(1) Let $\sigma\colon \mathbb{C} \to \mathbb{C}$ be complex conjugation. If W is a $\mathbb{C}$-vector space then we define

$$^\sigma W := W \otimes_{\mathbb{C},\sigma} \mathbb{C}\,.$$

Concretely this means that $^\sigma W$ is isomorphic to W as a *real* vector space, but we let a complex number z act on $^\sigma W$ as multiplication by $\bar{z}$.

Note that $^\sigma\bigl(^\sigma W\bigr)$ is canonically isomorphic to W. Further we have $^\sigma\bigl(W_1 \otimes_{\mathbb{C}} W_2\bigr) \cong {}^\sigma W_1 \otimes_{\mathbb{C}} {}^\sigma W_2$.

A $\mathbb{C}$-linear map $f\colon W_1 \to W_2$ induces a $\mathbb{C}$-linear map $^\sigma f = (f \otimes \mathrm{id})\colon {}^\sigma W_1 \to {}^\sigma W_2$.

(2) Definition. Let W be a complex vector space. Then a *hermitian form* on W is a map

$$h\colon W \times W \to \mathbb{C}$$

that is $\mathbb{C}$-linear in the first variable, $\mathbb{C}$-antilinear in the second variable, and such that $h(w', w) = \overline{h(w, w')}$ for all w, $w' \in W$.

Equivalent: A hermitian form is a $\mathbb{C}$-linear map $h\colon W \otimes_{\mathbb{C}} {}^\sigma W \to \mathbb{C}$ such that, writing $\mathrm{sw}\colon W \otimes {}^\sigma W \to {}^\sigma W \otimes W$ for the map that switches the factors, the diagram

$$\begin{tikzcd}
W \otimes_{\mathbb{C}} {}^\sigma W \arrow[r, "h"] \arrow[d, "\mathrm{sw}"'] & \mathbb{C} \arrow[d, "\sigma"] \\
{}^\sigma W \otimes_{\mathbb{C}} W \arrow[r, "^\sigma h"] & {}^\sigma\mathbb{C}
\end{tikzcd}$$

is commutative.

If h is a hermitian form on W then $h(w, w) \in \mathbb{R}$ for all w. We call h positive definite if $h(w, w) > 0$ for all $0 \neq w \in W$.

(3) Let h be a hermitian form on W. Easy to check:
(i) $S := \mathrm{Re}(h)\colon W \times W \to \mathbb{R}$ is a symmetric $\mathbb{R}$-bilinear form;
(ii) $A := \mathrm{Im}(h)\colon W \times W \to \mathbb{R}$ is an antisymmetric $\mathbb{R}$-bilinear form;
(iii) $A(iw, iw') = A(w, w')$ and $S(iw, iw') = S(w, w')$ for all w, $w' \in W$;
(iv) $A(w, w') = -S(iw, w')$.

Conclusion: to give a hermitian form h on W is equivalent to giving a symmetric $\mathbb{R}$-bilinear form S on W (or, more precisely: on the underlying real vector space) such that $S(iw, iw') = S(w, w')$ for all w, $w' \in W$. Correspondence: To h we associate $S := \mathrm{Re}(h)$; to S we associate $h = S + iA$ with A given by (iv).

Note that S is positive definite if and only if h is.

(4) What we did in (1)–(3) can also be done in families. For this, consider a complex manifold $(X, \mathscr{O}_X)$ of dimension n, and let $(X, \mathscr{A}_X)$ be the associated C^∞-manifold.

Write $\mathscr{E}$ for the real tangent bundle (a locally free $\mathscr{A}_X$-module of rank $2n$) and T for the holomorphic tangent bundle (a locally free $\mathscr{O}_X$-module of rank n). Further, let us write

$$C^\infty T := T \otimes_{\mathscr{O}_X} \mathscr{A}_{X,\mathbb{C}}$$

for the complex C^∞-bundle associated to T.

Let $\sigma\colon \mathscr{A}_{X,\mathbb{C}} \to \mathscr{A}_{X,\mathbb{C}}$ be complex conjugation. If $\mathscr{W}$ is a complex C^∞-bundle on X (i.e., a locally free $\mathscr{A}_{X,\mathbb{C}}$-bundle) then we define

$${}^\sigma\mathscr{W} := \mathscr{W} \otimes_{\mathscr{A}_{X,\mathbb{C}},\sigma} \mathscr{A}_{X,\mathbb{C}}\,.$$

If W is the fibre of $\mathscr{W}$ over a point $x \in X$ then the fibre of ${}^\sigma\mathscr{W}$ is ${}^\sigma W$.

(5) Definition. A hermitian metric on X is a homomorphism of $\mathscr{A}_{X,\mathbb{C}}$-modules

$$h\colon C^\infty T \otimes_{\mathscr{A}_{X,\mathbb{C}}} {}^\sigma(C^\infty T) \to \mathscr{A}_{X,\mathbb{C}}$$

that on each fibre gives a positive definite hermitian form.

(6) As we have seen, we have an isomorphism of $\mathscr{A}_{X,\mathbb{C}}$-modules

$$\mathscr{E}_\mathbb{C} \cong C^\infty T \oplus \overline{C^\infty T}\,, \tag{6.1}$$

where $\overline{}\colon \mathscr{E}_\mathbb{C} \to \mathscr{E}_\mathbb{C}$ is complex conjugation. Note:

- — complex conjugation on $\mathscr{E}_\mathbb{C}$ is of course not $\mathscr{A}_{X,\mathbb{C}}$-linear, only $\mathscr{A}_X$-linear;
- — $\overline{C^\infty T}$, which is by definition the image of $C^\infty T \subset \mathscr{E}_\mathbb{C}$ under this map, is again a complex subbundle of $\mathscr{E}_\mathbb{C}$, i.e., an $\mathscr{A}_{X,\mathbb{C}}$-submodule;
- — the map

  $$\overline{}\colon C^\infty T \to \overline{C^\infty T}\,,$$

  which by construction is bijective, gives an isomorphism of $\mathscr{A}_{X,\mathbb{C}}$-modules

  $${}^\sigma(C^\infty T) \xrightarrow{\sim} \overline{C^\infty T}\,. \tag{6.2}$$

Consider the composition

$$\mathscr{E} \hookrightarrow \mathscr{E}_\mathbb{C} \xrightarrow{\text{pr}} C^\infty T\,.$$

This gives an isomorphism $\mathscr{E} \xrightarrow{\sim} C^\infty T$ of *real* bundles, i.e., of $\mathscr{A}_X$-modules. We now define

$$J\colon \mathscr{E} \to \mathscr{E}$$

as the $\mathscr{A}_X$-linear endomorphism that under this isomorphism corresponds to the multiplication by i on $C^\infty T$. In particular, $J^2 = -\mathrm{id}$.

By what was explained in (3), to give a hermitian metric h on X is equivalent to giving a Riemann metric S (which is a symmetric homomorphism $\mathscr{E} \otimes_{\mathscr{A}_X} \mathscr{E} \to \mathscr{A}_X$) with the additional property that $S(Je, Je') = S(e, e')$ for all local sections e, e' of $\mathscr{E}$.

(7) Let $h\colon C^\infty T \otimes_{\mathscr{A}_{X,\mathbb{C}}} {}^\sigma C^\infty T \to \mathscr{A}_{X,\mathbb{C}}$ be a hermitian metric on X, and consider the corresponding Riemann metric $S := \mathrm{Re}(h)\colon \mathscr{E} \otimes_{\mathscr{A}_X} \mathscr{E} \to \mathscr{A}_X$, and the imaginary part $\beta := \mathrm{Im}(h)\colon \mathscr{E} \otimes_{\mathscr{A}_X} \mathscr{E} \to \mathscr{A}_X$, which is an antisymmetric form. Define

$$\omega \in \Gamma(X, \mathscr{A}^2) = \Gamma\bigl(X, \wedge^2(\mathscr{E}^\vee)\bigr)$$

as the image of $(-1/2) \cdot \beta \in \Gamma\bigl(X, (\mathscr{E}^{\otimes 2})^\vee\bigr)$ under the natural map

$$\Gamma\bigl(X, (\mathscr{E}^{\otimes 2})^\vee\bigr) = \Gamma\bigl(X, \mathscr{A}^1 \otimes \mathscr{A}^1\bigr) \longrightarrow \Gamma\bigl(X, \mathscr{A}^2\bigr)\,.$$

So ω is a real global 2-form on X.

Caution: In the literature one encounters several normalization constants in the definition of ω. I here take a factor $-1/2$ but in some books a different factor is used.

(8) Lemma. Consider the bilinear form

$$\beta_{\mathbb{C}}\colon \mathscr{E}_{\mathbb{C}} \otimes_{\mathscr{A}_{X,\mathbb{C}}} \mathscr{E}_{\mathbb{C}} \to \mathscr{A}_{X,\mathbb{C}}$$

obtained as the $\mathbb{C}$-linear extension of the form $\mathrm{Im}(h)$ above. Write

$$\mathscr{E}_{\mathbb{C}} \otimes_{\mathscr{A}_{X,\mathbb{C}}} \mathscr{E}_{\mathbb{C}} = (C^\infty T \otimes C^\infty T) \oplus (C^\infty T \otimes \overline{C^\infty T}) \oplus (\overline{C^\infty T} \otimes C^\infty T) \oplus (\overline{C^\infty T} \otimes \overline{C^\infty T})\,.$$

- (i) We have $\beta_{\mathbb{C}} = 0$ on $C^\infty T \otimes C^\infty T$ and on $\overline{C^\infty T} \otimes \overline{C^\infty T}$.
- (ii) Identifying $\overline{C^\infty T}$ with ${}^\sigma(C^\infty T)$ as in (6.2), the restriction of $\beta_{\mathbb{C}}$ to $C^\infty T \otimes \overline{C^\infty T}$ is just $(-i/2)$ times the form h.

(9) An example. Take $X = \mathbb{C}^n$, on which we use $(z_1, \ldots, z_n)$ as coordinates. We can globally trivialize the holomorphic tangent bundle, i.e., $T = X \times \mathbb{C}^n$, with coordinates $\bigl((z_1, \ldots, z_n), (\partial_{z_1}, \ldots, \partial_{z_n})\bigr)$. For h we take the hermitian metric that on each tangent space $T_x \cong \mathbb{C}^n$ is the hermitian form $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by $\bigl((a_1, \ldots, a_n), (b_1, \ldots, b_n)\bigr) \mapsto a_1\overline{b_1} + \cdots + a_n\overline{b_n}$. Put differently, $h\colon C^\infty T \otimes \overline{C^\infty T} \to \mathscr{A}_{X,\mathbb{C}}$ is given by

$$h(\partial_{z_i} \otimes \partial_{\bar{z}_j}) = \delta_{i,j} \qquad \text{(Kronecker delta).}$$

We want to view the imaginary part of h as an antisymmetric bilinear form on the real tangent bundle $\mathscr{E}$. Writing $z_j = x_j + iy_j$, the ∂_{x_j} and ∂_{y_j} give a frame for $\mathscr{E}$. In the decomposition (6.1) the subbundle $C^\infty T$ is the one spanned by the ∂_{z_j}, and $\overline{C^\infty T}$ is spanned by the $\partial_{\bar{z}_j}$.

We have $dz_j = dx_j + idy_j$ and $d\overline{z}_j = dx_j - idy_j$. Dually this gives

$$\partial_{z_j} = \frac{\partial_{x_j} - i\partial_{y_j}}{2} \quad \text{and} \quad \partial_{\overline{z}_j} = \frac{\partial_{x_j} + i\partial_{y_j}}{2}\,.$$

So

$$\partial_{x_j} = \partial_{z_j} + \partial_{\overline{z}_j} \quad \text{and} \quad \partial_{y_j} = i\partial_{z_j} - i\partial_{\overline{z}_j}$$

Our identification $\mathscr{E} \xrightarrow{\sim} C^\infty T$ (first embedding $\mathscr{E}$ into $\mathscr{E}_{\mathbb{C}}$ then projecting down to $C^\infty T$), is therefore given by $\partial_{x_j} \mapsto \partial_{z_j}$ and $\partial_{y_j} \mapsto i\partial_{z_j}$.

We then find that the form $\beta = \operatorname{Im}(h)\colon \mathscr{E} \otimes_{\mathscr{A}_X} \mathscr{E} \to \mathscr{A}_X$ is given by

$$\beta(\partial_{x_j}, \partial_{x_k}) = \beta(\partial_{y_j}, \partial_{y_k}) = 0 \quad \text{and} \quad \beta(\partial_{x_j}, \partial_{y_k}) = -\beta(\partial_{y_j}, \partial_{x_k}) = -\delta_{j,k}\,.$$

Now we extend this form $\mathbb{C}$-linearly to an alternating form $\beta_{\mathbb{C}}\colon \mathscr{E}_{\mathbb{C}} \otimes \mathscr{E}_{\mathbb{C}} \to \mathscr{A}_{X,\mathbb{C}}$, and we calculate:

(i) On $C^\infty T \otimes C^\infty T$:

$$\begin{aligned}\beta_{\mathbb{C}}(\partial_{z_j} \otimes \partial_{z_k}) &= (1/4)\cdot\beta_{\mathbb{C}}\big((\partial_{x_j} - i\partial_{y_j}) \otimes (\partial_{x_k} - i\partial_{y_k})\big)\\ &= (1/4)\cdot\{\beta(\partial_{x_j} \otimes -i\partial_{y_k}) + \beta(-i\partial_{y_j} \otimes \partial_{x_k})\\ &= (1/4)\cdot\{i\delta_{j,k} - i\delta_{j,k}\} = 0\,.\end{aligned}$$

In a similar way we see that $\beta_{\mathbb{C}} = 0$ on $\overline{C^\infty T} \otimes \overline{C^\infty T}$.

(ii) On $C^\infty T \otimes \overline{C^\infty T}$ and $\overline{C^\infty T} \otimes C^\infty T$:

$$\begin{aligned}\beta_{\mathbb{C}}(\partial_{z_j} \otimes \partial_{\overline{z}_k}) &= (1/4)\cdot\beta_{\mathbb{C}}\big((\partial_{x_j} - i\partial_{y_j}) \otimes (\partial_{x_k} + i\partial_{y_k})\big)\\ &= (1/4)\cdot\{\beta(\partial_{x_j} \otimes i\partial_{y_k}) + \beta(-i\partial_{y_j} \otimes \partial_{x_k})\\ &= (1/4)\cdot\{-i\delta_{j,k} - i\delta_{j,k}\} = (-i/2)\cdot\delta_{j,k}\,,\end{aligned}$$

and

$$\begin{aligned}\beta_{\mathbb{C}}(\partial_{\overline{z}_j} \otimes \partial_{z_k}) &= (1/4)\cdot\beta_{\mathbb{C}}\big((\partial_{x_j} + i\partial_{y_j}) \otimes (\partial_{x_k} - i\partial_{y_k})\big)\\ &= (1/4)\cdot\{\beta(\partial_{x_j} \otimes -i\partial_{y_k}) + \beta(i\partial_{y_j} \otimes \partial_{x_k})\\ &= (1/4)\cdot\{i\delta_{j,k} + i\delta_{j,k}\} = (i/2)\cdot\delta_{j,k}\,.\end{aligned}$$

In tensor notation, this means that $\beta_{\mathbb{C}}$, which is an element of $\mathscr{E}^\vee \otimes \mathscr{E}^\vee = \mathscr{A}^1 \otimes \mathscr{A}^1$, is the element

$$-\sum_{j=1}^n dx_j \otimes dy_j + \sum_{j=1}^n dy_j \otimes dx_j = (-i/2)\sum_{j=1}^n dz_j \otimes d\overline{z}_j + (i/2)\cdot\sum_{j=1}^n d\overline{z}_j \otimes dz_j\,.$$

The image of this element in $\mathscr{A}^2$ is

$$-2\sum_{j=1}^n dx_j \wedge dy_j = -i\cdot\sum_{j=1}^n dz_j \wedge d\overline{z}_j\,.$$

Hence:

$$\omega = \sum_{j=1}^n dx_j \wedge dy_j = \frac{i}{2}\cdot\sum_{j=1}^n dz_j \wedge d\overline{z}_j\,.$$