Robustly minimal iterated function systems on compact manifolds generated by two diffeomorphisms

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Abstract

An iterated function system on a manifold, generated by a finite number of diffeomorphisms, is called minimal if for each point its orbit under all possible compositions of the diffeomorphisms lies dense in the manifold. On each compact manifold we provide an iterated function system generated by two diffeomorphisms that is minimal and also C^1 robustly minimal (iterated function systems generated by two diffeomorphisms that are small perturbations in the C^1 topology are also minimal).

1 Introduction

As is well known, iterated function systems are a popular way to generate and explore a variety of fractals [1, 3]. The iterated function systems then consist of contractions on Euclidean space, the fractals occur as attractors for the iterated function systems. In larger generality one can consider iterated function systems, generated by a number of diffeomorphisms, on manifolds and study their dynamics. Fractals may appear as attractors for such iterated function systems. Other examples are known where points can move over the entire manifold under iteration by the iterated function system; such an iterated function system is termed minimal. We provide a result showing that this is not a pathological possibility: on each compact manifold we construct iterated function systems generated by just two diffeomorphisms, that are C^1 robustly minimal.

An earlier result contained in [5] by Gorodetskiĭ and Il'yashenko constructs robustly minimal iterated function systems on the circle. They provide an example of an iterated function system generated by two circle diffeomorphisms, that is robustly minimal in the C^1 topology. The example consists of an irrational rigid rotation and a diffeomorphism with an attracting and a repelling fixed point; we refer to [7, Proposition 12] for details of the construction. In [4] this one-dimensional example is generalized to iterated function systems on higher-dimensional manifolds, also C^1 robustly minimal, but with a number of generators depending on the dimension of the manifold. A somewhat related problem on minimality of an iterated function system generated by a generic pair of area preserving diffeomorphisms was recently raised in [8].

We begin to introduce definitions and notations of iterated function systems, and then formulate our main result. Throughout this paper, M will stand for an n-dimensional smooth compact manifold.

Definition 1.1. *The iterated function system* $\mathcal{G}(M; g_1, \ldots, g_m)$ *on* M *generated by a collection* $\mathcal{L} = \{g_1, \ldots, g_m\}$ *of diffeomorphisms on* M *is given by iterates* $g_{i_1} \circ \cdots \circ g_{i_k}$ *with* $i_j \in \{1, \ldots, m\}$*. If* $\triangle \subset M$ *is invariant under* $g_1, \ldots, g_m$*, i.e.* $g_i(\triangle) \subset \triangle$ *for* $i = 1, \ldots, m$*, we write* $\mathcal{G}(\triangle; g_1, \ldots, g_m)$ *for the iterated function system on* $\triangle$*.*

With $\mathcal{L} = \{g_1, \ldots, g_m\}$ and for a set $O \subset M$, write

$$\mathcal{L}(O) = g_1(O) \cup \ldots \cup g_m(O).$$

Iterating we get $\mathcal{L}^p(O) = \mathcal{L}(\mathcal{L}^{p-1}(O))$ for $p > 1$.

Definition 1.2. *An iterated function system* $\mathcal{G}(M; g_1, \ldots, g_m)$ *on* M *is minimal if for any* $x \in M$ *the collection of iterates* $g_{i_1} \circ \cdots \circ g_{i_k}(x)$*,* $i_j \in \{1, \ldots, m\}$*, is dense in* M*.*

Consider the space $\mathrm{Diff}^1(M)$ of C^1 diffeomorphisms of M, endowed with the C^1 topology.

Theorem 1.1. *Let* M *be a compact connected* n*-dimensional manifold. Then there exist diffeomorphisms* g_1, g_2 *on* M *and a neighborhood*

$$U \subset \mathrm{Diff}^1(M) \times \mathrm{Diff}^1(M)$$

of (g_1, g_2) *such that each element in* U *generates a minimal iterated function system on* M*.*

The theorem is proved in Section 3, using the material on iterated functions systems generated by affine maps, presented in the following section. For preparing this note, I acknowledge helpful discussions with Daniël Younis.

2 Iterated function systems generated by affine maps

We start with considerations on iterated function systems generated by affine maps on $\mathbb{R}^n$, where we assume $n \geq 2$. We will provide an affine contraction S and an affine expansion T (i.e. T^{-1} is a contraction) so that the iterated function system generated by S and T is minimal on all of $\mathbb{R}^n$ (compare [2] for minimal iterated functions systems on $[0, \infty)$ generated by affine maps). Moreover, $S \circ T$ will be a contraction and the iterated function system generated by the two affine contractions S and $S \circ T$ possesses an attractor with nonempty interior.

Consider the rotation R in $\mathbb{R}^n$, $n \geq 2$,

$$R(x_1, \ldots, x_n) = (\pm x_n, x_1, \ldots, x_{n-1}),$$

where the sign is such that $R \in SO(n)$, i.e., a minus sign for even n and a plus sign for odd n. Define further the translation $H(x_1, \ldots, x_n) = (x_1 + s, x_2, \ldots, x_n)$ and the affine map S on $\mathbb{R}^n$ by

$$S(x_1, \ldots, x_n) = H \circ rR(x_1, \ldots, x_n) = (\pm r x_n + s, r x_1, \ldots, r x_{n-1}), \tag{1}$$

for constants $0 < r < 1$, $s > 0$. Likewise, define an affine map T on $\mathbb{R}^n$ by

$$T(x_1, \ldots, x_n) = (-a x_1, a x_2, \ldots, a x_{n-1}, -a x_n - 2\frac{s}{r}) \tag{2}$$

with $a > 1$. Similar to S, T is the composition of a map from $SO(n)$ which is multiplied by a factor, a, and a translation. Note that S is a contraction by $r < 1$, while T is an expansion as $a > 1$. Compute

$$S \circ T(x_1, \ldots, x_n) = H^{-1} \circ arR(x_1, \ldots, x_n) = (\mp ar x_n - s, -ar x_1, ar x_2, \ldots, ar x_{n-1}).$$

The affine map $S \circ T$ is a contraction for $ar < 1$.

Lemma 2.1. *There are constants $0 < r < 1$, $s > 0$, $a > 1$ with $ar < 1$ so that the iterated function system generated by the contractions S and $S \circ T$ has an attractor $\triangle$ with nonempty interior. Moreover, this interior contains the fixed point of T.*

Proof. Define the box $B(1, v_2, \ldots, v_n)$ with corners $(\pm 1, \pm v_2, \ldots, \pm v_n)$. We will find r, s, a and $v_2, \ldots, v_n$ so that

$$S(B) \cup S \circ T(B) \supset B \tag{3}$$

This is a consequence of the following conditions,

$$rv_n + s > 1, -rv_n + s < 0, r > v_2, rv_2 > v_3, \ldots, rv_{n-1} > v_n. \tag{4}$$

In fact, (4) implies

$$arv_n + s > 1, -arv_n + s < 0, ar > v_2, arv_2 > v_3, \ldots, arv_{n-1} > v_n \tag{5}$$

and (4) and (5) together give (3).

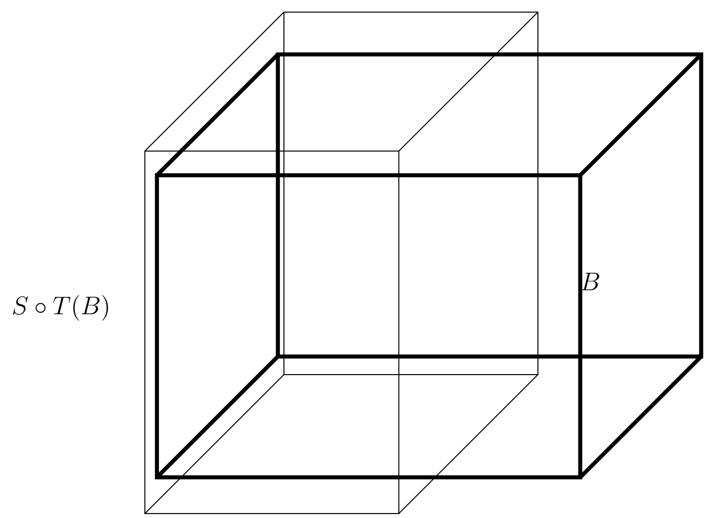

Figure 1: *The image $S \circ T(B)$ covers over half the box B. The image $S(T)$ covers the remaining part of B, so that $S(B) \cup S \circ T(B)$ contains B.*

The repelling fixed point of T is located at $(0, \ldots, 0, \frac{-2s}{r(a+1)})$. It lies in B if

$$2s < v_n r(a + 1). \tag{6}$$

Now, (4) and (6) can be satisfied by taking suitable $v_n < \ldots < v_2 < r$ all near 1 and $s = 1 - v_n^2$ near 0 so that (4) holds, and a with $ar < 1$.

Since S and $S \circ T$ are contractions, there is a ball O that is mapped into itself by both S and $S \circ T$, i.e., $\mathcal{L}(O) \subset O$. Thus

$$\triangle = \lim_{p \to \infty} \mathcal{L}^p(O)$$

is a nonempty compact set that is invariant for $\mathcal{L}$. Since S and $S \circ T$ are contractions, $\triangle$ is the unique compact set that is invariant for $\mathcal{L}$ [6]. Because $\mathcal{L}(B) \supset B$, the set $\triangle$ contains B. $\square$

Corollary 2.1. *The iterated function system $\mathcal{G}(\mathbb{R}^n; S, T)$ is minimal.*

The proof of Lemma 2.1 gives more than its statement as it includes arguments for C^1 robust occurrence of invariant sets with nonempty interior. Denote by $\mathrm{Diff}^1(\mathbb{R}^n)$ the set of diffeomorphisms on $\mathbb{R}^n$, endowed with the C^1 compact-open topology.

Corollary 2.2. *There exists a neighborhood $W \subset \mathrm{Diff}^1(\mathbb{R}^n) \times \mathrm{Diff}^1(\mathbb{R}^n)$ of $(S, S \circ T)$ such that each element $\mathcal{F} = (f_1, f_2)$ in this neighborhood admits an invariant set with non-empty interior.*

In rescaled coordinates $(y_1, \ldots, y_n) = h(x_1, \ldots, x_n) = (\delta x_1, \ldots, \delta x_n)$, $\delta > 0$, the attractor $\triangle$ is multiplied by a factor δ. The affine maps computed in the $(y_1, \ldots, y_n)$ coordinates become

$$h \circ S \circ h^{-1}(y_1, \ldots, y_n) = (\pm r y_n + \delta s, r y_1, \ldots, r y_{n-1})$$

and

$$h \circ T \circ h^{-1}(y_1, \ldots, y_n) = (-a y_1, a y_2, \ldots, a y_{n-1}, -a y_n - 2\frac{\delta s}{r});$$

the maps are unaltered except for the translation vector which is multiplied by δ. In other words, if S and T are affine maps as above so that $\mathcal{G}(\mathbb{R}^n; S, T)$ has an attractor $\triangle$, than replacing s in their expressions by δs yields an iterated function system with attractor $\delta\triangle$.

3 Proof of the main result

The proof of Theorem 1.1 combines Lemma 2.1, which will provide a set with nonempty interior in which the iterated function system has dense orbits, with a mechanism to move from this set to other parts of the manifold, and back, by iterating.

Proof of Theorem 1.1. We only treat $n \geq 2$; as mentioned in the introduction, the one-dimensional case of iterated function systems on the circle is found in [5].

Take a gradient Morse-Smale vector field $\dot{x} = \nabla F(x)$ on M with a unique hyperbolic repelling equilibrium q and a unique hyperbolic attracting equilibrium p (see e.g. [9, Theorem 3.35] for the existence of Morse functions F with unique extrema) and let f be its time-1 map. Below we use that the stable manifold of p and the unstable manifold of q lie dense in M.

First we consider f near p. Working in a coordinate chart on a small open neighborhood V of p, we may assume that f acts on $\mathbb{R}^n$ and f is a contraction on a ball B_o in $\mathbb{R}^n$. Recall the affine contraction S on $\mathbb{R}^n$ given by (1) and the affine expansion T on $\mathbb{R}^n$ given by (2), as in Lemma 2.1. Since f is homotopic to S on B_o, we may alter f to a diffeomorphism that remains a contraction on B_o and is equal to S on a smaller ball $B_i \subset B_o$. The resulting diffeomorphism on M will be g_1. Note that g_1 has finitely many hyperbolic fixed points, among which a unique hyperbolic attracting fixed point. The stable manifold of this attracting fixed point lies dense in M. Likewise, consider f^{-1} near p and alter it to a diffeomorphism that equals T on a neighborhood of p. Write f_2 for the resulting diffeomorphism on M. Note that f_2 has finitely many hyperbolic fixed points. Finally, with a small perturbation, perturb f_2 to a diffeomorphism g_2 so that each of the fixed points of g_2 is contained in the stable manifold of the attracting fixed point of g_1.

Applying Lemma 2.1, the iterated function system generated by g_1 and $g_1 \circ g_2$ has an attractor Δ with nonempty interior near p, with Δ containing the repelling fixed point of g_2 in its interior. The unstable manifold of the repelling fixed point of g_2 lies dense in M. Iterates of Δ under g_2 therefore cover a dense subset of M. Now, iterates of a single point under g_2 will converge to one of the fixed points of g_2, which lie in the stable set of the attracting fixed point of g_1 and are mapped into $\triangle$ under iteration by g_1. This implies that the iterated function system $\{g_1, g_2\}$ is minimal.

The construction is robust for perturbations in the C^1 topology, as is clear from the following observations. By Corollary 2.2, if h_1 and h_2 are diffeomorphisms C^1 close to g_1 and g_2, the iterated function system generated by h_1 and $h_1 \circ h_2$ possesses an attractor with nonempty interior that contains the repelling fixed point of h_2. Small C^1 perturbations of g_1 have nearby fixed points and bounded parts of its stable manifolds are also nearby. The same applies to g_2. For both diffeomorphisms the stable and unstable manifolds of their attracting respectively repelling fixed points are dense in M. In particular, the fixed points of h_2 are contained in the stable manifold of the attracting fixed point of h_1, for h_1 and h_2 sufficiently small C^1 perturbation of g_1 and g_2. $\square$

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