Minimale scheve produkten
Dynamische Systemen

For irrational $\alpha \in (0,1)$, consider the skew product map

$$T(x,y) = (x+\alpha, y+x) \tag{1}$$

on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We claim that this defines a minimal map. Note that the base system $x \mapsto x+\alpha$ on the circle $\mathbb{R}^2/\mathbb{Z}^2$ is minimal.

We first demonstrate that T is topogically transitive, so that a dense orbit exists. Take two nonempty open sets $U, V \subset \mathbb{T}^2$. We need to establish that there exists $n \in \mathbb{N}$ so that $T^n(U) \cap V \neq \emptyset$. Without loss of generality we may assume that U and V are product sets. Calculate

$$T^n(x,y) = (x+n\alpha, y+nx+\frac{1}{2}n(n-1)\alpha) \tag{2}$$

It follows that a horizontal line piece $I \times \{y\}$ is wrapped around the torus by a sufficiently high iterate of T (this is a consequence of the term nx in the second coordinate). Since the base space dynamics $x \mapsto x+\alpha$ is topologically transitive, this implies that T is topologically transitive. Hence, there is a point (q,p) with a dense forward orbit (see Proposition 2.2.1).

We continue by showing that the orbit of (q,p) is a minimal set, by estimating return times to a small neighborhood of it (recall from Proposition 2.1.3 that $\overline{\mathcal{O}^+(q,p)}$ is minimal precisely if (q,p) is almost periodic). Let W be a smal neighborhood of (q,p). Because the positive orbit of (q,p) is dense, there is a subset of integers $A \subset \mathbb{N}$ so that $T^i(q,p) \in W$ for $i \in A$. Assume by contradiction that A is not relatively dense. Then there is a sequence i_j with $T^{i_j}(q,p) \to (r,o) \in W$ and the positive orbit of (r,o) is outside W (compare the proof of Proposition 2.1.3). As the base system is minimal, the positive orbit of (r,o) does accumulate onto the fiber $\{q\} \times \mathbb{S}^1$. Let (q,n) be an accumulation point of $\mathcal{O}^+(r,o)$.

Now make the following observation. Let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ so that $(q,p), T(q,p), \ldots, T^N(q,p)$ is ε-dense in the torus. From (2) it is clear that the same holds for any point (q,m), in particular for (q,n): $(q,n), T(q,n), \ldots, T^N(q,n)$ is ε-dense in the torus. Chosing ε small enough, one ensures that one of these points lies in W. By continuity a small neighborhood of (q,n) ends up in W. This contradicts the statement that the positive orbit of (r,o), which accumulates onto (q,n), is disjunct from W.