

The Notion of Convexity in Music

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Abstract

It has been observed that tonal spaces can be mapped onto a two-dimensional lattice \mathbb{Z}^2 [1]. It has also been shown that various musical items, such as the major and minor diatonic scales, form convex subsets in this space [1, 5]. This triggers the question whether convexity is a more widespread concept in music. The current paper systematically investigates the convexity for a number of musical phenomena among which scales, chords, (harmonic) reduction and pitch spelling. From the four results, we hypothesize that the notion of convexity may be a covering concept of musical phenomena and that it may serve as a necessary condition for the 'goodness' of musical items.

1 Introduction

How do we perceive music? Are there general principles that govern our musical perception or that determine the "goodness" of musical items? While several (perceptual) principles have been proposed, ranging from the Gestalt laws [15] to preference rules [6], there is little or no empirical research on "goodness" principles for music. That is, what makes music music? When is a sequence of notes a musical scale? To explore such general questions, it is not enough to make recourse to West-

ern music history or to traditional notions of harmony: many more scales and chords have been proposed in the history of music than those treated in traditional textbooks. The current paper intends to make a start in finding general empirical principles for the "goodness" of musical items. The items we discuss range from ancient Greek scales to Chinese Zhou scales, and from the major triad to the eleventh and thirteenth chords. Together with other musical items (pitch spelling, reductions) we found that there is a highly persistent empirical principle holding for all well-formed musical objects: if represented in a tonal space, scales, chords, reductions and pitch spellings are virtually always convex or star-convex. But what is convexity and star-convexity of musical items? In the next section, we will first explain these two topological notions and how they apply to music. In section 3, we show how they apply to scales (in 5-limit just intonation). Next, in section 4, the convexity of chords is investigated. Section 5 deals with convexity in harmonic reduction, and in section 6 we argue how pitch spelling can make use of convexity.

2 Convexity in thirds-space

Convexity is a notion from topology. It is defined as follows: A subset Y of \mathbb{R}^n is said to be convex if $\alpha x + (1 - \alpha)y$ is in Y whenever x and y are in Y and $0 \leq \alpha \leq 1$.

or equal tempered midi numbers. A projection from the note names to these numbers can be made to obtain the thirds space of midi numbers (fig. 2).

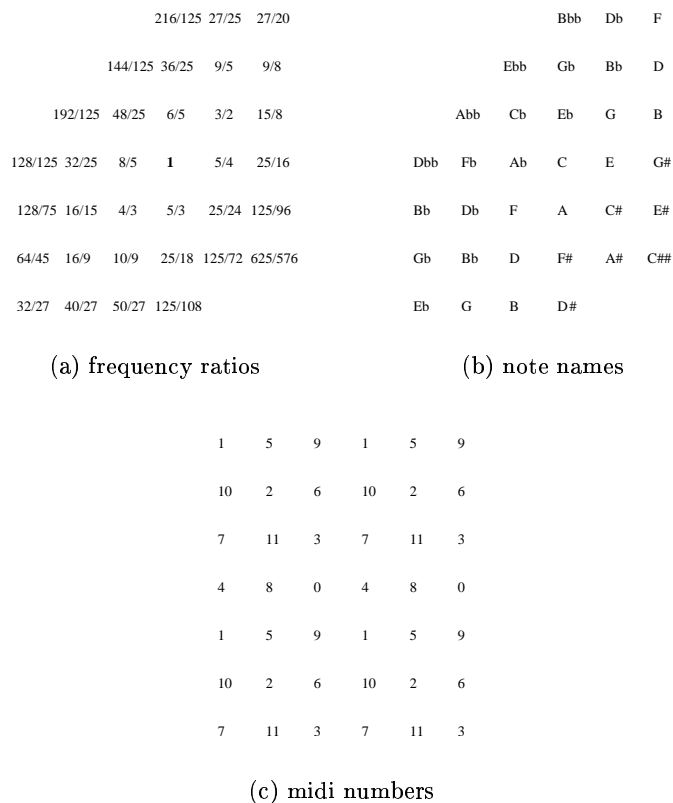


Figure 2: Three representations of thirds-space: intervals space, note-name space, and space of midi numbers. (We left out some complex frequency ratios and note names with many accidentals, however all three spaces are infinite in horizontal and vertical directions.)

It is clear that a convex object in the intervals space is also convex in the note name space and the midi number space. The other way round however, mapping the figures from right to left (in figure 2) is not necessarily true. Mapping figure 2c onto 2b there is for example the problem which note name ($C, B\#\#, \dots$) should be assigned to number '0', and mapping fig. 2b onto 2a we see that a D can refer to ratio $9/8$ as well as to $10/9$.

3 Convexity of scales

The major scale in 5-limit just intonation is defined as the scale in which each of the major triads I, IV and V is taken to have frequency ratios $4 : 5 : 6$. From this can be calculated that the ratios of the scale are given by: $1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8$. These are indicated in figure 3. In this figure, the (neutral) minor scale is indi-

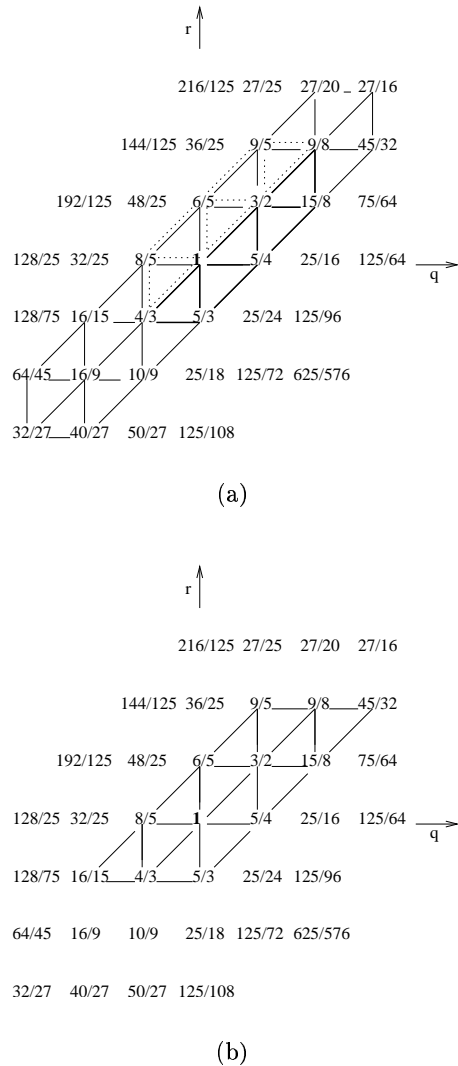


Figure 3: Thirds space of frequency ratios. (a): major scale is connected by thick lines, minor scale indicated by dashed lines, all internal intervals from major or minor scale indicated by normal lines. (b): chromatic scale.

cated as well. These scales both turn out to form convex regions. Considering all intervals internal in the major scale (which are the same intervals as the ones internal in the minor scale), and connecting these points in the thirds space, again a convex region is found. The chromatic scale as defined by Vogel [14] can also be found as a convex set in the thirds space (fig 3).

From the Scala website [11] a large collection of over 3000 scale files is available for downloads. From this collection we chose the 5-limit just intonation scales and checked whether they are convex in the thirds-space or not. The scales that are investigated are listed in table 1 and it is indicated whether these scales are respectively convex and star-convex.

scale	convex	star-convex
Ramis' Monochord	yes	yes
Erlangen Monochord	yes	yes
Erlangen Monochord revised	yes	yes
Fogliano's Monochord no. 1	yes	yes
Fogliano's Monochord no. 2	yes	yes
Agricola's Monochord	yes	yes
De Caus's Monochord	yes	yes
Kepler's Monochord no. 1	yes	yes
Kepler's Monochord no. 2	yes	yes
Mersenne's Spinnet Tuning no.1	yes	yes
Mersenne's Spinnet Tuning no.2	no	yes
Mersenne's Lute Tuning no.1	no	yes
Mersenne's Lute Tuning no.2	yes	yes
Marpurg's Monochord no.1	yes	yes
Marpurg's Monochord no.3	no	yes
Marpurg's Monochord no.4	yes	yes
Malcolm's Monochord	yes	yes
Euler's Monochord	yes	yes
Montvallon's Monochord	yes	yes
Romieu's Monochord	yes	yes
Kinberger I	yes	yes
Rousseau's Monochord	yes	yes

Table 2: List of 5-limit just intonation scales from Barbour.

Notice that all scales but four are convex. Differentiating between 'original' scales like the 'Ancient Greek Aeolic' and the 'Indian shrutti scale', and 'constructed' scales we observe that all original scales are convex. Moreover, all scales are *star-convex*. Several of the scales from table 1 are symmetric around the prime

interval 1/1 as well, meaning that both an interval and its inverse are present in the scale. In his book, Barbour [2] gives several examples of 5-limit just intonation 12-note systems. They are listed in table 2. Of the 26 scales, 23 are convex. Again, all of them are star-convex.

4 Convexity of chords

Now that we observed some scales in the thirds space, we can also look at smaller musical items, like chords. Considering different kinds of chords, a distinction can be made between chords that are built from harmonic notes, which are notes that are present in the scale of the specific key, and chords that contain non-harmonic notes, the so-called altered chords. All chords built from harmonic notes discussed by Piston [9] are convex (therefore also star-convex) in the note-name space. The chords are listed in table 3.

harmonic chords	convex	star-convex
major triad	yes	yes
minor triad	yes	yes
diminished triad	yes	yes
augmented triad	yes	yes
dominant seventh chord	yes	yes
major seventh chord	yes	yes
minor seventh chord	yes	yes
half-diminished seventh chord	yes	yes
major-minor seventh chord	yes	yes
augmented seventh chord	yes	yes
diminished seventh chord	yes	yes
triad with added sixth	yes	yes
complete dominant ninth chord	yes	yes
tonic/dominant eleventh chord	yes	yes
tonic/dominant thirteenth chord	yes	yes

Table 3: Chords built from harmonic notes taken from Piston [9].

Altered chords are difficult to study since it is possible, through the process of chromatic alteration, to create a very large number of altered chords. Therefore, in this paper we reduce the number of these chords to the ones composers have used. In table 4 the altered chords as discussed by Piston [9] are listed and in the ta-

name	description	convex	star-convex
aeolic.scl	Ancient Greek Aeolic	yes	yes
chin_5.scl	Chinese pentatonic from Zhou period	yes	yes
cifariello.scl	F. Cifariello Ciardi, ICMC 86 Proc. 15-tone 5-limit tuning	yes	yes
cluster.scl	13-tone 5-limit Tritriadic Cluster	yes	yes
cons_5.scl	Set of consonant 5-limit intervals within the octave	yes	yes
coul_13.scl	Symmetrical 13-tone 5-limit just system	no	yes
coul_27.scl	Symmetrical 27-tone 5-limit just system	yes	yes
danielou5_53.scl	Danielou's Harmonic Division in 5-limit, symmetrized	no	yes
darreg.scl	set of 19 ratios in 5-limit JI is for his megalyra family	no	yes
fokker-h.scl	Fokker-H 5-limit per.bl. synt.comma small & diesis, KNAW B71, 1968	yes	yes
fokker-k.scl	Fokker-K 5-limit per.bl. of 225/224 & 81/80 & 10976/10935, KNAW B71, 1968	yes	yes
harrison_5.scl	From Lou Harrison, a pelog style pentatonic	yes	yes
harrison_min.scl	From Lou Harrison, a symmetrical pentatonic with minor thirds	yes	yes
hirajoshi2.scl	Japanese pentatonic koto scale	yes	yes
indian.scl	Indian shruti scale	yes	yes
indian_12.scl	North Indian Gamut, modern Hindustani gamut out of 22 or more shrutis	yes	yes
ionic.scl	Ancient greek Ionic	yes	yes
ji_13.scl	5-limit 12-tone symmetrical scale with two tritones	yes	yes
ji_19.scl	5-limit 19-tone scale	yes	yes
ji_22.scl	5-limit 22-tone scale	yes	yes
ji_31b.scl	A just 5-limit 31-tone scale	yes	yes
johnston_81.scl	Johnston 81-note 5-limit scale of Sonata for Microtonal Piano	no	yes
kayolonian_81.scl	19-tone 5-limit scale of the Kayenian Imperium on Kayolonia (reeks van Sjauriek)	yes	yes
kring1.scl	Double-tie circular mirroring of 4:5:6 and Partch's 5-limit tonality Diamond	yes	yes
lumma5.scl	Carl Lumma's 5-limit version of lumma7, also Fokker 12-tone just	yes	yes
mandelbaum5.scl	Mandelbaum's 5-limit 19-tone scale	yes	yes
monzo-sym-5.scl	Monzo symmetrical system: 5-limit	yes	yes
pipedum_15.scl	126/125, 128/125 and 875/864, 5-limit, Paul Erlich, 2001	yes	yes
turkish.scl	Turkish, 5-limit from Palmer on a Turkish music record, harmonic minor inverse	yes	yes
wilson5.scl	Wilson's 22-tone 5-limit scale	yes	yes
wilson_17.scl	Wilson's 17-tone 5-limit scale	yes	yes

Table 1: List of 5-limit just intonation scales from Scala archive

ble is indicated whether these chords are (star-)convex. Remarkably, most altered chords are not convex, and some altered chords are not even star-convex. Thus we hypothesize that convexity may roughly distinguish between harmonic and altered chords.

The reason that the convexity of the chords is consid-

ered in the note name space rather than in the frequency ratio space is because it is difficult to say something a priori about the intonation of the chords as there is no established theory about the intonation of all chords. Future research is planned to find out whether chords that are convex in the note name space present the best

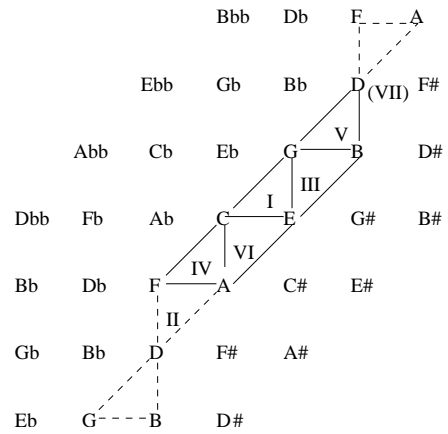
altered chords	convex	star-convex
non dominant diminished seventh chord	yes	yes
Neapolitan sixth	yes	yes
augmented sixth (Italian)	no	yes
augmented six-five-three (German)	no	yes
augmented six-four-three (French)	no	no
doubly augmented fourth	no	yes
chords with raised fifth - major	yes	yes
- minor	no	yes
- with minor seventh	no	yes
dominant chord with lowered fifth	no	yes
-with seventh	no	no
dominant chord with lowered and raised fifth	no	yes

Table 4: Chords containing non-harmonic notes, taken from Piston [9].

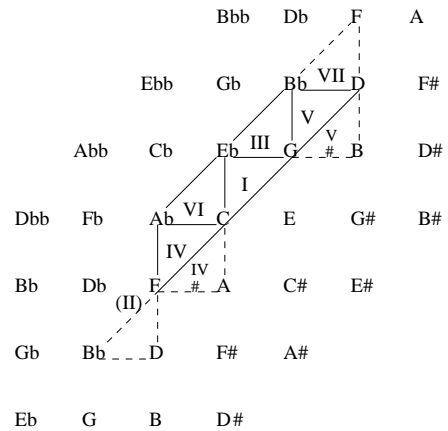
intonation in the frequency ratio space.

5 Convexity of Harmonic reduction

Harmonic reductions of music are known to be useful to discover the harmonic structure of a piece allowing an easier analysis. In this process a score can be reduced to chords and ultimately to triads[12, 10]. We saw in section 4 that all triads are convex. In the major scale only major, minor and one augmented triad are naturally present. It is easy to check from figure 4a that every two chords following each other can form a convex set². Therefore, whatever segmentation of music in a major key used (segmentation per chord, bar, phrase, etc), the reduction of the music to triads is always a convex set in the thirds space. For music in a minor key this is somewhat more complicated. From figure 4b one can see that the neutral minor and the harmonic minor scale form convex regions in the thirds space. In the harmonic minor scale the seventh note of the scale is raised by a half tone, such that the *III*, *V* and *VII* triad are changed. In the ascending melodic minor scale the



(a) major scale



(b) minor scale

Figure 4: (a) Triads from the major scale situated in the note-name space, *VII* is the diminished triad $B-D-F$. (b) Triads from the minor scale, *II* is the diminished triad $D-F-Ab$.

sixth and the seventh tone of the scale are raised. Consequently the triads on *II*, *IV* and *VI* are adjusted as well. Therefore, music in a minor key and thus the harmonic reduction thereof takes into account many more triads than music in a major key. Considering the harmonic progressions for the minor mode as given by Pis-

²With dotted lines is indicated how the scales proceed to both sides, so that more than one region (convex or not convex) of two adjacent chords can be chosen. For example, for triads *IV* and *V* it is not directly clear that they can form a convex set, but if *V* is chosen as indicated by the dotted lines, they do.

ton [9] all progressions³ form convex sets except for the progression: *VII*(harmonic)-*I*. At least, this progression is not a convex set (but it does form a star-convex set) if the triads *VII*(harmonic) and *I* are chosen such that the triads themselves are convex. But the notes of the triads (in *C* minor: *B - D - F*, *C - E \flat - G*) can also be chosen in the note name space, such that this progression does form a convex set. The way the location of the notes is chosen depends on the intonation of the notes as the mapping in figure 2 indicates. In just intonation intervals are tuned to simple number ratios. That fact that interval can be harmonic as well as melodic can cause some problems. It is not always possible to make sure that two adjacent chords are tuned to lowest number ratios in harmonic as well as melodic form. In the case of the *VII*(harmonic)-*I* progression, we could argue that the melodic just intonation overrules the harmonic intonation of the *VII* chord, and that therefore the *F* of the *VII* chord should not be tuned as 27/20 but as 4/3. In this case the *VII - I* progression does form a convex set.

6 Pitch spelling using convexity

Reading figure 2 from *c* to *b* to *a*, it represents a mapping from MIDI to the spelling of the pitches, to the preferred tuning, respectively. It is easy to map figure 2a to 2b to 2c, but the other way round is difficult since information is missing. Studying a MIDI-input it is not always clear what notes the composer meant to write, and reading a score, it is up to the musician how to intone the notes. We are challenged to find algorithms following the mapping:

$$\text{MIDI} \rightarrow \text{note names} \rightarrow \text{preferred tuning} \quad (5)$$

Several pitch spelling algorithms have been made, see for example [4, 13, 7, 3], the best performing a correct spelling of the notes of 99.8%. This is already very high and the question arises whether it is worth searching for a better algorithm. If, however, we manage to find a

³leaving the progressions involving *III*(harmonic) and *VI*(ascending melodic) out, since these are rarely used according to Piston [9].

visual mapping in the thirds-space that maps the MIDI numbers to the correctly spelled pitches, we can directly apply this visual mapping to derive a preferred tuning. The methods above for pitch spelling [4, 13, 7, 3] in a way all first search for the right key context. By finding a good mapping from figure 2c to figure 2b, we can determine the key context of a chosen piece as well. As shown in section 3, the major and minor diatonic scale are found in a specific region, that can indicate the key context. If we have now as input the MIDI numbers from one bar of a piece of music and mark these numbers in figure 2c, we can search for the right shape of a major or minor scale so that most of the notes of the scale are filled. For example, the first bar of Beethoven's piano Sonata Op. 109 is encoded as indicated in figure 5. Matching the typical shaped region of the major and

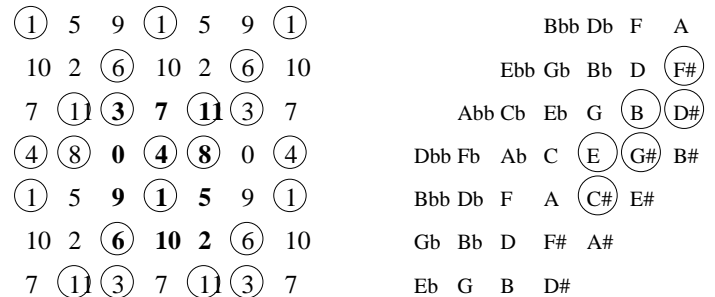


Figure 5: Pitch spelling method. Encoding of first bar of Beethoven's piano Sonata Op. 109

minor scale and looking for a fit, the notes in the note name space in figure 5 are the result. At equally good matches the key with the least number of sharps and flats is chosen. In this case '4' major is translated into *E* major, as opposed to *F \flat* major. This visual mapping procedure has been tested on a couple of pieces and seems to work as well as the methods mentioned above.

7 Conclusion

In this paper the property of (star-)convexity is used to describe and explain the goodness of musical items. We noted that several musical items are (star-)convex but that this process is not reversible: convex sets do not

per se make good chords/scales/etc. Convexity is used to describe 'underlying' features in music but no 'surface' features such as rhythm and melody. We observed that all 5-limit just intonation scales and all chords built from harmonic notes are either convex or star-convex. This suggests that (star-)convexity may serve as a condition for the 'goodness' of musical items. Of course, star-convexity is a less straight notion than convexity, but it is intriguing that all the items discussed here follow this property.

We have also shown that the tonal coherence which forms the reduction of a piece is a convex body. With this, we could hypothesize that people listening to a piece of music strive to perceive convex structures. Finally, convexity can be used in a pitch spelling algorithm, and ultimately in a preferred tuning algorithm.

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