

Cohomology of Varieties—homework assignment

Notes: (1) You may use the results that have been discussed in the course, as well as results in the literature. However, in each case you should give a precise statement of the result you use, and if possible give a reference.

(2) Please hand in before the end of January 2009. If you will not be able to hand in by then, please let me know. Please keep a copy of what you hand in. In case you have your solutions available electronically (preferably in pdf format) then you can email them to bmoonen@uva.nl.

(3) To err is human, so if you think there is a mistake in the formulation of an exercise, please contact me.

(4) Unless specified otherwise, “sheaf” means “sheaf of abelian groups”. If X is a topological space, we write \mathbf{Ab}_X for the category of sheaves of abelian groups on X . We write \mathbb{Q}_X for the constant sheaf on X with stalks \mathbb{Q} .

Exercise 1.

- (i) Let $f: X \rightarrow Y$ be a continuous map of topological spaces. Prove that the functor $f_*: \mathbf{Ab}_X \rightarrow \mathbf{Ab}_Y$ is left exact.
- (ii) Suppose $f^{-1}: \mathbf{Ab}_Y \rightarrow \mathbf{Ab}_X$ is exact. Prove that for an injective sheaf \mathcal{S} on X the push-forward $f_*\mathcal{S}$ is an injective sheaf on Y . [*Hint:* Use that f^{-1} is left adjoint to f_* .]
- (iii) Let Z be a subset of a topological space Y . Let $i: Z \hookrightarrow Y$ be the inclusion map. Prove that $i^{-1}: \mathbf{Ab}_Y \rightarrow \mathbf{Ab}_Z$ is an exact functor. If Z is closed in Y , prove that also $i_*: \mathbf{Ab}_Z \rightarrow \mathbf{Ab}_Y$ is exact. [*Hint:* This is something you can check on stalks.]

Note: You will need the above results in the next exercises!

Remark: Let U be an open subset of a topological space Y . Let $j: U \hookrightarrow Y$ be the inclusion map. Then j_* is not exact, in general.

Exercise 2.

- (i) Prove that for any continuous map $f: X \rightarrow Y$ and any $i \geq 0$, there exist morphisms of functors

$$P_f^i: H^i(Y, f_*-) \rightarrow H^i(X, -)$$

that have the following properties:

- (a) For $i = 0$ we get the canonical isomorphisms $\Gamma(Y, f_*-) \xrightarrow{\sim} \Gamma(X, -)$.
- (b) If $f = \text{id}_X$ then P_f^i is the identity for all i .
- (c) If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous, $P_{g \circ f}^i = P_f^i \circ P_g^i$ for all i . (Note that this makes sense.)
- (d) Let $f: X \rightarrow Y$ be continuous, and suppose

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \tag{*}$$

is a short exact sequence on X . Let $\mathcal{A} \subset f_*\mathcal{F}''$ be the image of the morphism $f_*\mathcal{F} \rightarrow f_*\mathcal{F}''$, and note that by (i) of Exercise 1 the sequence

$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow \mathcal{A} \longrightarrow 0 \quad (\#)$$

is again short exact. Then we have a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^i(Y, f_*\mathcal{F}) & \longrightarrow & H^i(Y, \mathcal{A}) & \xrightarrow{\partial} & H^{i+1}(Y, f_*\mathcal{F}') & \longrightarrow & \dots \\ & & P_f^i(\mathcal{F}) \downarrow & & \downarrow Q_f^i & & \downarrow P_f^{i+1}(\mathcal{F}') & & \\ \dots & \longrightarrow & H^i(X, \mathcal{F}) & \longrightarrow & H^i(X, \mathcal{F}'') & \xrightarrow{\partial} & H^{i+1}(X, \mathcal{F}') & \longrightarrow & \dots \end{array}$$

where the horizontal lines are the long exact sequences associated to the short exact sequences $(\#)$ and $(*)$, respectively, and where the map $Q_f^i: H^i(Y, \mathcal{A}) \rightarrow H^i(X, \mathcal{F}'')$ is defined as the composition of the map $H^i(Y, \mathcal{A}) \rightarrow H^i(Y, f_*\mathcal{F}'')$ given by the inclusion $\mathcal{A} \hookrightarrow f_*\mathcal{F}''$ and the map $P_f^i(\mathcal{F}'')$.

Note: Saying that P^i is a morphism of functors means that for every sheaf \mathcal{F} on X we have a homomorphism

$$P^i(\mathcal{F}): H^i(Y, f_*\mathcal{F}) \rightarrow H^i(X, \mathcal{F})$$

such that for every homomorphism of sheaves $\varphi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ on X we have a commutative diagram

$$\begin{array}{ccc} H^i(Y, f_*\mathcal{F}_1) & \xrightarrow{P^i(\mathcal{F}_1)} & H^i(X, \mathcal{F}_1) \\ H^i(f_*\varphi) \downarrow & & \downarrow H^i(\varphi) \\ H^i(Y, f_*\mathcal{F}_2) & \xrightarrow{P^i(\mathcal{F}_2)} & H^i(X, \mathcal{F}_2) \end{array}$$

where we write $f_*\varphi: f_*\mathcal{F}_1 \rightarrow f_*\mathcal{F}_2$ for the push-forward of φ via f .

Hint for this exercise: Given a sheaf \mathcal{F} on X , choose an injective resolution \mathcal{I}^\bullet . Note that $f_*\mathcal{I}^\bullet$ is again a complex. Now choose an injective resolution \mathcal{J}^\bullet of $f_*\mathcal{F}$, and construct a morphism of complexes $\mathcal{J}^\bullet \rightarrow f_*\mathcal{I}^\bullet$. Finally remark that $\Gamma(Y, f_*\mathcal{G}) = \Gamma(X, \mathcal{G})$ for any sheaf \mathcal{G} on X .

Remark: Given a continuous map $f: X \rightarrow Y$ and a sheaf \mathcal{G} on Y , we have a canonical homomorphism $\gamma: \mathcal{G} \rightarrow f_*f^{-1}\mathcal{G}$, by adjunction of the functors f^{-1} and f_* . In particular, we get functorial homomorphisms

$$P_f^{i,i}(\mathcal{G}): H^i(Y, \mathcal{G}) \rightarrow H^i(X, f^{-1}\mathcal{G})$$

by composing $H^i(\gamma)$ with $P_f^i(f^{-1}\mathcal{G})$. The functors P_f^i (and the functors $P_f^{i,i}$ obtained from them) give the cohomology theory of sheaves a (contravariant) functorial character.

- (ii) Let Z be a closed subset of a topological space Y . Let $i: Z \hookrightarrow Y$ be the inclusion map. Prove that the maps $P_i^n(\mathcal{F}): H^n(Y, i_*\mathcal{F}) \rightarrow H^n(Z, \mathcal{F})$ are isomorphisms for all \mathcal{F} and n . [*Hint:* Use the results of Exercise 1.]
- (iii) Let $X := \mathbb{R}^2$ with the Euclidean topology. Let $O := (0, 0) \in X$, and take $U := X \setminus \{O\}$. Let $j: U \hookrightarrow X$ be the inclusion map. Prove that the natural map $\mathbb{Q}_X \rightarrow j_*\mathbb{Q}_U$ is an

isomorphism. Conclude that for an inclusion $j: U \hookrightarrow X$ of an open subset, the maps $P_i^n(\mathcal{F}): H^n(Y, j_*\mathcal{F}) \rightarrow H^n(U, \mathcal{F})$ are not, in general, isomorphisms.

- (iv) Let \tilde{C} be a compact connected topological surface of genus g . (A 2-sphere, a doughnut, etc.) Let P and P' be two distinct points of \tilde{C} , and define C as the quotient of \tilde{C} obtained by identifying the points P and P' . (In other words, we glue P to P' and otherwise leave \tilde{C} unaltered.) Let $\pi: \tilde{C} \rightarrow C$ be the quotient map, and let $Q := \pi(P) = \pi(P') \in C$. Write $i: \{Q\} \hookrightarrow C$ for the inclusion map. Prove that we have a short exact sequence

$$0 \longrightarrow \mathbb{Q}_C \longrightarrow \pi_*\mathbb{Q}_{\tilde{C}} \longrightarrow i_*\mathbb{Q}_{\{Q\}} \longrightarrow 0.$$

Now prove that

$$H^0(C, \mathbb{Q}) \cong H^0(\tilde{C}, \mathbb{Q}) \cong \mathbb{Q}, \quad H^2(C, \mathbb{Q}) \cong H^2(\tilde{C}, \mathbb{Q}) \cong \mathbb{Q},$$

and

$$\dim H^1(C, \mathbb{Q}) = 1 + \dim H^1(\tilde{C}, \mathbb{Q}).$$

Exercise 3. In this exercise we work on $\mathbb{P} := \mathbb{P}^r(\mathbb{C})$ with $r > 1$. Let $\mathcal{O}_{\mathbb{P}}$ be the structure sheaf of \mathbb{P} . On \mathbb{P} we have a unique line bundle $\mathcal{O}_{\mathbb{P}}(1)$ with $\dim H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) = r + 1$, and in fact any line bundle on \mathbb{P} is of the form

$$\mathcal{O}_{\mathbb{P}}(n) := \begin{cases} \mathcal{O}(1)^{\otimes n} & \text{for } n \geq 0 \\ \mathcal{O}(-n)^\vee & \text{for } n \leq 0, \end{cases}$$

where, for a line bundle \mathcal{L} , we denote by $\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_{\mathbb{P}})$ the dual of \mathcal{L} . Note that $\mathcal{O}_{\mathbb{P}}(m) \otimes \mathcal{O}_{\mathbb{P}}(n) \cong \mathcal{O}_{\mathbb{P}}(m+n)$ for all $m, n \in \mathbb{Z}$.

Let $X \subset \mathbb{P}$ be a complex submanifold defined by an equation $F = 0$, where F is a homogeneous polynomial of degree $d > 0$ in $\mathbb{C}[x_0, x_1, \dots, x_r]$. (We say that X is a hypersurface of degree d .) Let $i: X \hookrightarrow \mathbb{P}$ be the inclusion. Define $\mathcal{O}_X(n) := i^*\mathcal{O}_{\mathbb{P}}(n)$.

You may use in this exercise the following basic facts:

- (A) We know the cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}}(n)$. (See for instance HAG, Chap. III, § 5. Essentially the same calculation is valid in the holomorphic context.) Namely, we have:
- $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) = 0$ if $i \notin \{0, r\}$.
 - $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ has dimension $\binom{n+r}{r}$. (In particular, it is zero for $n < 0$.)
 - $H^r(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ has dimension $\binom{-n-1}{r}$. (In particular, it is zero for $n > -r - 1$.)
- (B) There is an exact sequence of sheaves

$$0 \longrightarrow \Omega_{\mathbb{P}}^1 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1)^{\oplus r+1} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0.$$

- (C) The kernel of the homomorphism $i^\#: \mathcal{O}_{\mathbb{P}} \rightarrow i_*\mathcal{O}_X$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(-d)$, and by consequence we have a short exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}}$ -modules

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0.$$

(D) There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow i^* \Omega_{\mathbb{P}^1}^1 \longrightarrow \Omega_X^1 \longrightarrow 0$$

of sheaves on X .

(E) If \mathcal{F} is a sheaf of $\mathcal{O}_{\mathbb{P}^1}$ -modules on \mathbb{P}^1 then $i_* i^* \mathcal{F} \cong \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^1}} i_* \mathcal{O}_X$.

Remark: While for this exercise you may simply use these facts, it is a good test to see if you actually understand (and perhaps can prove) these facts.

(i) Take $r = 2$, so X is a curve of degree d in $\mathbb{P}^2(\mathbb{C})$. Show that the Hodge numbers of X are given by the following Hodge diamond:

$$\begin{array}{ccc} & & 1 \\ & \binom{d-1}{2} & \binom{d-1}{2} \\ & & 1 \end{array}$$

(ii) Take $r = 3$, so X is a surface of degree d in \mathbb{P}^3 . Show that the Hodge numbers of X are given by the following Hodge diamond:

$$\begin{array}{ccccc} & & & & 1 \\ & & & & \\ & & 0 & & 0 \\ & \binom{d-1}{3} & \binom{2d-1}{3} - 4\binom{d}{3} + 1 & & \binom{d-1}{3} \\ & & 0 & & 0 \\ & & & & 1 \end{array}$$

(iii) Now assume $r \geq 4$. Calculate all Hodge numbers $h^{0,q} := \dim H^q(X, \mathcal{O}_X)$ and $h^{1,q} := \dim H^q(X, \Omega_X^1)$ of X .

Exercise 4. The goal of this exercise is to go through some multilinear algebra, and to apply this to the operators we encounter in the study of compact Kähler manifolds. We begin with some reminders about hermitian forms. All vector spaces in this exercise are assumed to be finite dimensional.

If W is a complex vector space, we write $W^\vee := \text{Hom}_{\mathbb{C}}(W, \mathbb{C})$ for the dual space. For any $n \geq 0$ we have a natural isomorphism $(W^\vee)^{\otimes n} \xrightarrow{\sim} (W^{\otimes n})^\vee$. Further we define

$$\sigma W := W \otimes_{\mathbb{C}, \sigma} \mathbb{C},$$

where the map $\sigma: \mathbb{C} \rightarrow \mathbb{C}$ is complex conjugation. There is a natural isomorphism $\sigma(\sigma W) \cong W$; we shall take this as an identification.

There are natural isomorphisms

$$(\sigma W)^\vee \xrightarrow{\sim} \sigma(W^\vee) \quad \text{and} \quad \wedge^n(\sigma W) \xrightarrow{\sim} \sigma(\wedge^n W).$$

In particular, we may use the notation ${}^\sigma W^\vee$ without ambiguity.

Let $h: W \times W \rightarrow \mathbb{C}$ be a hermitian form. Recall that such a form can also be written as a \mathbb{C} -linear map $h: W \otimes_{\mathbb{C}} {}^\sigma W \rightarrow \mathbb{C}$. Define

$$\alpha_h: W \rightarrow {}^\sigma W^\vee$$

to be the \mathbb{C} -linear map given by $\alpha_h(w)(x) = h(w, x)$. We say that h is non-degenerate if α_h is an isomorphism. (In particular this is the case if h is positive definite.) We call α_h the adjoint of h .

If we have hermitian forms on some vector spaces, we can form the induced hermitian forms on direct sums, tensor products, etc. Some examples:

- (1) If h_1 is a hermitian form on W_1 and h_2 is a hermitian form on W_2 then we can define $h_1 \otimes h_2$ on $W_1 \otimes W_2$ by the rule

$$(h_1 \otimes h_2)(w_1 \otimes w_2, w'_1 \otimes w'_2) = h_1(w_1, w'_1) \cdot h_2(w_2, w'_2).$$

The adjoint form $(W_1 \otimes W_2) \rightarrow \sigma(W_1 \otimes W_2)^\vee \cong \sigma W_1^\vee \otimes \sigma W_2^\vee$ is $\alpha_{h_1} \otimes \alpha_{h_2}$.

- (2) If h is a hermitian form on W then we get a hermitian form \bar{h} on ${}^\sigma W$ by $\bar{h}(w \otimes 1, w' \otimes 1) = \overline{h(w, w')}$. The adjoint is the map ${}^\sigma \alpha_h$.
- (3) If h is a *nondegenerate* hermitian form on W then we can define a hermitian form ${}^\sigma h^\vee$ on ${}^\sigma W^\vee$ by the requirement that the adjoint of ${}^\sigma h^\vee$ is the inverse of α_h . If $\{e_1, \dots, e_d\}$ is an orthonormal basis for W with respect to the form h and if $\{e_1^\vee, \dots, e_d^\vee\}$ is the dual basis for W^\vee then $\{e_1^\vee \otimes 1, \dots, e_d^\vee \otimes 1\}$ is an orthonormal basis for ${}^\sigma W^\vee$ with regard to the form ${}^\sigma h^\vee$.
- (4) Combining (2) and (3): If h is a nondegenerate hermitian form on W then we get an induced hermitian form h^\vee on W^\vee .

Slightly more subtle is the procedure to obtain an induced hermitian form on the exterior powers of W . Recall that $\wedge^n W$ is a quotient of $W^{\otimes n}$; more precisely: In the tensor algebra $T(W) := \bigoplus_{m \geq 0} W^{\otimes m}$ we consider the 2-sided ideal $A(W) \subset T(W)$ that is generated by all elements $w \otimes w$. Then $A(W)$ is a graded ideal, which means that $A(W) = \bigoplus A^m(W)$ with $A^m(W) := A(W) \cap T^{\otimes m}$, and by definition $\wedge^n W := W^{\otimes n} / A^n(W)$.

If $\{e_1, \dots, e_d\}$ is a basis for W and $I \subset \{1, \dots, d\}$ is a subset with n elements, say $I = \{i_1, \dots, i_n\}$ with $i_1 < i_2 < \dots < i_n$, write $f_I := e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n} \in \wedge^n W$. Then the elements f_I form a basis for $\wedge^n W$. (In particular, $\wedge^n W$ has dimension $\binom{d}{n}$.)

- (i) Consider the map

$$W^{\otimes n} \times W^{\otimes n} \rightarrow \mathbb{C} \quad \text{given by} \quad (w_1 \otimes \dots \otimes w_n, y_1 \otimes \dots \otimes y_n) \mapsto \det(h(w_i, y_j)).$$

This map is \mathbb{C} -linear in the first variable and \mathbb{C} -antilinear in the second variable; hence it defines a \mathbb{C} -linear map

$$\tilde{h}_n: W^{\otimes n} \otimes {}^\sigma W^{\otimes n} \rightarrow \mathbb{C}.$$

Show that, with notation as above, \tilde{h}_n is zero on $A^n(W) \otimes \sigma W^{\otimes n}$ and on $W^{\otimes n} \otimes A^n(\sigma W)$, and conclude that it induces a \mathbb{C} -linear map

$$h_n: \wedge^n W \otimes \sigma(\wedge^n W) \rightarrow \mathbb{C}.$$

If $\{e_1, \dots, e_d\}$ is an orthonormal basis for W with respect to the form h and if we define a basis $\{f_I\}$ for $\wedge^n W$ as above, show that we have

$$h_n(f_I, f_J) = \begin{cases} 0 & \text{if } I \neq J; \\ 1 & \text{if } I = J. \end{cases}$$

(Here I and J are subsets of $\{1, 2, \dots, d\}$ with $\#I = \#J = n$.)

From now on we fix a \mathbb{C} -vector space T of dimension d with a positive definite hermitian form h on it. Let $V := T^\vee$ and define

$$A^{p,q} := \left(\wedge^p(T^\vee) \right) \otimes_{\mathbb{C}} \left(\wedge^q(\sigma T^\vee) \right) = (\wedge^p V) \otimes_{\mathbb{C}} (\wedge^q \sigma V).$$

We have natural product maps $A^{p,q} \otimes A^{p',q'} \rightarrow A^{p+p',q+q'}$, which we denote by $a \otimes a' \mapsto a \wedge a'$.

Following the procedures discussed above, the form h on T induces hermitian forms on the spaces $A^{p,q}$. By abuse of notation, we shall again write h for these induced hermitian forms.

Note that $A^{d,d}$ is 1-dimensional. Let $\omega \in A^{1,1}$ and $v \in A^{d,d}$ be the elements that under the natural isomorphisms

$$A^{1,1} \xrightarrow{\sim} (T \otimes \sigma T)^\vee \quad \text{and} \quad A^{d,d} \xrightarrow{\sim} ((\wedge^d T) \otimes \sigma(\wedge^d T))^\vee$$

correspond to the hermitian forms $h: T \otimes \sigma T \rightarrow \mathbb{C}$ and $h_d: (\wedge^d T) \otimes \sigma(\wedge^d T) \rightarrow \mathbb{C}$.

(ii) Prove that there is a complex number b such that

$$v = b \cdot \underbrace{(\omega \wedge \dots \wedge \omega)}_d$$

and calculate this number.

(iii) Prove that there is a unique isomorphism $*$: $A^{p,q} \xrightarrow{\sim} A^{d-p,d-q}$ such that

$$\beta \wedge * \gamma = h(\beta, \gamma) \cdot v$$

for all $\beta, \gamma \in A^{p,q}$. Show that the composition

$$A^{p,q} \xrightarrow{*} A^{d-p,d-q} \xrightarrow{*} A^{p,q}$$

is the scalar multiplication by some complex number, and calculate this number.

(iv) Define

$$L: A^{p,q} \rightarrow A^{p+1,q+1}$$

to be the map given by $L(\beta) = \omega \wedge \beta$. Define

$$L^*: A^{p+1,q+1} \rightarrow A^{p,q}$$

to be the adjoint of L with regard to the hermitian forms on the spaces $A^{p,q}$. In other words, for all $\beta \in A^{p,q}$ and $\gamma \in A^{p+1,q+1}$ we have

$$h(\beta, L^*(\gamma)) = h(L(\beta), \gamma).$$

Show that $L^* = *^{-1} \circ L \circ *$. Further, prove that $[L, L^*]$ acts as scalar multiplication by $p + q - d$ on $A^{p,q}$.

Let now X be a d -dimensional compact Kähler manifold with Kähler metric h . Let $\omega \in \Gamma(X, \mathcal{A}_X^2) \cap \Gamma(X, \mathcal{A}^{1,1})$ be the 2-form associated to h , and let $v_X \in \Gamma(X, \mathcal{A}_X^{2d})$ be the volume form of X . Define the Lefschetz operator $L: \Gamma(X, \mathcal{A}_X^n) \rightarrow \Gamma(X, \mathcal{A}_X^{n+2})$ by $L(\beta) = \omega \wedge \beta$.

(v) Define an operator H on $\bigoplus_{n=0}^{2d} \Gamma(X, \mathcal{A}_X^n)$ by letting H act on $\Gamma(X, \mathcal{A}_X^n)$ as multiplication by $n - d$. Prove that there exists an operator L^* such that we get a representation of the Lie algebra \mathfrak{sl}_2 on $\bigoplus_{n=0}^{2d} \Gamma(X, \mathcal{A}_X^n)$ by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto L^*, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto L.$$

Also prove that the \mathbb{C} -linear extension of L^* maps $\Gamma(X, \mathcal{A}^{p,q})$ to $\Gamma(X, \mathcal{A}^{p-1,q-1})$.