

Corrections to: A.S. Troelstra (editor), Metamathematical Investigation of Intuitionistic Arithmetic and Analysis June 22, 2009

This text contains corrections and additions to “Metamathematical Investigation of Intuitionistic Arithmetic and Analysis” which appeared in 1973 as number 344 in the “Springer Lecture Notes in Mathematics” series.

The original edition ran out of print many years ago. A small run of a corrected edition has been produced in 1993 as a report (X-93-05) of the Institute of Logic, Language and Information (ILLC) of the University of Amsterdam, but this is now also out of print.

In the mean time Springer Verlag has decided to make the volumes of the series Lecture Notes in Mathematics which are out of print available as “publish-on-demand” editions.

The text has been typeset in Latex. Wavy underlining in the original text is now interpreted as boldface, underlining as italics. Double wavy underlining has been interpreted by a sans serif font. However, we have retained double underlining and did not replace it by Fraktur.

A first list of Errata appeared in 1974 as a report of the Mathematical Institute of the University of Amsterdam; many more errata have been discovered since then. In particular I should like to thank Marc Bezem, Susumu Hayashi, Jane Bridge Kister, Jaap van Oosten and Jeffery Zucker.

The counting of lines includes the lines in displayed formulas; for indications, e.g. a name or a number for a group of displayed lines, which are between lines so to speak, an ad hoc indication will be chosen.

Underlining in the original text has been rendered as italics in these correction; double underlining has been rendered as such, but a double wavy underlining corresponds to a sans serif letter in these corrections.

XIII Add below the summary of §6:

- §7 *Applications: proof theoretic closure properties* 258
List of rules (3.7.1) — closure under ED, DP, CR₀, ECR₀, ED', ACR, IPR'^ω (3.7.2–5) — closure under CR (3.7.6) — closure under ECR₁ (3.7.7–8) — extensions to analysis (3.7.9)
- 6 In 1.1.7, interchange “ $\forall I_r$ ” and “ $\forall I_1$ ”.
- 7₁₃ Read “ $\exists E$ ” for “ $\exists I$ ”.
- 8⁵ Read “essentially”.
- 8 In the first proof tree, “ $B \rightarrow \lambda$ ” should be “ $B \rightarrow \lambda (2)$ ” and “(2)” should be repeated at the lowest horizontal line.
- 16¹³ Read “ A ” for “ F ”.
- 18 In 1.3.3 the axiom

$$x_i = x'_i \rightarrow \phi(x_1, \dots, x_i, \dots, x_n) = \phi(x_1, \dots, x'_i, \dots, x_n)$$

(for any n -ary function constant ϕ , $1 \leq i \leq n$) can be replaced by the corresponding axiom for S only:

$$x = y \rightarrow Sx = Sy,$$

since the general case can be established by induction (since all ϕ except S are introduced by schemas for primitive recursive functions).

18₁₁ Read “*Defining*” for “*Definining*”.

19 Add at the bottom of the page rules expressing the functional character of the F_k :

$$\frac{F_k t_1 \dots t_{n-1} t_n \quad F_k t_1 \dots t_{n-1} t'_n}{t_n = t'_n}.$$

20⁸ Replace “ F'_{k_m} ” by “ F_{k_m} ”.

25 Addition to second paragraph of (D) : “Canonical” essentially means that the arithmetization provably satisfies the “same” inductive closure conditions as the predicate itself.

26³ Read “ $\ulcorner t \urcorner$ ” for “ t ”.

26⁵ Read “that $\ulcorner A(\bar{x}_1, \dots, \bar{x}_n) \urcorner$ stands for ...”.

27⁸ Read “ \simeq ” for “ $=$ ”.

27 Add at the bottom of the page a paragraph:

We follow *Kleene 1952* and use $\Lambda x.t$, t a p-term, to indicate a gödelnumber for t as partial recursive function of x ; if t contains, besides the free variables x, x_1, \dots, x_n , $\Lambda x.t$ is a (primitive) recursive function of x_1, \dots, x_n .

29₂ Read “ y_0 ” for “ y ”.

29₁ Read “ y_1 ” for “ y ”.

33₁₀ Read “We put $\ulcorner \xi t_1 \dots t_n \urcorner = \dots$ ”.

41₂ Read “ $t' \neq x^\sigma$ ” for “ $t' \neq x^\sigma$ ”.

44¹⁷ Replace “slightly ... is” by “seemingly stronger (but in fact equivalent) variant is”.

44¹⁹⁻²¹ Delete “In ... EXT-R'.”

44₆–45⁵ Replace these lines by the following:

The following two propositions are due to M. Bezem (Equivalence of Bar Recursors in the Theory of Functionals of Finite Type, *Archive for Mathematical Logic* 27 (1988), 149–160).

PROPOSITION. The rule EXT-R' is derivable in qf-**WE-HA** ^{ω} .

PROOF. Assume EXT-R, and let $\vdash P \rightarrow s_1 = s_2, \vdash Q[x/t_1]$ Here $s_1 = s_2$ as usual is shorthand for an equation between terms of type 0 $s_1x_1x_2 \dots x_n = s_2x_1x_2 \dots x_n$, where x_1, x_2, \dots, x_n are variables not free in P, s_1, s_2 . Without loss of generality we can assume $P \equiv (t_1 = 0), Q[x] \equiv (t[x] = 0)$ (x not free in P, s_1, s_2). Below we shall abbreviate $t[x/s]$, for arbitrary s , as $t[s]$. So we have

$$(1) \quad \vdash t_1 = 0 \rightarrow s_1 = s_2, \quad \vdash t[s_1] = 0.$$

Define

$$s'_i := \mathbf{R}_\sigma s_i 0^{(\sigma)(0)\sigma}, \quad s_i \in \sigma.$$

Then, with $x \notin \text{FV}(s_i), i = 1, 2$:

$$\vdash x = 0 \rightarrow s'_i x = s_i, \quad \vdash x \neq 0 \rightarrow s'_i x = 0^\sigma.$$

Applying EXT-R to $s'_i 0 = s_i$ yields $\vdash t[s_i] = t[s'_i 0]$. By replacement (i.e. $x = y \rightarrow t[x] = t[y]$) we obtain

$$\vdash t_1 = 0 \rightarrow t[s'_i 0] = t[s'_i t_1].$$

Since also (1) holds, and $t_1 = 0$ is decidable, $\vdash s'_1 t_1 = s'_2 t_1$, so again using EXT-R

$$\vdash t[s'_1 t_1] = t[s'_2 t_1],$$

hence

$$\vdash t_1 = 0 \rightarrow t[s_2] = t[s'_2 0] = t[s'_2 t_1] = t[s'_1 t_1] = t[s'_1 0] = t[s_1] = 0.$$

Q.e.d.

PROPOSITION. The deduction theorem holds for qf-**WE-HA** $^\omega$ + EXT-R', hence also for qf-**WE-HA** $^\omega$.

PROOF. It suffices to prove the deduction theorem for the system with EXT-R', and in this case the deduction theorem is easy.

45_{16,17} Delete these lines.

55₁ Read “ $z < x$ ” for “ $z < z$ ”.

56₁₂ Read “ $Q(x, \underline{v})$ ” for “ $Q(0, \underline{v})$ ”.

56₆ Read “ \top ” for “ T ”.

58₁₂ Read “ \top ” for “ T ”.

59 Add at the bottom: “Cf. also *Luckhardt 73*, pp. 66–67.”

63¹⁰ Delete the first equation.

- 67₁₁ At end of line we add “assumed to be provably linear in **HA**”.
- 71¹⁵ Read “ $\sigma((Q_2 V_n)A) \equiv (Q_1 v_{m+n+1})\sigma(A), \dots$ ”.
- 74 In comparing section 1.9.14 with more recent literature (such as A.S. Troelstra, D. van Dalen, *Constructivism in Mathematics*, Amsterdam 1988), it is to be noted that definedness of a term containing functions and numbers with partial application is here supposed to be defined in the sense of Kleene 1969, that is to say a function applied to an argument is defined if we can sufficiently many values of the function to find its value at the argument; this convention does not agree with the logic of partial terms with its strictness condition.
- 75⁴ Read “ $U(\text{lth}(u)) = y$ ” for “ $U(\text{lth}(u) = y)$ ”.
- 75₅ Read “ x ” for “ α ”.
- 80₁₃ Addition to 1.9.23: “Cf. also Kreisel 1967, page 249, where the role of generalized bar induction in proving the continuous functionals to be a model of bar recursion is mentioned.”
- 83_{18,13} Read “ 0_σ ” for “ 0^σ ”.
- 83₁₀ Read “ $B_\sigma y z u(c * \hat{v})c$ ” for “ $B_\sigma y z u c(c * \hat{v})c$ ”.
- 91¹² Add “ In Friedman *B* it is shown that for r.e. axiomatizable extensions of **HA**, DP implies ED. ”.
- 94_{4,5} Replace by: “In Luckhardt *A* it is shown that the principle is equivalent to M. ”
- 95 Add at the end of 1.11.6:
 It has been noted by C.A. Smorynski that, for theories with decidable prime formulas, IP + M together amount to the principle of the excluded third. E.g. for **HA**, **HA** + IP + M = **HA**^c, which is seen as follows. Assume $A \vee \neg A$ to be proved already in **HA** + IP + M, and consider $\exists x A x$. By M, $\forall x (A x \vee \neg A x) \& \neg \neg \exists x A x \rightarrow \exists x A x$; by the induction hypothesis and IP, this implies $\exists x A x \vee \neg \exists x A x$. Application of propositional operators preserves decidability, and $\forall x A x \leftrightarrow \neg \exists \neg A x$ by the decidability of A , hence $(\forall x A x \vee \neg \forall x A x) \leftrightarrow (\neg \exists x \neg A x \vee \neg \neg \exists x \neg A x) \leftrightarrow \neg \exists x \neg A x \vee \exists x \neg A x$, hence $\forall x A x \vee \neg \forall x A x$.
- 95₄ Replace “ $\alpha x = U z$ ” by “ $\alpha y = U z$ ”.
- 98⁶ Read “ $x_1^{(\sigma)\tau}$ ” for “ $x^{(\sigma)\tau}$ ”.
- 111⁹ Read “ $t_1 \equiv t'_1$ ” for “ $t_1 \equiv t'$ ”.
- 113₄ Read “ **H** $\vdash t = \bar{n}$ ”.
- 114^{21,22} Delete the sentence beginning “For yet another ...”.
- 117₆ Read “and t is” for “and τ is”.
- 119⁴ Read “... representing αn).”.

125₁ Read “ $(\sigma)(\tau)\sigma$ ” for “ $(\sigma)(\tau), \sigma$ ”.

126¹⁰ Read “of” for “if”.

128^{12,13} The open problem has been solved by M. Bezem, in the sense that the two structures are isomorphic: J.S.L. 50 (1985), pp. 359–371.

129 Add between lines 6 and 7:

If we replace in the right hand side of this equivalence A by a predicate letter X , we have the inductive condition $B(X, x, y)$ characterizing A .

132^{18,20} Replace “**I-HA** $^\omega$ ” by “**I-HA** $^\omega + \text{IE}_0$ ”.

133⁴ Delete “, CTM’, CTNF’”.

133^{10,11} Replace final comma by stop in line 10 and delete “CTM’, CTNF’” in line 11.

133_{5,4} Read “CTNF’” for “CTNF”.

141¹⁰ Read “ W_σ^1 ” for “ I_σ^1 ”.

144⁸ Read “(4)” for “(1)”.

147_{1,148}¹ Read “ p_q ” for “ p_k ”.

148¹ Insert before comma “ $\& \{z\}(p_q) = \{z\}(y)$ (since $z \in E(V)$, $y \in V^*$)”.

158¹³ Read “ $t =_e s$ ” for “ $t = s$ ”.

158¹⁶ Read “CTNF” for “CTNF’”.

158₁₁ Read “ $x^2[\Sigma(\Pi x^2)\Pi]$ ”.

158₉ Read “, that $\Sigma(\Pi x_1^2)\Pi$ and”.

158₈ Read “ $\Sigma(\Pi x_2^2)\Pi$ in the model”.

159¹² Read “... which $x^1 \bar{n}_i$ contr $\overline{\alpha n_i}$ has”.

159₁₄ Read “ $t' \in 2$ ” for “ $t' \in z$ ”.

159_{1,2} – 160¹⁻⁴ Replace these lines by:

$$t^2 \alpha = \begin{cases} 0 & \text{if } \exists i(1 \leq i \leq k \ \& \ \bar{\alpha}_1(k+1) = \bar{\alpha}(k+1)) \\ m+1 & \text{otherwise, where } m = \max\{\alpha_i(y) \mid 1 \leq i \leq k, 0 \leq y \leq k\}. \end{cases}$$

Now $Ft^2 \neq 0$; for, $\overline{(\pi_{0,0}0)}(k+1), \dots, \overline{(\pi_{0,0}k)}(k+1)$ are all distinct, hence one of them, say $\overline{(\pi_{0,0}k_0)}(k+1)$ ($0 \leq k_0 \leq k$) is distinct from all $\bar{\alpha}_1(k+1), \dots, \bar{\alpha}_k(k+1)$, and therefore $t^2(\pi_{0,0}k_0) = m+1$; but then $\overline{(\lambda x^0.t^2(\pi_{0,0}x))}(k+1)$ differs from all $\bar{\alpha}_1(k+1), \dots, \bar{\alpha}_k(k+1)$, and thus Ft^2 takes the value $m+1$.

160₁ Read “=” for “ \equiv ”.

- 161¹² Delete “not”.
- 164 Remark to be added at the end of 2.8.5: J.M.E. Hyland showed in his thesis that Scarpellini’s model coincides with the model ECF ”.
- 167₇ Read “establishing” for “establish”.
- 173 Remark to be added in 2.9.10:
 If a coding by functions is given for the elements of σ^\smile , such that there are continuous Φ_0, Φ with $\Phi_0\xi$ the length of the sequence coded by ξ , $\Phi(n, \xi)$ the n -th component extracted from ξ , then one can construct a bijection between two codings of this kind.”
- 179¹⁷ Read “ **HA**^c + G₁.”
- 180⁶ Read “ | F holds ” for “ | holds ”.
- 180₁₃ read “ P ” for “ D ”.
- 183₂ Read “ PCA). ” for “ P A). ”.
- 184¹⁷ Add after “ terms ”: “satisfying ($t \in V$ and $t' = t \Rightarrow t' \in V$) ”.
- 188¹⁷ Read “mathematical”.
- 189¹² Read “ & $P(B(j_1x))$ ” for “ & $P(A(j_1x))$ ”.
- 190₁₇ Delete “, and $P(F_1^*), \dots, P(F_n^*)$ ”.
- 190₁₂ Delete “ Also $\forall xP(Bx)$ ”.
- 190₄ Delete “It follows that ... hence $P(C)$.”, and replace “Also” by “Then”.
- 192¹ Add after “hence”: “ ! $t \& t r_P A$ is an abbreviation for ($\exists x(t \simeq x \& x r_P A)$).”.
- 192₁₋₇ The argument given in the first edition is not correct. The result is a consequence of the unprovability in **HA** of the DP, which has been proved by J. Myhill (A note on indicator functions, Proc. Amer. math. Soc. 29 (1973), 181–183) and by Friedman in a stronger form (On the derivability of instantiation properties, J.S.L. 42 (1977), 506–514).
- 194₁₀ Insert “ **HA** \vdash ” between “ (ii) ” and “ $A(\underline{a}) \rightarrow !\psi(\underline{a}) \& \dots$ ”.
- 194₉ Replace this line by
Proof. The “only if” part is established as follows. Assume $\vdash A_{\underline{a}} \leftrightarrow B_{\underline{a}}$, B almost negative. Then there is a recursive ϕ such that $\vdash \forall u(u r A_{\underline{a}} \rightarrow !\{j_1 \phi_{\underline{a}}\}(u) \& \{j_1 \phi_{\underline{a}}\}(u) r B_{\underline{a}})$, and $\vdash \forall u(u r B_{\underline{a}} \rightarrow !\{j_2 \phi_{\underline{a}}\}(u) \& \{j_2 \phi_{\underline{a}}\}(u) r A_{\underline{a}})$, which together with 3.2.11 for B readily yields the desired conclusion.
- 194₂ Read “ $Uv r A_{\underline{a}}$ ” for “ $v r A_{\underline{a}}$ ”.

198 Add after 3.2.22:

Remark. In the writings of the Russian constructivist school (cf. e.g. Dragalin 1969) one finds the following extension of CT_0 :

$$\text{CT}' \quad \forall x(\neg Ax \rightarrow \exists yBxy) \rightarrow \exists u\forall x(\neg Ax \rightarrow \exists v(Tuxv \& B(x, Uv))).$$

However, in the presence of M this is equivalent to ECT_0 , i.e.

$$\mathbf{HA} + \text{ECT}_0 + M = \mathbf{HA} + \text{CT}' + M.$$

To see this, let us first assume CT' , M , and let Ax be almost negative. then by M $Ax \leftrightarrow A'x$, A' negative, and hence $\neg\neg A'x \leftrightarrow Ax$ (1.10.8); thus an instance of ECT_0 can be interpreted as an instance of CT' .

Conversely, if ECT_0 and M are assumed, and we let $\forall x(\neg Ax \rightarrow \exists yBxy)$, then by ECT_0 , 3.2.8 $\neg Ax \leftrightarrow \exists z(zr\neg Ax) \leftrightarrow \forall z(zr\neg Ax) \leftrightarrow 0r\neg Ax$; $0r\neg Ax$ is almost negative. Replacing $\neg Ax$ by $0r\neg Ax$ we have $\forall x(0r\neg Ax \rightarrow \exists yBxy)$ to which we can apply ECT_0 etc.

200–201 The claim of 3.2.26 is correct, but the proof given is incorrect. The attempted proof in list of Errata from 1974 is also flawed, but J. van Oosten presented me with a proof that IP_0 is derivable from ECT_0 and MP .

Here follows the proof: Assume

$$\forall x(Ax \vee \neg Ax), \quad \forall x Ax \rightarrow \exists y B.$$

By ECT_0 there is a number n such that

$$\forall x(\{n\}x = 0 \leftrightarrow Ax), \quad \text{hence } \forall x(\{n\}x = 0) \rightarrow \exists y B.$$

Again by ECT_0 there is a number m such that

$$\forall x(\{n\}x = 0) \rightarrow !\{m\}0 \wedge B(\{m\}0).$$

Let k be a number such that

$$\{k\}j(a, b) = \min_x[\{a\}x \neq 0 \vee Tb0x].$$

Since

$$\begin{aligned} &\neg\neg(\exists x(\{n\}x \neq 0) \vee \forall x(\{n\}x = 0)), \quad \text{it follows that} \\ &\neg\neg(\exists x(\{n\}x \neq 0) \vee !\{m\}0), \quad \text{therefore} \\ &\neg\neg!\{k\}j(n, m); \quad \text{hence with MP } !\{k\}j(n, m). \end{aligned}$$

From this we see, that by the definition of k

$$\begin{aligned} &\{n\}(\{k\}j(n, m)) \neq 0 \rightarrow \neg\forall x Ax, \\ &T(m, 0, \{k\}j(n, m)) \rightarrow (\forall x Ax \rightarrow B(U(\{k\}j(n, m)))). \end{aligned}$$

Hence

$$\exists y(\forall x Ax \rightarrow By).$$

- 202⁶⁻¹³ The alternative argument for the non-realizability of formula (1) is erroneous.
- 203 Add after 3.2.29: “Friedman has shown (*Friedman B*) how to extend q-realizability by a similar trick.”.
- 203₄ Read “*Cellucci*”.
- 214₅ Replace “negative” by “ \exists -free (i.e. not containing \vee, \exists)”.
- 214_{4,3} Delete “on the convention . . . omitted,”.
- 215¹ Replace “negative” by “ \exists -free”.
- 215⁹ Add “**N- \mathbf{HA}^ω ,**” after “ **\mathbf{HA}^ω ,**”.
- 215¹¹ Read “. . . sequence \underline{t} of . . .”.
- 215₁₅ Replace “negative” by “ \exists -free”.
- 216²⁰ Read “ $\underline{y} \text{mr}_P A$ ” for “ $y \text{mr}_P A$ ”.
- 217^{1,2,17} Read “ \underline{T} ” for “ T ” (4 times).
- 217¹³ Replace this line by:
- $$\text{IP}^- \quad (A \rightarrow \exists y^\sigma B) \rightarrow \exists y^\sigma (\neg A \rightarrow B)$$
- (y^σ not free in A , A \exists -free, i.e. not containing \vee, \exists)
- 217_{16,15} Delete “, taking for . . . into account”. Add after 3.4.7:
- Remark.* The schema
- $$\text{IP}^\omega \quad (\neg A \rightarrow \exists y^\sigma B) \rightarrow \exists y^\sigma (\neg A \rightarrow B) \quad (y^\sigma \text{ not free in } B).$$
- is readily seen to be modified-realizable, hence $\mathbf{H} + \text{IP}^- + AC \vdash \text{IP}^\omega$. Since in systems with decidable prime formulae negative and \exists -free formulas coincide, and for negative A $\neg\neg A \leftrightarrow A$, we have in such cases also that IP^ω implies IP^- .
- 217_{10,9,4,2,1} Replace “ IP^ω ” by “ IP^- ”.
- 217₁₀ Read “ $\mathbf{H} +$ ” for “ $\mathbf{H} \vdash$ ”.
- 217₈ Add after line: “For $\mathbf{H} = \mathbf{HA}^\omega$, $\mathbf{I-HA}^\omega$, \mathbf{HRO}^- , $\mathbf{E-HA}^\omega$, IP^- may be replaced by IP^ω .”.
- 217₅ Read “ \exists -free” for “negative”.
- 221₇ Read “ M_{PR} ” for “ MP_{PR} ”.
- 221₂ Read “3.4.14” for “3.4.4”.

222 Add after 3.4.14:

Remark. V.A. Lifschitz has shown (Proceedings of the American Mathematical Society 73 (1979), 101–106) that also $\mathbf{HA} + \mathbf{CT}_0! \not\vdash \mathbf{CT}_0$, where

$\mathbf{CT}_0! \quad \forall x \exists! y A(x, y) \rightarrow \exists u \forall x \exists v (Tuxv \ \& \ A(x, Uv)).$ ”.

222₂ Read “ $\forall \alpha \neg \neg \exists x$ ” for “ $\forall \alpha \neg \neg \exists z$ ”.

223¹ Read “... was suggested by results contained in”.

224₁, 225¹ Read “ \mathbf{IP}^- ” for “ \mathbf{IP}^ω ”.

226₁₆ Replace “for” by “. For”.

226₈ Insert after “... numbers” “(provably linear in \mathbf{HA})”.

227 Add “(\prec provably linear in \mathbf{HA})”.

228₁₆ Read “ $U_{j(n,i)}^1 x$ ” for “ $U_{j(n,i)}^1$ ”.

228_{7,6} These lines must read respectively “ $\dots \equiv \forall X^* \forall D_X (x \text{ mr } A(X))$ ” and “ $\dots \equiv \exists X^* \exists D_X (x \text{ mr } A(X))$ ”.

229₁₁ Read “of s^1 in HRO”.

230⁵ Read “eliminating” for elementary”.

233₅ Read “ Π_2^0 ” for “ Π_z^0 ”.

238^{4,6,7} Delete “ $]^D$ ”.

239₄ Read “ $(\underline{x}, \underline{v}, \underline{Zv})$ ” for “ $(\underline{x}, \underline{vZ}, \underline{v})$ ”.

240³ Read “ \underline{y} ” for “ \underline{Yx} ”.

242¹² Read “ $\vdash F^D$ ” for “ $+F^D$ ”.

242¹⁷ Read “now” for “not”.

244_{7,6} Replace these lines by:

If we take everywhere X to be identically 1, we obtain the Dialectica interpretation.

245¹⁰ “Shoenfield” should be underlined.

251₄ Delete “ $\mathbf{N-HA}^\omega$,” and add “; $(\mathbf{N-HA}^\omega + \mathbf{IP}^- + \mathbf{AC}) \cap \Gamma_1 = \mathbf{N-HA}^\omega \cap \Gamma_1$ ” (cf. the corrections to page 217).

255₆ Read “... & A^*y ”].”.

264¹⁶ Read “ \mathbf{HA} ” for “HA”.

264₈ Read “ $\rightarrow \exists x^\sigma A$ ” for “ $\rightarrow \neg \exists x^\sigma A$ ”.

- 265¹ Read “ $(z \neq 0 \rightarrow \neg A)$ ” for “ $(z \neq 0 \rightarrow A)$ ”.
- 267₂ Read “ $\exists y(y p$ ” for “ $\exists(y p$ ”.
- 273₁₇ Read “ 3.9.13 ” for “ 3.9.11 ”.
- 274¹ Read “ 3.9.14 ” for “ 3.9.12 ”.
- 274₄ Read “ 3.9.15 ” for “ 3.9.13 ”.
- 275₆ Delete “ (”.
- 278 Second proof tree under 4), read “ Π ” for the highest “ Π_i ”.
- 279 Replace in the first four proof trees exhibited the occurrences of “ A ” (but not the A in “ A_1 ”, “ A_2 ”, “ Aa ” or “ $\exists xAx$ ”) by “ B ” (7 occurrences).
- 280 Replace under “ 13) ” “ $A0$ ” by “ Aa ”.
- 282⁷ Read “ $\Pi' \succ_1 \Pi''$, (without ... ”.
- 282²³ Add “ of λ_1 ” at the end.
- 282 In the display at the bottom of the page, the first two lines should be

$$\frac{\Pi}{A} \quad 0 = 0 \qquad \frac{\Pi}{A} \quad sa = sa$$

- 283 In the displayed proof tree read “ $\&_1E$ ” for “ $\&E$ ”.
- 284¹ read “form (*Prawitz*” for “(from *Prawitz*”.
- 285 Immediately above the paragraph starting with “This makes it ...” read

$$\frac{\Pi'}{[A]} \quad \text{for} \quad \frac{\Pi}{[A]}$$

$$\frac{\Pi}{B} \qquad \qquad \frac{\Pi}{B}$$

- 286 Replace last paragraph of 4.1.7. by:
 For applications, we need only a normalization theorem (not a strong normalization theorem) relative to $\mathcal{R}_{c\lambda}$; so if the reader wishes, he may use the preceding remark and delete everything in the proof below referring to λ -contractions.
- 287¹⁶ Read “ $\text{Prd}_1(\Pi), \text{Prd}_2(\Pi), \dots$ ”.
- 287₅ Read “ $\text{SV}(\text{Sub}(A, \Pi, \text{Prd}(\Pi), \text{Ass}(\Pi)))$ ”.
- 287₂ Read “ Π ” for “ Π_1 ”.
- 288⁹ Insert “, $\text{Ass}(\Pi)$ ” after “ $\text{Prd}_2(\Pi)$ ”.

288 Directly above footnote, read

$$\frac{\Pi'_i}{A} \text{ for } \frac{\Pi_i}{A}.$$

290¹⁷ Read “ $\Pi' \in \text{PRD}(\Pi)$ ”.

290 In the first displayed proof tree, replace “ At ” by “ At' ”.

291 The second displayed proof tree should read:

$$\frac{\Pi'_1 \dots \Pi'_n}{A}.$$

292¹ Read “condition IV ” for “ condition II ”.

293 Second line of paragraph starting with “Condition IV for Π' ...”, read “for Π ” instead of “for \exists ”.

294 Replace in the second display “ Π_3 ” by “ Π_4 ”, and in the line under this display, replace “ Π_4 ” by “ Π_5 ”.

294 Replace in the third display “ Π_3 ” by “ Π_4 ”. In the line under the third display, insert after “reduces to”: “the left subdeduction of”.

295⁶ Read “ Π_6 ” for “ Π_3 ”.

295 In the third display, place in the second proof tree “ Π_0 ” above “ $\exists xBx$ ”.

295₁₂ Insert between “condition” and “I”: “IV, and hence”.

295 In the line below the third display, insert before “is SV”:
“and also

$$\begin{array}{c} \Pi_4 \\ [B_1t'] \\ \Pi'_1(t') \\ [Dt] \\ \Pi_1(t) \\ A \end{array}$$

”

297¹⁷ Read “ 2.2.25 ” for “ 2.2.31 ”.

299¹⁵ Add “ (major) ” at the end.

300⁶ Add after the comma “which may be empty,”.

300^{9,10} Replace “preceding” by “succeeding”.

301² Read “were” for “would be”.

301¹⁰ Add before “) ”: “ ; also, A_i cannot be discharged by IND, since no application of IND lies below A_1 ”.

301₆ read “ 4.2.8 ” for “ 2.8 ”.

302⁵ Read “normal” for “formal”.

302¹⁴ Add “ σ ” at the end.

304³ Insert “(by 4.2.7)” before “ ; ”.

304¹⁰ Read “were” for “would be”.

304¹¹ Read “or IND-application occurring” for “occurring”.

304¹⁴ Replace “ IND-application ” by “ begin with a conclusion of an IND-application”.

304₁₉ Read “Let Φ denote the ”.

304 The proof in subsection 4.2.16 in the first edition contains a gap. Much simpler is the following argument:

PROOF. Note that Ψ is equivalent to a set of Harrop formulas: if $\exists xPx \in \Psi$, then we may replace this formula by $P\bar{n}$ for some \bar{n} such that $\mathbf{HA} \vdash P\bar{n}$. Then we can apply 4.2.12.

305₁ delete “ , or A_1 is a basic axiom ”.

305⁵ Read “ $A_1 \in \Psi$, i.e. B is prime ”.

305 Remark at 4.2.17: instead of referring to 3.6.7(ii), it suffices to note that only true closed Σ_1^0 -formulae are provable in \mathbf{HA} and \mathbf{HA}^c .

306, proof of 4.2.18. This proof is incorrect as it stands, since the conclusion of an IND-application is not necessarily atomic, only quantifier-free. The proof is correct if we replace in the statement of the theorem \mathbf{H} , qf- \mathbf{HA} by the corresponding systems with induction for atomic formulas only.

To establish the theorem as stated, we can e.g. proceed as follows: define a path of order 0 to be a path A_1, \dots, A_n with A_n conclusion of the deduction, and define a path of order $m + 1$ to be a path A_1, \dots, A_n such that either A_n is minor premiss of an \rightarrow E-application the major premiss of which belongs to a path of order m , or premiss of an IND-application the conclusion of which belongs to a path of order m .

In a strictly normal derivation, every formula occurrence belongs to some path of order m , for suitable m (since redundant applications of \forall E, \exists E have been removed). Then one readily proves, by induction on m , that for a strictly normal derivation of a quantifier-free formula in \mathbf{H} all formula occurrences on a path of order m are quantifier-free. (Note that here normalization also w.r.t. permutative reductions is necessary, in contrast to other applications. This could have been avoided by reduction of qf- \mathbf{HA} to a logic-free calculus, which is not a very elegant solution, however.)

306₁₆ Read “ 2.5.7 ” for “ 2.5.6 ”.

307, line 2 below second display. Read “ \mathcal{R}_c ” for “ \mathcal{R}_C ”.

307₂ Read “ 4.1.16 ” for “ 4.1.15 ”.

308 Last line of first display, replace in proof tree Π'' “ $\neg A \rightarrow Bx$ ” by “ $\neg A \rightarrow \exists x Bx$ ”.

309 Replace second sentence of the statement of 4.3.5 by:

Then a spine of Π not ending with an introduction does not contain IP-applications, and ends with an application of a basic rule or an atomic instance of λ_I .

Delete the third sentence.

The proof of 4.3.5. should be reformulated as follows:

Let A_1, \dots, A_n be a spine of Π not ending with an introduction. Then, by 4.3.4(iii) there are two cases:

(1°) A_1 is a basic axiom. So the spine coincides with its minimum part, hence A_n is atomic.

(2°) A_1 is of the form $\neg B$, to be discharged by $\rightarrow I$, followed by IP. this case is excluded, for the sort of inference following A_1 can be (not IP, or $\rightarrow I + IP$, but) $\rightarrow E$ only, leaving us with $A_2 \equiv \lambda$, and a minimum part A_2, \dots, A_n .

311⁸ read “any one”.

311¹⁷ Read “Red₁” for “Red”.

311₁₉ Read “of” for “from”.

311₁₂ Read “ $\forall I_r$ ” for “ $\forall I_1$ ”.

311₁₁ Read “ $\forall I_1$ ” for “ $\forall I_r$ ”.

311₇ Read “ $SV_{d-1}(\text{Subst}(\text{Param}(\Pi), x, \text{Prd}_1(\Pi)))$ ”.

313¹ Read “ (ii) ” for “ (iii) ”.

313₁₂ Read “ 4.4.3 ” for “ 4.4.2 ”.

313₁₀ Insert “(1.5.6)” before “ : ”.

314⁴ Read “ Proof_n ” for “ Proof ”.

314₅ Read “ 4.4.3 ” for “ 4.4.2 ”.

314₃ Read “ $\mathbf{HA} \vdash \forall x \exists y z (\text{Proof}_n(y, \ulcorner A(\bar{x}, \bar{z}) \urcorner) \& A(x, z))$ ”.

321 As observed by S.Hayashi, (On derived rules of intuitionistic second order arithmetic, *Commentarii Mathematici Universitatis Sancti Pauli* 26 (1977), 77–103), the proof of 4.5.8 indicated in the text of the first edition establishes a result which is too weak, e.g.

$$\forall n \forall A \in \text{Fm}^{(n)} (\vdash \text{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \leftrightarrow \forall x \text{Sat}^{(n)}(X, \ulcorner A(\bar{x}) \urcorner))$$

instead of

$$\forall n(\vdash \forall A \in \text{Fm}^{(n)}(\text{Sat}^{(n)}(X, \ulcorner \forall x Ax \urcorner) \leftrightarrow \forall x \text{Sat}^{(n)}(X, \ulcorner A(\bar{x}) \urcorner)).$$

Following Hayashi, the desired stronger conclusion can be established as follows.

We first define the notion of a formation sequence of a formula A in $\text{Fm}^{(n)}$.

DEFINITION. A *formation sequence* (fs) of $A \in \text{Fm}^{(n)}$ is a finite sequence of quadruples $\langle a_0, b_0, c_0, t_0 \rangle, \dots, \langle a_m, b_m, c_m, t_m \rangle$ such that

- (1) $t_m = \ulcorner A \urcorner$; t_0 , and c_i for $1 \leq i \leq m$ are codes of formulas of complexity $\leq n$.
- (2) $a_i \in \mathbb{N}$ for $0 \leq i \leq m$, $a_{i+1} \leq i$ for $0 \leq i < m$.
- (3) $b_i, c_i \in \mathbb{N}$ for $0 \leq i \leq m$; t_{i+1} is the code of the term which is the result of substituting the term with code $t_{a_{i+1}}$ for the second-order variable $V_{b_{i+1}}^p$ in the formula (with index) c_{i+1} and logical complexity $\leq n$, where p is the number of free variables in $t_{a_{i+1}}$ (end of definition).

Now $\text{Sat}_n(X, \ulcorner A \urcorner)$ is constructed as before. Let f, g, h range over formation sequences. We then define, similar to $\text{Sat}^{(n)}(X, \ulcorner A \urcorner)$ of the text, and with help of Sat_n , the formula $\text{Sat}_f^{(n)}(X, \ulcorner A \urcorner)$, where f is an fs for A with $t_m = \ulcorner A \urcorner$, and $\text{Sat}_f^{(n)}$ is constructed parallel to the substitutions of f . Then one proves

LEMMA. In **HAS**

- (i) $\forall f \forall A, B \in \text{Fm}^{(n)} \exists g, h \forall X (\text{Sat}_f^{(n)}(X, \ulcorner A \circ B \urcorner) \leftrightarrow \text{Sat}_g^{(n)}(X, \ulcorner A \urcorner) \circ \text{Sat}_h^{(n)}(X, \ulcorner B \urcorner))$
for $\circ \in \{\rightarrow, \&, \vee\}$.
- (ii) $\forall f \forall A \in \text{Fm}^{(n)} \exists g \forall X (\text{Sat}_f^{(n)}(X, \ulcorner Qv_i A(v_i) \urcorner) \leftrightarrow (Qv_i) \text{Sat}_g^{(n)}(X, \ulcorner A(\bar{v}_i) \urcorner))$
for $Q \in \{\forall_1, \exists_1\}$.
- (iii) $\forall f \forall A \in \text{Fm}^{(n)} \exists g \forall X (\text{Sat}_f^{(n)}(X, \ulcorner QV_i^p A(V_i^p) \urcorner) \leftrightarrow (QY^p) \forall Z^1 (\forall y_1, y_2 (j(y_1, y_2) \neq j(p, i) \rightarrow Z_{(y_1, y_2)}^1 = X_{(y_1, y_2)}) \wedge Z_{(p, i)}^1 = Y \rightarrow \text{Sat}_g^{(n)}(Z, \ulcorner A(V_i^p) \urcorner))$
for $Q \in \{\forall_2, \exists_2\}$.
- (iv) $\forall X, f, g, n (\text{FS}(f, n) \wedge \text{FS}(g, n) \rightarrow \text{Sat}_f^{(n)}(X, n) \leftrightarrow \text{Sat}_g^{(n)}(X, n))$, where $\text{FS}(f, n)$ expresses “ f is a formation sequence of a formula A with $\ulcorner A \urcorner = n$ ”.

Proof. The proof of (i)–(iii) by induction on the length of f ; the proof of (iv) uses (i)–(iii) and induction on n .

We may then put

$$\text{Sat}^{(n)}(X, \ulcorner A \urcorner) \leftrightarrow \exists f \text{Sat}_f^{(n)}(X, \ulcorner A \urcorner)$$

and can then establish a stronger version of 4.5.8, namely

$$\forall n(\mathbf{HAS} \vdash \forall A \in \text{Fm}^{(n)}(\text{Sat}^{(n)}(X, \ulcorner \forall x Ax \urcorner) \leftrightarrow \forall x \text{Sat}^{(n)}(X, \ulcorner A\bar{x} \urcorner))$$

etc. etc.

- 322¹³ read “choice” for “hoice”.
- 375₁₇ Read “ $\alpha > \alpha_0$ &” for “ $\alpha > \alpha_0$ ”.
- 389 Subsection 5.7.3: more information about Kripke models for second-order intuitionistic arithmetic may be found in: D.H.J. de Jongh, C.A. Smoryński, Kripke models and the intuitionistic theory of species, *Annals of Mathematical Logic* 9 (1977), 157–186.
- 391₃ read “ $(\alpha x \neq 0)$ ” for “ $(\alpha x \neq 0)$ ”.
- 391₁ Add after “)” “in the presence of continuity axioms”.
- 398₁₂ Read “ $\mathcal{L}[x]$ ” for “ \mathcal{L} ”.
- 414₁₁ Read “ $S_2 f \in Y$ ” for “ $S_1 f \in Y$ ”.
- 422₁₀ Read “ \mathbf{ID}_ν^c ” for “ \mathbf{ID}_ν^C ”.
- 435¹⁰ Read “ $|\mathbf{ID}_\nu^c|$ ” for “ $|\mathbf{ID}^c|$ ”.
- 438⁵ Read “ $P_1(Xn, n)$ ” for “ $P_1(Xn, x)$ ”.
- 439₁₆ Read “type-0-valued”.
- 440₆ Read “ $\text{Max}_1(\alpha, 0) = \alpha$ ” for “ $\text{Max}_1(\alpha, 0)$ ”.
- 448_{13,14} The equality in (7) of 6.9.1 was proved for all recursive ν by 1977, independently by Buchholz, Pohlers and Sieg, using various sophisticated proof-theoretic techniques (see W. Buchholz, S. Feferman, W. Pohlers, and W. Sieg, *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*. Springer Verlag, Berlin 1977). Hence the equalities
- $$|\mathbf{ID}_2^c| = |\mathbf{ID}_2| = |\mathbf{T}_2|$$
- hold (end of 6.8.9). Hence also the equalities (5) and (6) of 6.9.1 are true.
- 451^{12–17} See the remark to page 448.
- 457₁₄ Read “ $(\lambda n.X_1 \dots X_k)$ ” for “ $(\lambda X_1 \dots X_k)$ ”.
- 462¹ Insert before the second line “Corrections in the bibliography consist sometimes in replacements, sometimes in added information between square brackets.”
- 462^{8,9} Replace by “LMPS IV: P.Suppes, L.Henkin, Gr.C. Moisil, A. Joja (eds.), North-Holland Publ. Co., Amsterdam 1973.”
- 462¹⁷ Read “Cambridge Summer School in Mathematical Logic”
- 462¹⁸ Read: H.Rogers, A.R.D. Mathias (eds.), 1973.
- 462¹⁹ Add at the end “, 1973”.

- 462²⁰ Delete.
- 462₁₅ Add “[Zeitschrift für mathematische Logik und Grundlagen der Mathematik 20 (1974), 289–306.]”
- 462₁₄ Add: “[cf. H.P. Barendregt, Combinatory logic and the axiom of choice, *Indagationes Mathematicae* 35(1973), 203–221.]”
- 462_{10,11} Replace by: “ *Theoretical Computer Science* 3 (1977), 225–242.
- 462₅ Add “[J.S.L. 41 (1976), 328– 336]”.
- 462₃ Add “[J.S.L. 40 (1975), 321– 346]”.
- 462₁ Add “[J.S.L. 41 (1976), 18–24]”.
- 463¹ Read “ Cellucci”.
- 463¹⁵ Read “*Cambr. Proc.* 1–94.”
- 463₂₅ Add “[Did not appear]”
- 463₂₁ Read “*Archiv für mathematische Logik* 16 (1974), 49–66.”
- 464⁵ Read “*Cambr. Proc.* 113–170.”
- 464₂₃ Read “Schliessen.”
- 464₁ Add “, 232–252”.
- 465²¹ Read “*Cambr. Proc.* 274–298.”
- 465²⁴ Read “*J.S.L.* 38 (1973), 453–459.”
- 465²⁶ Add “[Did not appear]”
- 465²⁸ Add “[Did not appear]”
- 465³⁰ Read “*J.S.L.* 41 (1976), 574–582.”
- 466²⁶ Read “Section VI” for “Section IV”.
- 466₁₄ Add: “[Appeared in: J.P.Seldin, J.R.Hindley (eds.), *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism.* Academic Press, New York 1980, 480–490.]”
- 467²² Add: “[Never published]”
- 467²⁴ Add: “[Appeared in: A.S. Troelstra, D. van Dalen (eds.), *The L.E.J Brouwer Centenary Symposium.* North-Holland Publ. Co., Amsterdam 1982, 51–64.]”
- 468⁸ Read “*Philosophica*”.
- 469⁹ Read “in” for “in:”.

- 469¹² Read “Zentralblatt”.
- 469²⁶ Read “IPT” for Oslo Proc.”
- 469₂₄ Read “ Π_1^1 ” for “ Π ”.
- 469₈ Read “1” for “I”.
- 470₁₂ Add: “[Cf. paper under this title in: S. Kanger (ed.), Proceedings of the 3rd Scandinavian Logic Symposium, North-Holland Publ. Co., Amsterdam 1975, 81–109.]”
- 471¹⁰ Read: “Compositio Mathematica 26 (1973), 261–275.”
- 472⁵ Insert before stop: “, 225–250”.
- 472₁ Replace by “Archiv für Mathematische Logik 16 (1974), 147–158.”
- 473¹¹ Add “[Unpublished]”.
- 474₁₁ Read “1970” for “170”.
- 475¹³ Read “Cambr. Proc. 171–205.”