Atomic disintegrations for partially hyperbolic diffeomorphisms

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Abstract

Shub & Wilkinson and Ruelle & Wilkinson studied a class of volume preserving diffeomorphisms on the three dimensional torus that are stably ergodic. The diffeomorphisms are partially hyperbolic and admit an invariant central foliation of circles. The foliation is not absolutely continuous, in fact, Ruelle & Wilkinson established that the disintegration of volume along central leaves is atomic. We show that in such a class of volume preserving diffeomorphisms the disintegration of volume along central leaves is a single delta measure.

MSC 37C05, 37D30

1 Introduction

We consider volume preserving perturbations of the following diffeomorphisms on the three dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$:

$$(x, y, z) \mapsto (A(x, y), z), \tag{1}$$

where $A \in GL(2, \mathbb{Z})$ is a hyperbolic torus automorphism. The diffeomorphism (1) has an invariant foliation of circles $\{(x, y) = \text{constant}\}$. By classical work on normal hyperbolicity [15], the perturbations admit an invariant foliation of central leaves that are circles close to $\{(x, y) = \text{constant}\}$.

Shub & Wilkinson [21] establish the existence, arbitrarily close to (1), of a C^1 open set of C^2 volume preserving diffeomorphisms that are ergodic with respect to volume. They show that the diffeomorphisms from the open class admit a positive central Lyapunov exponent (computed for vectors tangent to central leaves) Moreover, by Ruelle & Wilkinson [20], a set of full Lebesgue measure for which the Lyapunov exponents exist, intersects almost every circle from the invariant foliation in k points for some finite integer k.

The arguments followed by Ruelle & Wilkinson involve Pesin theory, in particular the construction of local unstable manifolds in nonuniformly hyperbolic systems. With such methods it is not clear how to obtain further information on the number of atoms k. We strengthen the result by showing that the theorem holds with $k = 1$, so, combined with [20] (see also [6, Section 7.3]), we arrive at the following result.

Theorem 1.1. *In any neighborhood of* (1) *there is a* C^1 *open set* $\mathcal{U}$ *of* C^2 *volume preserving diffeomorphisms on* $\mathbb{T}^3$*, so that for each* $F \in \mathcal{U}$*,*

1. *F is ergodic with respect to Lebesgue measure,*
2. *there is an invariant foliation of C^2 circles $W^c(p)$, $p \in \mathbb{T}^3$, so that for Lebesgue almost all p, if $v \in T_pW^c(p)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \ln |DF^n(p)v| = \lambda$$

   for some $\lambda < 0$,

3. *the disintegrations of Lebesgue measure along central leaves are delta measures (in particular, there is an invariant set of full Lebesgue measure in $\mathbb{T}^3$ that intersects almost every central leaf in a single point).*

Note that the central Lyapunov exponent in the formulation of the theorem is negative, the inverse diffeomorphisms possess a positive central Lyapunov exponent as in [20].

The regularity properties of invariant foliations play a crucial role. The foliations of strong stable manifolds and of strong unstable manifolds are absolutely continuous; that is, the holonomy between cross sections maps sets of measure zero to sets of measure zero. Also, a local set that intersects leaves of local strong (un)stable manifolds in sets of leaf measure zero, has Lebesgue measure zero in $\mathbb{T}^3$. See [18] for more information. The properties in the theorem express the statement that the central foliation of circles is not absolutely continuous.

To prove the above theorem, we augment the arguments from [20] with roughly the following type of reasoning, as worked out in the following section. Suppose there is a central leaf that is periodic under F of period ℓ, say, such that F^ℓ has a unique hyperbolic attracting and a unique hyperbolic repelling fixed point on this leaf. Iterates near this leaf pick up much contraction, mapping the leaf measure on the leaf to a measure close to a delta measure. It is also needed that iterates are sufficiently often near the periodic central leaves to pick up enough contraction. Making use of topological transitivity, we show that for almost all $p \in \mathbb{T}^3$, pushing forward leaf measure on $W^c(p)$ by high iterates F^n, yields subsequences where the pushed forward leaf measure gets close to a delta measure. Additional reasoning establishes convergence to a delta measure. Combined with the results contained in [20] this suffices to prove the theorem. In our arguments we make use of absolute continuity of strong stable and strong unstable foliations, and of bounded distortion properties for iterates of points in local strong stable and local strong unstable manifolds.

The diffeomorphisms we consider are partially hyperbolic diffeomorphisms; we refer to [6, 18] for background. Various of the dynamical properties of the diffeomorphisms occur in this more general context. There is further a similarity with techniques to prove synchronization in random or forced circle diffeomorphisms [13, 16, 23].

2 Proof of the theorem

We first introduce some notation for invariant manifolds. Write $W^i(p)$, $i = ss, c, uu$, for the strong stable, center or strong unstable manifold containing p. Further, $W^{ss,c}(p)$ is the center stable manifold and $W^{c,uu}(p)$ is the center unstable manifold containing p.

Recall that a foliation on a manifold is minimal if all its leaves lie dense in the manifold. Minimal strong stable or strong unstable foliations are abundant in the context of partially hyperbolic diffeomorphisms [5].

Lemma 2.1. *In any neighborhood of* (1) *there is a diffeomorphism F with the following properties:*

1. *there is a central leaf, fixed for F, with a unique hyperbolic attracting fixed point P and a unique hyperbolic repelling fixed point Q,*
2. *the strong unstable and strong stable foliations are minimal.*

Moreover, these properties are robust.

Proof. The map given as the inverse $(j \circ h)^{-1}$ with

$$h(x, y, z) = (2x + y, x + y, x + y + z + b\sin(2\pi y)),$$
$$j(x, y, z) = (x + (1 + \sqrt{5})a\cos(2\pi z), y + 2a\cos(2\pi z), z) \tag{2}$$

for a, b small, which is the inverse of the map used in [20], is one example of a map for which the first item holds. In the first item the fixed central leaf can be replaced by a periodic central leaf with only notational changes in the following.

For the second item, [5] discusses minimal strong unstable and strong stable foliations in general, not necessarily conservative (volume preserving) diffeomorphisms. But as the basic tool of blenders [4, 6] is available for conservative diffeomorphisms [19], their construction can be followed and thus the second item holds.

For convenience of the reader we spend a few words on clarifying the use of blenders. Consider a hyperbolic periodic point P with one dimensional unstable manifold $W^{uu}(P)$. A blender associated with P is an open set V near P so that $W^{uu}(P)$ intersects each center-stable strip that stretches through V (see the references mentioned above).

Consider a diffeomorphism possessing a periodic point P with one dimensional unstable manifold and a periodic point with two dimensional unstable manifold Q, such as (2). In [19] it is established that there are arbitrarily small perturbations of such diffeomorphisms that admit a heterodimensional cycle. Blenders are found in further arbitrarily small perturbations from here, and hence blenders occur arbitrarily close to (2).

We note that a blender associated with P gives a hyperbolic set, containing a dense set of periodic points with one dimensional unstable manifold, close to P. Again resorting to [19], an arbitrarily small perturbation ensures that $W^{ss}(Q)$ intersects V. Then $W^{c,uu}(Q) \subset \overline{W^{uu}(P)}$: high iterates of a small neighborhood O of a point in $W^{c,uu}(Q)$ under F^{-1} accumulate onto $W^{ss}(Q)$ by the λ-lemma and hence contain points accumulating onto $W^{uu}(P)$ due to the blender associated with P.

Since center unstable leaves are dense in $\mathbb{T}^3$ and hence $W^{c,uu}(Q)$ is dense in $\mathbb{T}^3$, we get that $W^{uu}(P)$ is dense in $\mathbb{T}^3$. Since strong unstable manifolds accumulate onto $W^{uu}(P)$, all strong unstable manifolds are dense in $\mathbb{T}^3$, that is, the strong unstable foliation is minimal. Similarly one obtains a minimal strong stable foliation. □

We remark that a conservative perturbation of (2) with minimal strong stable and strong unstable foliations is ergodic [10].

We use a partition of $\mathbb{T}^3$ which is perhaps easiest explained by making use of a topological conjugacy to a skew product system, as in the following result from Gorodetskiĭ [14].

Lemma 2.2. *There is a homeomorphism h on $\mathbb{T}^3$ with $h \circ F = S \circ h$, where S is a skew product diffeomorphism*

$$S(x, y, z) = (A(x, y), F_{x,y}(z)),$$

for a hyperbolic torus automorphism A and with $z \mapsto F_{x,y}(z)$ a diffeomorphism depending continuously on (x, y).

Take a Markov partition $\mathcal{R} = \{R_1, \dots, R_n\}$ for the base dynamics. Recall that a partition element R_i is a rectangle, bounded by segments in local stable and local unstable manifolds. Consider the partition of $\mathbb{T}^3$ with partition elements $R_i \times \mathbb{T}$. The image under the topological conjugacy h is a partition $\{S_1, \dots, S_n\}$ of $\mathbb{T}^3$. The partition elements are diffeomorphic to a product of a rectangle and a circle, its boundaries are parts of center stable and center unstable manifolds of F. Note that the boundaries of the partition elements (and their forward and backward orbits) are of zero Lebesgue measure.

We write $W^{ss}_{loc}(p)$ for the local strong stable manifold containing p with boundary points in the boundary of a partition element S_i. Likewise other local invariant manifolds such as $W^{ss,c}_{loc}(p)$ have their boundary in the boundary of a partition element S_i.

The following lemma contains the key argument for the proof of the theorem. Its proof relies on minimality of the strong unstable foliation. We denote the leaf measure (Lebesgue measure) on central leaves by λ.

Lemma 2.3. *For Lebesgue almost all $(x, y, z) \in \mathbb{T}^3$, $F^n|_{W^c(F^{-n}(x,y,z))}\lambda$ contains a delta measure in its limit points in the weak star topology.*

Proof. Consider the equivalence relation $p \sim q$, for $p, q \in S_i$, if $p \in W^{ss}_{loc}(q)$. Identifying equivalent points will map each S_i to a side of the boundary of S_i in a center unstable manifold diffeomorphic to $I^u_i \times \mathbb{T}$ for an interval I^u_i. By using coordinates and combining all the intervals I^u_i a single interval I^u is obtained. The torus automorphism A induces a piecewise expanding Markov map $f : I^u \to I^u$, monotone and continuous on a finite collection of subintervals of I^u. See Figure 1 for an illustration.

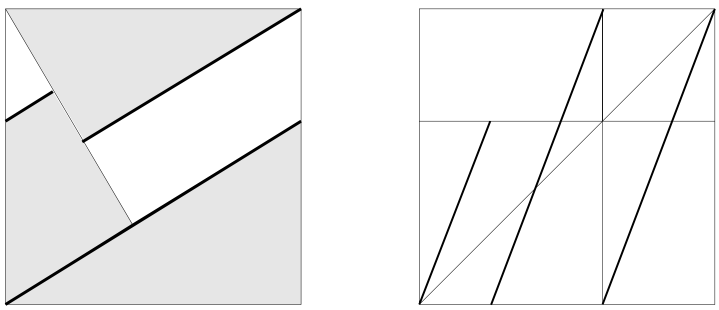

Figure 1: *As illustration a Markov partition with two partition elements for the torus automorphism* $(x, y) \mapsto (2x+y, x+y) \mod 1$ *is shown. The interval I^u is built by taking an unstable side for each partition element, as indicated by the bold line pieces. The resulting piecewise expanding Markov map $f : I^u \to I^u$ is depicted on the right.*

A coordinate x^u in I^u identifies a local center stable manifold $W^{ss,c}_{loc}(x^u)$. Likewise, $x^u \in I^u, z \in \mathbb{T}$ define a local strong stable manifold $W^{ss}_{loc}(x^u, z)$, which is parameterized by a coordinate x^s. A pair (x^s, x^u) identifies a central leaf and we may therefore write $W^c(x^s, x^u)$.

We prove that on each $W^{ss}_{loc}(x^u_0, z_0)$, the set of x^s_0 values for which the statement of the lemma holds has full Lebesgue measure. Since the strong stable foliation is absolutely continuous this proves the lemma. For intervals I inside central leaves we also write $|I| = \lambda(I)$ to denote their length. Note that the length is measured according to the z coordinate in a central leaf; the length of an interval however goes to zero precisely if its Lebesgue measure goes to zero.

For a point $(x^s_0, x^u_0, z_0) \in \mathbb{T}^3$, write $(x^s_{-i}, x^u_{-i}, z_{-i}) = F^{-i}(x^s_0, x^u_0, z_0)$. Fix $\varepsilon > 0$. Define

$$\Delta_m = \{x^s_0 \in W^{ss}_{loc}(x^u_0, z_0) \mid \text{ for each } i \leq m \text{ and each interval } I \subset W^c(x^s_{-i}, x^u_{-i}) \text{ with } |I| > 1 - \varepsilon, \\ |F^i(I)| > \varepsilon\}.$$

As a consequence of the construction in the following steps, we establish that Δ_m has Lebesgue measure going to zero as $m \to \infty$.

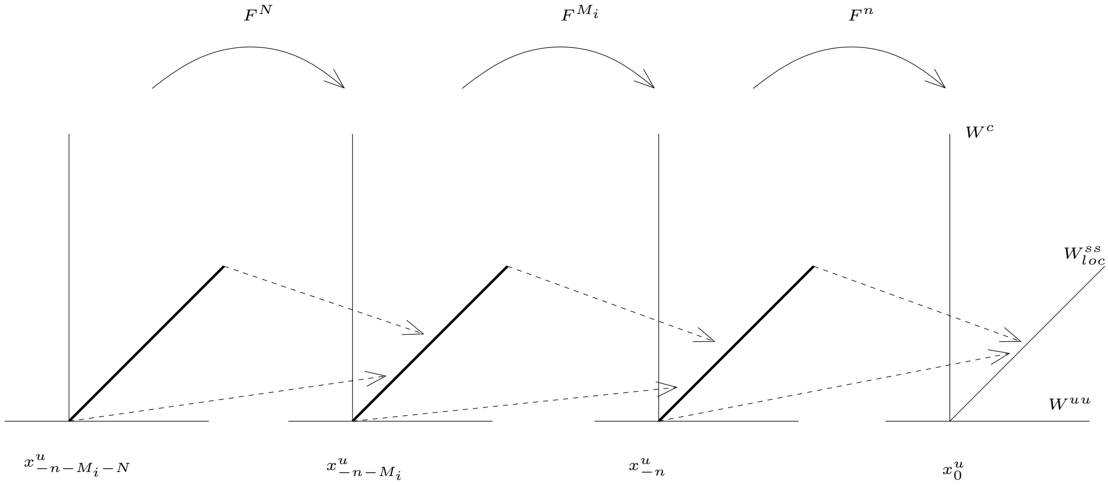

Figure 2: *The intuitive idea behind the construction in Step 1-4 in the proof of Lemma 2.3 is the following: given x^u_0 and x^u_{-n}, a bounded number $M_i + N$ of additional inverse iterates are found so that F^{N+M_i+n} maps some large interval V (of length larger than $1 - \varepsilon$) in a central leaf $W^c(x^s_{-n-M_i-N}, x^u_{-n-M_i-N})$ to a small interval (of length smaller than ε). The construction makes use of iterates near a fixed central leaf with a unique attracting fixed point. There is a bound on the required additional number of inverse iterates that is independent of n. This means that it is common to find iterates that map a large interval in a central leaf to a small interval.*

Let $x^u_{-n} \in f^{-n}(x^u_0)$, thus $F^n(W^{ss,c}_{loc}(x^u_{-n})) \subset W^{ss,c}_{loc}(x^u_0)$. The local strong stable manifold $W^{ss}_{loc}(x^u_{-n}, z_{-n})$ is mapped to an interval inside $W^{ss}_{loc}(x^u_0, z_0)$ (the image being exponentially small in n) by F^n. Figure 2 attempts to illustrate elements of the construction below.

Step 1. Recall from Lemma 2.1 the existence of a central leaf, fixed by F, containing an attracting fixed point P and a repelling fixed point Q. Note that any interval in $W^c(P) \setminus Q$ is contracted under iterating by F. The existence of strong stable and strong unstable foliations near $W^c(P)$ shows that a similar contraction occurs on central leaves near $W^c(P)$ as long as iterates remain near $W^c(P)$. So, we can choose an interval $K \subset I^u$, so that for $x^u \in K$ and $W^c(x^s, x^u)$ close to $W^c(P)$, there are an interval $V \subset W^c(x^s, x^u)$ with $|V| > 1 - \varepsilon$ and $N \in \mathbb{N}$, so that

$$|F^N(V)| < \varepsilon.$$

Here, considering K as part of $W^{uu}_{loc}(P)$, one can take K to be a fundamental domain in $W^{uu}_{loc}(P)$ with $K, F(K), \ldots, F^N(K)$ near $W^c(P)$. For use in the following steps we note that a stronger contraction is obtained (the image $F^N(V)$ can be made smaller) when replacing K by $f^{-i}(K)$ and N by $N+i$.

Step 2. Let $q > 1/\varepsilon$. The strong unstable manifold $W^{uu}(P)$ lies dense in $\mathbb{T}^3$, in fact, iterates of any compact subinterval of $W^{uu}(P)$ lie dense in $\mathbb{T}^3$. One can therefore take points $(y^s_{M_i}, y^u_{M_i}, a_{M_i}) \in F^{M_i}(F^N(K))$, $1 \le i \le q$ (the points being written in coordinates in $I^s \times I^u \times \mathbb{T}$), so that

1. $y^u_{M_i} = x^u_{-n}$ (that is the points $(y^s_{M_i}, y^u_{M_i}, a_{M_i})$ lie in the same local center stable manifold $W^{ss,c}_{loc}(x^u_{-n})$),
2. the leaf coordinates a_{-M_i} are mutually different (that is, the points $(y^s_{M_i}, y^u_{M_i}, a_{M_i})$ lie in different local strong stable manifolds).

The numbers M_i, $1 \le i \le q$, are bounded independent of n.

Step 3. Consider inverse images $x^u_{-N-M_1} \in f^{-N-M_1}(x^u_{-n}), \ldots, x^u_{-N-M_q} \in f^{-N-M_q}(x^u_{-n})$. For N large enough (independent of n), $F^{N+M_i}(V)$ are mutually disjoint intervals (when projecting along strong stable manifolds; all intervals lie in the same center unstable manifold). As $q > 1/\varepsilon$, for one of the intervals $F^{N+M_i}(V)$ the image under F^n has length smaller than ε. There is $D > 0$, so that for any $z \in \mathbb{T}$, with F^{N+M_i} mapping $W^{ss}_{loc}(x^u_{-n-N-M_i}, z)$ into $W^{ss}_{loc}(x^u_{-n}, \hat{z})$, we have

$$|W^{ss}_{loc}(x^u_{-n-N-M_i}, z)| / |W^{ss}_{loc}(x^u_{-n}, \hat{z})| > D.$$

Step 4. The construction in the steps above depends on ε, but not on n. So there are bounds on N and M_i as functions of ε, but uniformly in n. We repeat that, writing $l = N + M_i$, each $W^{ss}_{loc}(x^u_{-n-l}, z)$ for varying z is mapped into some local strong stable manifold $W^{ss}_{loc}(x^u_{-n}, y)$, where the quotient of the lengths is bounded away from zero. By bounded distortion, iterating under F^n does not change too much relative length of intervals. Bounded distortion of F^n on W^{ss}_{loc} means there is $C > 0$ so that

$$\frac{1}{C} \le \frac{|DF^n(p)e^{ss}|}{|DF^n(q)e^{ss}|} \le C$$

where e^{ss} is a unit tangent vector to W^{ss}_{loc}. Bounded distortion applies since strong stable (and strong unstable) manifolds are $C^{1+\alpha}$ for some $\alpha > 0$, with a uniform bound for the Hölder constant for all manifolds. See e.g. [12, Section V.2].
For any n, with F^l mapping $W^{ss}_{loc}(x^u_{-n-l}, z_{-n-l})$ into $W^{ss}_{loc}(x^u_{-n}, z_{-n})$, and F^n mapping $W^{ss}_{loc}(x^u_{-n}, z_{-n})$ into $W^{ss}_{loc}(x^u_0, z_0)$, we have

$$|F^{n+l}(W^{ss}_{loc}(x^u_{-n-l}, z_{-n-l}))| / |F^n(W^{ss}_{loc}(x^u_{-n}, z_{-n}))| > D$$

for some $D > 0$, independent of n.

The above construction steps prove that Δ_{tl} has Lebesgue measure less than $(1-\delta)^t$ for some $\delta > 0$, $l \in \mathbb{N}$. So

$$\Delta = \{x^s_0 \in W^{ss}_{loc}(x^u_0, z_0) \mid \text{ for each } i \text{ and for each interval } I \subset W^c(x^s_{-i}, x^u_{-i}) \text{ with } |I| > 1 - \varepsilon, \; |F^i(I)| > \varepsilon\}$$

has zero Lebesgue measure. This in turn proves the lemma, by varying $x^u_0 \in I^u$, $z_0 \in \mathbb{T}$, noting the absolute continuity of the strong stable foliation, and letting ε go to zero. □

The next step is to obtain convergence of the push forwards of leaf measures to a delta measure.

Lemma 2.4. *For Lebesgue almost all $(x, y, z) \in \mathbb{T}^3$, $F^n|_{W^c(F^{-n}(x,y,z))}\lambda$ converges to a delta measure in the weak star topology.*

Proof. Recall the partition $\{S_1, \ldots, S_n\}$ of $\mathbb{T}^3$ from the proof of Lemma 2.3, and consider F acting on the union $S = \cup_i S_i$ of partition elements. Note that F acting on $\mathbb{T}^3$ is obtained by gluiing partition elements along boundaries.

The lemma is proved by applying [11, Proposition 3.1] (see also [2, Theorem 1.7.2]) that treats relations between invariant measures for endomorphisms and their natural extensions. These results are formulated for skew product diffeomorphisms and translate to our setting by Lemma 2.2.

For a point p from a partition element S_i, write $\pi^{ss}(p)$ for its projection along the leaf $W^{ss}_{loc}(p)$ onto a center unstable side, which we denote by T_i, of S_i. Write F^+ for the dynamical system on $T = \cup_i T_i$, obtained by composing F with π^{ss}. We have the following properties, implying that F is the natural extension of F^+, see [2, Appendix A]:

1. F^+ is a factor of F,
2. With $\mathcal{F}$ the Borel σ-algebra on $\mathbb{T}^3$, $\mathcal{F}^+$ the Borel σ-algebra on T, and $\mathcal{G} = (\pi^{ss})^{-1}(\mathcal{F}^+)$, we have $\sigma(F^n(\mathcal{G}), n \in \mathbb{N}) = \mathcal{F} \mod 0$. Here $\sigma(F^n(\mathcal{G}), n \in \mathbb{N})$ is the σ-algebra generated by $F^n(\mathcal{G})$.

Under the homeomorphism h that provides the topological conjugacy $h \circ F = G \circ h$ from Lemma 2.2, Lebesgue measure λ on $\mathbb{T}^3$ is pushed forward to the measure $h\lambda$ with a marginal Λ on $\mathbb{T}^2$. Let $G^+ = h \circ F^+ \circ h^{-1}$. Note that $\nu^+ = h\pi^{ss}\lambda$ is an invariant measure for G^+. Interpret ν^+ as a measure on S with σ-algebra $h(\mathcal{G})$. Now [11, Proposition 3.1] provides convergence of measures

$$F^n_{A^{-n}(x,y)}\nu^+_{A^{-n}(x,y)} \to \nu_{x,y}, \tag{3}$$

in the weak star topology, for Λ almost all x, y.

If $C \subset \mathbb{T}^2$ is a set of full Λ measure, then $h\lambda(C \times \mathbb{T}) = 1$, and therefore $\lambda(h^{-1}(C \times \mathbb{T})) = 1$. We hence obtain the following statement. Take Lebesgue measure λ and consider the corresponding invariant measure $\mu^+ = \pi^{ss}\lambda$ for F^+. While λ is ergodic, by [11] also μ^+ is ergodic. Write μ^+_p for its disintegrations on $W^c(p)$. Interpreting μ^+ as a measure on S with σ-algebra $\mathcal{G}$, one has

$$F^n|_{W^c(F^{-n}(x,y,z))}\mu^+_{F^{-n}(x,y,z)} \to \mu_{x,y,z}, \tag{4}$$

for Lebesgue almost all $(x, y, z) \in \mathbb{T}^3$, with convergence in the weak star topology.

By Lemma 2.5 below and compactness of the state space, given $\varepsilon > 0$ there is $\delta > 0$ so that uniformly in $p \in T$, $\mu^+_p(I) > \delta$ for each interval I with $|I| > \varepsilon$. As a consequence, the reasoning of Lemma 2.3 applies with Lebesgue measure on central leaves $W^c(p)$ replaced by μ^+_p and therefore gives convergence of $F^n|_{W^c(F^{-n}(x,y,z))}\lambda$ and $F^n|_{W^c(F^{-n}(x,y,z))}\mu^+_{F^{-n}(x,y,z)}$ to a delta measure. □

The following lemma is applied in the proof above, and contains properties on support and regularity of the disintegrations μ^+_p introduced in the proof above. We remark that [22] contains stronger results on absolute continuity of center stable manifolds, and hence of the disintegrations μ^+_p (see the arguments in the proof below). Recall that a measure is diffuse if it has no atoms.

Lemma 2.5. *The disintegrations μ^+_p are diffuse, have support equal to $W^c(p)$ and vary continuously with p in the weak star topology.*

Proof. Recall the Markov partition of $\mathbb{T}^3$ with partition elements $R_i \times \mathbb{T}$. Note that in this setup,

$$\mu_p^+ = \mathbb{E}(\lambda \mid \mathcal{G})_p, \tag{5}$$

see the proof of Lemma 2.4 and [2, Theorem 1.7.2]. Lebesgue measure has disintegrations λ_p along center leaves. We claim that λ_p is u-invariant, meaning that the disintegrations λ_p are invariant under the holonomy along strong unstable leaves.

Take a measure m which is u-invariant inside partition elements and has absolutely continuous disintegrations along strong unstable leaves it is supported on. One may take for m a measure supported on a single strong unstable leaf inside a partition element. A Césaro accumulation point of push forwards $F^i m$, $i \geq 0$, is a Gibbs u-measure [9, 17]. As F admits a negative central Lyapunov exponent, the Gibbs u-measure is unique [7] and therefore equal to Lebesgue measure. So $F^i m$ converges to Lebesgue measure as $i \to \infty$.

To get that Lebesgue measure is u-invariant, consider coordinates as in Lemma 2.2 in which the center foliation is affine and in which further the strong unstable foliation is affine (locally, inside a partition element). Take m supported on a single strong unstable leaf inside each partition element, equal to the absolutely continuous measure that is invariant under the piecewise expanding Markov map f, on that leaf. In these coordinates the measures $F^i m$, $i \geq 0$, are product measures. This makes clear that Lebesgue measure appears as a product measure, i.e. the disintegrations of $F^i m$ converge to the disintegrations of Lebesgue measure and Lebesgue measure is u-invariant. See [8, Section 4] for a detailed treatment in a similar situation, and also [3, Remark 4.1].

If an open set has positive measure, also the image under F has positive measure. Since λ is invariant and the strong unstable lamination is minimal, with (5) this shows that the support of μ_p^+ equals $W^c(p)$. Continuous dependence of μ_p^+ on p is implied by (5) and u-invariance of λ. By ergodicity μ_p^+ is diffuse. □

We quote from [20] (see also [11, Proposition 2.6] or [18, Lemma 10.5]):

Lemma 2.6. *Let $0 < \kappa < 1$. There is $R > 0$ and a set $\Lambda \subset \mathbb{T}^3$ of positive Lebesgue measure, intersecting central leaves in disintegrated measure at least κ, so that for each pair $p \in \Lambda$, $q \in W^c(p)$ that are within distance R of each other, p and q are in the same stable manifold inside the central leaf.*

The Poincaré recurrence theorem yields that Lebesgue almost all points in Λ revisit Λ infinitely often under iteration by F and F^{-1}. Now, since F is a diffeomorphism, Lemma 2.3 says that $F^{-n}|_{W^c(x,y,z)}\lambda$ is close to a delta measure on $W^c(F^{-n}(x,y,z))$ for n large. This would contradict Lemma 2.6 unless there is a single delta measure in central leaves. This ends the proof of Theorem 1.1 and the paper.

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