Algebraic Number Theory. Additional exercises, 4 Sept. 2009.

1. Let K be a field, and let $f \in K[X]$ be an *irreducible* polynomial of degree $n > 0$ with coefficients in K. Let L be an algebraically closed field that contains K.

(i) Assume $\mathrm{char}(K) = 0$. Show that all roots of f in L are simple (i.e., have multiplicity 1). In particular, f has exactly n distinct roots in L.

(ii) Show, by means of an example, that the conclusion of (i) is not, in general, valid over fields of characteristic p.

(iii) Generalise (i) as follows: If K is a perfect field, then all roots of f in L are simple. [If necessary, first look up the definition of a perfect field.]

2. Let $K \subset L$ be an extension of fields. Let f, $g \in K[X]$. Let $h \in K[X]$ be a gcd of f and g. Is it true that h is still a gcd of f and g in the bigger ring $L[X]$? If "yes", prove this; if "no", show this by means of an example.

3. Let F be a number field. Write $\mu_F \subset F^*$ for the subgroup of roots of unity, i.e.,

$$\mu_F = \left\{ x \in F^* \;\middle|\; x^m = 1 \quad \text{for some integer } m > 0 \right\}.$$

Prove that μ_F is a finite cyclic group.