

# $C^1$ robustly minimal iterated function systems

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## Abstract

We construct iterated function systems on compact manifolds that are  $C^1$  robustly minimal. On the  $m$ -dimensional torus and on two dimensional compact manifolds, examples are provided of  $C^1$  robustly minimal iterated function systems that are generated by just two diffeomorphisms.

**Keywords:** iterated function systems, robust property, minimal systems.

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## 1 Introduction

Our motivation for this paper comes from a result contained in [3] by Gorodetskiĭ and Il'yashenko on iterated function systems on the circle. They provide an example of an iterated function system generated by two circle diffeomorphisms, that is robustly minimal in the  $C^1$  topology. The example consists of an irrational rigid rotation and a diffeomorphism with an attracting and a repelling fixed point; we refer to [6, Proposition 12] for details of the construction.

We generalize this example to iterated function systems on  $m$ -dimensional compact Riemannian manifolds  $M$  by constructing examples of  $C^1$  robustly minimal iterated function systems on  $M$ . On the  $m$ -dimensional torus  $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$  and on

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two dimensional compact manifolds, i.e. compact surfaces, robust minimal iterated function systems with two generators exist. A somewhat related problem on minimality of an iterated function system generated by a generic pair of area preserving diffeomorphisms was recently raised in [7].

We begin to introduce definitions and notations of iterated function systems, and then formulate our main results. Consider a collection of diffeomorphisms  $\mathcal{L} = \{g_1, \dots, g_n\}$  on  $M$ . The iterated function system  $\mathcal{G}(M; g_1, \dots, g_n)$  on  $M$  generated by  $\mathcal{L}$  is given by iterates  $g_{i_1} \circ \dots \circ g_{i_k}$  with  $i_j \in 1, \dots, n$ . As is well known, iterated function systems are a popular way to generate and explore a variety of fractals [1, 2]. Consider the space  $\text{Diff}^1(M)$  of  $C^1$  diffeomorphisms of  $M$ , endowed with the  $C^1$  topology. Recall that a map  $f : M \rightarrow M$  is minimal if each closed subset  $X \subset M$  such that  $f(X) \subset X$  is empty or coincides with  $M$ . An iterated function system  $\mathcal{G}(M; g_1, \dots, g_s)$  on  $M$  is minimal if each closed subset  $A \subset M$  such that  $g_i(A) \subset A$  for all  $i$  is empty or coincides with  $M$ . Equivalently, for a minimal iterated function system  $\mathcal{G}(M; g_1, \dots, g_s)$ , for any  $x \in M$  the collection of iterates  $g_{i_1} \circ \dots \circ g_{i_k}(x)$ ,  $i_j \geq 0$ , is dense in  $M$ .

**Theorem 1.1.** *Let  $M$  be a compact connected  $m$ -dimensional manifold. Then there exist diffeomorphisms  $T_1, \dots, T_{m+3}$  on  $M$  and a neighborhood*

$$U \subset \underbrace{\text{Diff}^1(M) \times \dots \times \text{Diff}^1(M)}_{m+3 \text{ times}}$$

*of  $(T_1, \dots, T_{m+3})$  such that each element in  $U$  forms a minimal iterated function system on  $M$ .*

This result raises the question of the minimal number of generators of  $C^1$  robustly minimal iterated function systems. The following two results provide answers in two cases, iterated function systems on tori and compact surfaces.

**Theorem 1.2.** *There exists two diffeomorphisms  $T_1, T_2$  on the  $m$ -dimensional torus  $\mathbb{T}^m$  and a neighborhood  $U \subset \text{Diff}^1(M) \times \text{Diff}^1(M)$  of  $(T_1, T_2)$  such that each element in  $U$  forms a minimal iterated function system on  $\mathbb{T}^m$ .*

**Theorem 1.3.** *Let  $M$  be a compact connected surface. There exists two diffeomorphisms  $T_1, T_2$  on  $M$  and a neighborhood  $U \subset \text{Diff}^1(M) \times \text{Diff}^1(M)$  of  $(T_1, T_2)$  such that each element in  $U$  forms a minimal iterated function system on  $M$ .*

These theorems are proved in the following section.

## 2 Robust minimal iterated function systems

For  $x \in M$ , define  $\Gamma(x) \subset \text{Diff}^1(M)$  by

$$\Gamma(x) = \{g \in \text{Diff}^1(M) \mid 1 < \|Dg^{-1}(g(x))\| < 2 \text{ and } \frac{1}{2} < \|Dg(x)\| < 1\}.$$

Given a small open neighborhood  $V$  of a point  $a \in M$ , put

$$C_V = \{g \in \text{Diff}^1(M) \mid g(\bar{V}) \subset V \text{ and } \forall x \in \bar{V}, g \in \Gamma(x)\}.$$

Note that in the  $C^1$ -topology,  $C_V$  is open and the map  $\alpha : C_V \rightarrow V$  that takes each map to its fixed point in  $V$  is continuous.

We start with some lemmas.

**Lemma 2.1.** *For  $\mathcal{L} = \{g_1, \dots, g_s\}$  with  $g_i \in C_V$ ,  $i = 1 \dots s$ , there exists a unique non-empty compact set  $\Delta$  such that the iterated function system  $\mathcal{G}(\Delta; g_1, \dots, g_s)$  is minimal.*

*Proof.* Take  $\mathcal{L}^0(\bar{V}) = \bar{V}$ ,  $\mathcal{L}^1(\bar{V}) = \mathcal{L}(\bar{V}) = \bigcup_{i=1}^n g_i(\bar{V})$ ,  $\mathcal{L}^p(\bar{V}) = \mathcal{L}(\mathcal{L}^{p-1}(\bar{V}))$  for  $p > 1$ . Since  $\mathcal{L}(\bar{V}) \subset V$ ,

$$\bar{V} \supset \mathcal{L}(\bar{V}) \supset \mathcal{L}^2(\bar{V}) \supset \dots \supset \mathcal{L}^p(\bar{V}) \supset \dots$$

Now  $\Delta = \lim_{p \rightarrow \infty} \mathcal{L}^p(\bar{V})$  is a nonempty compact set that is invariant for  $\mathcal{L}$ . Since  $g_i$  are contractions on  $\bar{V}$ ,  $\Delta$  is the unique compact set that is invariant for  $\mathcal{L}$  [4]. Thus the iterated function system  $\mathcal{G}(\Delta; g_1, \dots, g_s)$  is minimal.  $\square$

An ordered set of points  $\{p_1, \dots, p_{m+1}\} \subset \mathbb{R}^m$  is called affine independent if  $\{\overrightarrow{p_1 p_2}, \overrightarrow{p_1 p_3}, \dots, \overrightarrow{p_1 p_{m+1}}\}$  is linearly independent.

**Lemma 2.2.** *In Lemma 2.1, it is possible to choose  $\mathcal{L} = \{g_1, \dots, g_{m+1}\}$ ,  $g_i \in C_V$  for  $1 \leq i \leq m+1$ , so that the interior of  $\Delta$  is nonempty.*

*Proof.* In coordinates we may assume that  $V \subset \mathbb{R}^m$ . Choose  $\{g_1, \dots, g_{m+1}\} \subset C_V$  such that the subset  $\{\alpha(g_1), \dots, \alpha(g_{m+1})\}$  is affine independent. Moreover, choose  $g_i$ ,  $i = 1, \dots, m+1$ , so that  $Dg_i(\alpha(g_i))$  is a multiple of the identity. Take a linear system  $\tilde{\mathcal{L}} = \{k_1, \dots, k_{m+1}\}$ , where  $k_i(x) = Dg_i(\alpha(g_i))(x - \alpha(g_i)) + \alpha(g_i)$ ,  $i = 1, \dots, m+1$ . By shrinking  $V$ , if necessary,  $k_i$  is arbitrary close to  $g_i$  on  $\bar{V}$ . It is clear that the set  $\tilde{\Delta} = \text{conv}\{\alpha(g_1), \dots, \alpha(g_{m+1})\}$  is an invariant set for  $\tilde{\mathcal{L}}$ , where  $\text{conv}\{a_1, \dots, a_{m+1}\}$  is the convex hull spanned by  $\{a_1, \dots, a_{m+1}\}$ .

Take  $\alpha'_i \subset \text{int}(\tilde{\Delta})$  close to  $\alpha(g_i)$ ,  $i = 1, \dots, m+1$ . Let  $\Delta^2 = \text{conv}\{\alpha'_1, \dots, \alpha'_{m+1}\}$ . Then

$$\Delta^2 \subset \tilde{\mathcal{L}}(\Delta^2) \subset \dots \subset \tilde{\mathcal{L}}^n(\Delta^2) \subset \dots$$

which implies that  $\lim_{n \rightarrow \infty} \tilde{\mathcal{L}}^n(\Delta^2) = \bigcup_{n \geq 0} \tilde{\mathcal{L}}^n(\Delta^2) = \text{int}\tilde{\Delta}$ . Since  $g_i$  is close to  $k_i$  on  $\bar{V}$ ,

$$\Delta^2 \subset \mathcal{L}(\Delta^2) \subset \dots \subset \mathcal{L}^n(\Delta^2) \subset \dots \tag{1}$$

and hence  $\text{int}\Delta \supset \bigcup_{i \geq 0} \mathcal{L}^i(\Delta^2)$ .  $\square$

The proof of Lemma 2.2 gives more than its statement as it includes arguments for  $C^1$  robust occurrence of invariant sets with nonempty interior.

**Corollary 2.3.** *Let  $\{g_1, \dots, g_{m+1}\}$ ,  $g_i \in C_V$  for  $1 \leq i \leq m+1$ , be such that  $\{\alpha(g_1), \dots, \alpha(g_{m+1})\}$  is affine independent (assuming, as in the proof of Lemma 2.2, that  $V \subset \mathbb{R}^m$ ). Then there exists a neighborhood  $W \subset \underbrace{\text{Diff}^1(M) \times \dots \times \text{Diff}^1(M)}_{m+1 \text{ times}}$  of  $(g_1, \dots, g_{m+1})$  such that each element  $\mathcal{F} = (f_1, \dots, f_{m+1})$  in this neighborhood admits an invariant set with non-empty interior.*

The above lemmas are ingredients in the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We prove the theorem by establishing the following: there exist an open neighborhood  $V$  of a point  $q \in M$ , an iterated function system  $\mathcal{L} = \{g_1, \dots, g_{m+1}\}$  with  $g_i \in C_V$  for  $1 \leq i \leq m+1$ , a diffeomorphism  $T$  on  $M$ , and a neighborhood  $U \subset (\text{Diff}^1(M))^{m+3}$  of  $(T, T^{-1}, g_1, \dots, g_{m+1})$  such that each element in  $U$  forms a minimal system on  $M$ .

Take a gradient Morse-Smale vector field on  $M$  with a unique hyperbolic repelling equilibrium  $q$  (see e.g. [8, Theorem 3.35] for the existence of Morse functions with unique extrema) and let  $T$  be its time-1 map. Let  $V$  be a small open neighborhood of  $q$ . By following the argument in the proof of Lemma 2.2 for  $\mathcal{L} = \{g_1, \dots, g_{m+1}\}$ , we can choose  $\alpha'_i \in \text{int}(\Delta)$  sufficient close to  $\alpha(g_i)$  for  $i = 1, \dots, m+1$  such that  $\Delta^2 = \text{conv}\{\alpha'_1, \dots, \alpha'_{m+1}\} \subset \text{int}\Delta$ . We may assume that  $q$  lies in the interior of  $\Delta^2$ . The unstable manifold of  $q$  for  $T$  lies dense in  $M$ . Iterates of  $\Delta$  under  $T$  therefore cover a dense subset of  $M$ . Finally, choose  $g_1, \dots, g_{m+1}$  so that each of the finitely many critical points of  $T$  is mapped into the unstable manifold of  $q$  for  $T$  by at least one of the maps  $g_i$ . It is now easily seen that the iterated function system  $\{T, T^{-1}, g_1, \dots, g_{m+1}\}$  is minimal.

For  $\tilde{g}_i$   $C^1$ -close to  $g_i$ , the system  $\{\tilde{g}_1, \dots, \tilde{g}_{m+1}\}$  is minimal on a compact set  $\tilde{\Delta}$  containing  $\Delta^2$ . A diffeomorphism  $\tilde{T}$  that is  $C^1$ -close to  $T$  has its fixed points near those of  $T$ , in particular a unique hyperbolic repelling equilibrium  $\tilde{q}$  inside  $\Delta^2$  with dense unstable manifold [9]. It follows that a system  $\{\tilde{T}, \tilde{T}^{-1}, \tilde{g}_1, \dots, \tilde{g}_{m+1}\}$  whose generators are sufficiently  $C^1$ -close to those of  $\{T, T^{-1}, g_1, \dots, g_{m+1}\}$  is minimal.  $\square$

We finish with the examples of robust minimal systems on tori and compact surfaces. The proofs use arguments similar to the ones above.

*Proof of Theorem 1.2.* Let  $T_1$  be a  $C^1$  diffeomorphism of  $\mathbb{T}^m$  possessing an attracting fixed point  $a = (a_1, \dots, a_m)$ , so that  $T_1 \in \Gamma(a)$  and  $DT_1(a)$  is a diagonal matrix. Let  $T_2$  be a minimal translation on  $\mathbb{T}^m$ , see e.g. [5, Section 1.4].

Since  $T_2$  is a minimal translation and  $C_V$  is open, there exist  $n_i \in \mathbb{N}$  such that

$$g_i = T_2^{n_i} T_1 \in C(V)$$

for  $i = 1, \dots, m+1$  and the set  $\{\alpha(g_1), \dots, \alpha(g_{m+1})\}$  is affine independent. We can apply the arguments in the proof of Lemma 2.2 for  $\mathcal{L} = \{g_1, \dots, g_{m+1}\}$ ; let  $\Delta^2$  be as in that proof so that (1) applies.

Iterates of  $\Delta^2$  under  $T_2$  cover  $\mathbb{T}^m$ ; thus there exists a finite subcover  $\Omega^+$ . Moreover, since also  $T_2^{-1}$  is minimal, it follows that there exists a finite subcover  $\Omega^-$  of  $\mathbb{T}^m$  by the images of  $\Delta^2$  under the iterates of  $T_2^{-1}$ . The same applies when  $T_2$  is replaced by  $\tilde{T}_2$  from a small neighborhood of  $T_2$  in  $\text{Diff}^1(\mathbb{T}^m)$ .

By Corollary 2.3, there exists a neighborhood  $W$  of  $(g_1, \dots, g_{m+1})$  such that for each element  $(f_1, \dots, f_{m+1})$  in this neighborhood, the iterated function system  $\{f_1, \dots, f_{m+1}\}$  has a unique compact set  $\Delta_{\mathcal{F}}$  with non-empty interior on which it is minimal. The interior of  $\Delta_{\mathcal{F}}$  contains  $\Delta^2$ .

We conclude that there is a small neighborhood  $U$  of  $(T_1, T_2)$  so that for each  $(\tilde{T}_1, \tilde{T}_2)$  in  $U$ ,  $\{f_1, \dots, f_{m+1}\}$  with  $f_i = \tilde{T}_2^{n_i} \tilde{T}_1$  acts minimally on a set that contains  $\Delta^2$ , and forward and backward iterates under  $\tilde{T}_2$  of  $\Delta^2$  cover  $\mathbb{T}^m$ . This implies that  $(\mathbb{T}^m, T_1, T_2)$  is  $C^1$  robustly minimal.  $\square$

*Proof of Theorem 1.3.* Let  $T_1$  be the time-1 map of a gradient Morse-Smale flow on  $M$ , with a unique attracting fixed point  $q_1$  and a unique repelling fixed point  $p_1$ . Let  $T_2$  likewise be the time-1 map of a gradient Morse-Smale flow on  $M$ , with a unique attracting fixed point  $q_2$  and a unique repelling fixed point  $p_2$ , with  $q_2$  close to  $p_1$  and  $p_2$  close to  $q_1$ . Further, choose  $T_1, T_2$  so that  $T_2$  maps fixed points of  $T_1$  into the stable manifold of  $q_1$  (which lies dense in  $M$ ).

Take  $T_1$  with  $DT_1(q_1)$  close to identity and with complex conjugate eigenvalues. Take  $T_1$  and  $T_2$  so that further  $T_2 T_1^k$  for some  $k \in \mathbb{N}$  possesses a hyperbolic attracting fixed point  $q_{12}$  near  $q_1$ , with  $DT_2 T_1^k(q_{12})$  close to identity and with complex conjugate eigenvalues. Moreover, we can ensure that  $T_1$  and  $T_2 T_1^k$  are linear in coordinates near  $\{q_1\} \cup \{q_{12}\}$ , and time-1 maps of flows  $\varphi_1^t, \varphi_{12}^t$ .

We claim that  $\mathcal{L} = \{T_1, T_{12}\}$  is minimal on a unique compact set  $\Delta$  with a nonempty interior that contains  $q_1, q_{12}$ . To see this, write  $l$  for the closed line piece connecting  $q_1$  and  $q_{12}$  and consider the set  $U$  which is the union of  $\cup_{t \geq 0} \varphi_1^t(l)$  and  $\cup_{t \geq 0} \varphi_{12}^t(l)$ . Since  $q_1$  and  $q_{12}$  are attracting fixed points for  $T_1$  and  $T_1 T_2$ ,  $U$  is a closed set that contains  $q_1$  and  $q_{12}$  in its interior. Since  $DT_1(q_1)$  and  $DT_1 T_2(q_{12})$  are near the identity, one has  $\mathcal{L}(U) \supset U$ . Hence  $U \subset \Delta$ . Noting that  $U$  contains a slightly longer linepiece that extends  $l$ , one sees that  $U \subset \text{int } \mathcal{L}^n(U)$  for some  $n \in \mathbb{N}$ . Then, for  $\tilde{T}_1, \tilde{T}_2$   $C^1$ -close to  $T_1, T_2$ , also  $U \subset \text{int } \tilde{\mathcal{L}}^n(U)$  for  $\tilde{\mathcal{L}} = \{\tilde{T}_1, \tilde{T}_2\}$ . Thus  $\tilde{\mathcal{L}}$  is minimal on a compact set that contains  $U$  in its interior.

Finally, choose  $T_1, T_2$  so that the repelling fixed point  $p_2$  of  $T_2$  will be contained in  $U$ . It is easily checked that this can be done: in complex coordinates near  $\{q_1\} \cup \{q_{12}\}$  one may take  $T_1(z) = az$ ,  $T_2(z) = bz + 1$  and  $T_2 T_1^k(z) = ba^k z + 1$ , and by varying  $a, b, k$  while keeping  $ba^k$  fixed one may assume that the fixed point of  $T_2$  is near 0 and hence lies in  $U$ . The properties of  $T_1, T_2$  now imply that  $\mathcal{G}(M; T_1, T_2)$  is  $C^1$  robustly minimal, compare the proof of Theorem 1.1.  $\square$

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## References

- [1] M. F. Barnsley. *Fractals everywhere*. Academic Press Professional, Boston, MA, second edition, 1993.
- [2] K. Falconer. *Techniques in fractal geometry*. John Wiley & Sons Ltd., Chichester, 1997.
- [3] A. S. Gorodetskiĭ and Yu. S. Il'yashenko. Certain properties of skew products over a horseshoe and a solenoid. Grigorchuk, R. I. (ed.), *Dynamical systems, automata, and infinite groups*. Proc. Steklov Inst. Math. 231, 90-112 (2000).
- [4] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.
- [5] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995.
- [6] V.A. Kleptsyn and M.B. Nalskii. Contraction of orbits in random dynamical systems on the circle. *Funct. Anal. Appl.*, 38(4):267–282, 2004.
- [7] A. Koropeccki and M. Nassiri. Transitivity of generic semigroups of area-preserving surface diffeomorphisms. Preprint, 2009.
- [8] Y. Matsumoto. *An introduction to Morse theory*, volume 208 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2002.
- [9] J. Palis. On Morse-Smale dynamical systems. *Topology*, 8:385–404, 1968.