filtration $F^\bullet_K$ induced by the filtration $F^\bullet$; it is given by $F^n_K := K \cap F^n$. (Intersection taken inside V.) Prove that the obvious morphism $(K, F^\bullet_K) \to (V, F^\bullet)$ is a kernel of f in the category FVec_k.
(ii) Let $Q := W/f(V)$ be the cokernel of f in the usual sense of linear algebra. On Q we have the filtration $F^\bullet_Q$ induced by $G^\bullet$; it is given by $F^n_Q :=$ image of G^n in Q. Prove that the obvious morphism $(W, G^\bullet) \to (Q, F^\bullet_Q)$ is a cokernel of f in the category FVec_k.

(iii) The linear map f is said to be *strictly compatible* with the given filtrations if we have

$$F^n = \{v \in V \mid f(v) \in G^n\}$$

for all n. (Compatibility means that we always have "$\subseteq$".) Give examples of morphisms in FVec_k (necessarily compatible with the given filtrations) that are not strictly compatible. Show, by considering such morphisms, that FVec_k is not an abelian category.

SHEAF THEORY.

Do as many exercises as possible from HAG (=Hartshorne's book on Algebraic Geometry), Chap. II, § 1. Note: this section does not require any knowledge of Algebraic Geometry.

Exercise 3. Let X be a topological space. Let A be an abelian group. Consider the presheaf P on X with $P(U) = A$ for all non-empty open U and with restriction homomorphisms $\rho_{U,V}\colon P(U) \to P(V)$, for non-empty open $V \subseteq U$, all equal to the identity. Let F be the sheaf on X associated to this presheaf.

(i) Prove that we have

$$F(U) = \{\text{continuous maps } U \to A\},$$

where we give A the discrete topology.

(ii) The continuous maps $U \to A$ (with discrete topology on A) are also sometimes referred to as the locally constant maps. Explain why. Also show that $F(U) \cong A$ if U is connected.

Exercise 4. Let X be a topological space. Let Ab_X denote the category of sheaves of abelian groups on X. Show that this is an abelian category. (Cf. HAG, Chap. II, Exercise 1.7.)