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Convexity and compactness as models for the preferred intonation of chords

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ABSTRACT

It has been argued that convexity in the Euler lattice can be interpreted in terms of consonance (Honingh and Bod 2005). In this paper, a second hypothesis is presented that states that compactness in the Euler-lattice is an indication of consonance. It will be investigated if, and to which degree convexity and compactness are in agreement with the consonance of chords according to Euler's measure of consonance. First, some diatonic chords are observed, after which the compactness, convexity and consonance according to Euler, is calculated for all possible sets (chords) of 2, 3 and 4 notes within a bounded note name space, such that the relation between these three measures can be obtained.

Keywords

Intonation, convexity, compactness, Euler.

INTRODUCTION

The preferred intonation of an interval or chord in isolation is usually given by the most consonant performance of the chord. Many functions have been constructed to measure the consonance of an interval or chord, for example Helmholtz's roughness function (Helmholtz 1863), Euler's Gradus Suaventatis (Euler 1739; Fokker 1945), Parncutt's pitch distance (Parncutt 1994) or Sethares' dissonance curve (Sethares 1993) based on Plomp and Levelt's model (Plomp and Levelt 1965). These functions can be used to put musical intervals in an order of most consonant to dissonant. However, few of these functions have been used (and are difficult to use) to decide about different into- -nations of the same chord. For some chords, like for example a major triad, the intonation may be obvious, but for others there is no consensus. Consider for example a dominant seventh chord C-E-G-Bb. It can be tuned choosing the ratios: 1,5/4,3/2,9/5 such that the minor seventh is tuned as minor third 6/5 above the fifth; or tuned as 1,5/4,3/2,16/9 such that the minor seventh is chosen to be two fourths above the tonic, and many other possibilities are possible.

Regener (1973) stated the ambiguity involving just intonation frequency ratios as follows: Each notated interval actually corresponds to an infinite number of frequency ratios, since multiplication of a frequency ratio by any integer power of 81/80 leaves the interval unchanged. Regener (1973) describes furthermore two criteria that are commonly used or assumed in determining which are the ``preferential" frequency ratios in just intonation corresponding to a given interval:

1. Preferred ratios are those involving the lower numbers when in lowest terms.
2. Preferred ratios are those that can be derived by linear combination from known preferred values for other intervals (beginning with the ratios 3/2 for a perfect fifth and 5/4 for a ma-

jor third), possibly with a certain use of intervals in mind from some musical context.

It may be clear that these two criteria are not always in agreement and do not constitute a full intonation theory for chords in isolation.

A MODEL FOR INTONATION

It has been observed that the diatonic major and minor scale as well as all diatonic chords form convex and compact sets in the Euler-lattice (Honingh and Bod, 2005). The Euler-lattice can be presented in several forms (Honingh, 2003). In figure 1, the Euler-lattices constructed from frequency ratios and note names are shown. It can be understood from figure 1 that different locations of a note name exist in the note name space, all giving rise to a different ratio and therefore different intonation. The difference between two ratios having the same note name is 81/80, known as the syntonic comma. It can therefore be understood that a chord defined as a set of note names (for example C, E, G) has several possibilities for intonation. In terms of the tone spaces from fig. 1, the problem is which note names should be chosen to represent a chord in order to reflect the right frequency ratios that can be projected from one figure to the other. It has been explained in Honingh and Bod (2005) that convexity in the Euler lattice can be interpreted in terms of consonance. Therefore, a convex set could possibly represent the preferred set. A convex set has been defined as the set where, if drawing lines between all points in the set, all elements which lie within the spanned area are elements of the set. A set of notes can have more than one convex configuration in the note name space. For example, the two-note set C-G tuned as 1-3/2 is convex, but the tuning as 1-40/27 is also convex (see figure 1). In cases like this, a choice has to be made between the two configurations to present the preferred tuning, and a possibility is to chose the most compact one. Compactness is intuitively understood as the gradation to which elements of a set are close to the center of gravity of the set. In a three dimensional space the most compact object would be shaped like a ball. In this paper, the compactness is calculated by summing the distances between all pairs of points (notes in the tone space); the lower the resulting value, the more compact is the set. The decision to choose the most compact set is not a random choice. If two notes are close together in the tone space, they have many prime factors in common, as the tone space was built from powers of the primes 2,3 and 5 (see Honingh and Bod 2005). Therefore, the closer together two notes are in the tone space, the smaller are the integers forming the ratio that represents the interval between the two notes. According to just intonation ratios with small integer ratios are preferred. Generalizing this for chords consisting of more than two notes, the intonation of a chord whose notes are the most close together in the tone space should be preferred. Now we have motivated why to use compactness to decide which of the possible convex sets represents the preferred intonation, we can actually make two hypotheses:

	25/18	25/24	25/16	75/64	225/128		
40/27	10/9	5/3	5/4	15/8	45/32	135/128	
32/27	16/9	4/3	1	3/2	9/8	27/16	81/64
256/135	64/45	16/15	8/5	6/5	9/5	27/20	81/80
	256/225	128/75	32/25	48/25	36/25	27/25	

	F#	C#	G#	D#	A#		
G	D	A	E	B	F#	C#	
Eb	Bb	F	C	G	D	A	E
Cb	Gb	Db	Ab	Eb	Bb	F	C
	Ebb	Bbb	Fb	Cb	Gb	Db	

Figure 1: Tone space built from frequency ratios and note names. In the right figure, the note names are chosen to be in the key of C.

1. the preferred intonation of a chord is represented by the most compact of the possible convex configurations of that chord
2. the preferred intonation of a chord is represented by the most compact configuration of that chord.

To test these hypotheses, we need a consonance measure that is generally applicable for all types of sounds, and has frequency ratios as input. Since we consider chords in isolation, not in a sequence, we only have to deal with `vertical' intonation. Euler's Gradus function (is the only measure that) fulfills all these requirements. Euler developed his Gradus Suaventatis (degree of softness) Γ. The function is defined as a measure of the simplicity of a number or ratio.

Any positive integer α can be written as a unique product $\alpha = p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$ of positive integer powers e_i of primes $p_1 < p_2 < \dots < p_n$. Euler's formula is then defined as:

$$\Gamma(\alpha) = 1 + \Sigma_k e_k (p_k - 1),$$

$\Gamma(\alpha)$ is a number that expresses the simplicity of α. The lower the number the simpler is α. For intervals and chords a so-called exponent needs to be calculated to obtain Γ from. For an interval (frequency ratio) x/y the exponent is x·y so Γ(x·y) expresses the simplicity of the interval x/y. For chords where the frequency ratios are expressed as a:b:c, the exponent is given by the Least Common Multiple (LCM) of these a, b and c. The Gradus Suaventatis is then calculated as Γ(LCM(a,b,c)). Euler connected the simplicity of chords and intervals with the consonance thereof. This can be understood by thinking in terms of frequency periodicity. If one hears for example a tone of 300 (=5 × 60) Hz and one of 420 (=7 × 60) Hz, then per second 60 repeated patterns can be heard in which each pattern can be subdivided in 5 x 7=35 pieces. The more 'repetition' can be heard, the simpler or the more consonant is the sound, was the argument by Euler. Therefore, the lower the value Γ(LCM(a,b,c)), the more consonant is the chord a:b:c. Here is an example to calculate the Gradus Suaventatis. A major triad 1:5/4:3/2 can be written as 4:5:6 which can in turn be written as 2^2:5:2·3 (to make the calculation of the LCM easier). The LCM of these numbers is then 2^2 ·3 · 5=60 and the Gradus Suaventatis of 60 is Γ(60)=1+2·1+1·2+1·4=9. According to the tonal space that maps frequency ratios to note names (see figure 1), this chord can also be tuned differently, for example as 1:5/4:40/27, the fifth of the triad is then changed by the syntonic comma: (3/2)/(81/80)=40/27. The ratios 1:5/4:40/27 can be written differently as 108:135:160=2^2 ·3^3:3^3·5:2^5·5. Then the LCM equals 2^5·3^3·5=4320 which results in Γ(4320)=16. This is obviously higher than the value for the 4:5:6 chord and this means that this chord is less consonant than the 4:5:6 chord according to this function.

COMPOSITIONS IN THE TONE SPACE INDICATING THE INTONATION

In the same way as above we can compare other chords in different tuning to see which tuning is most preferable. We compare different configurations of a chord. By configuration we mean the locations of all elements in the tone lattice. The configuration of a set can be changed by moving one or more elements of the set by a syntonic comma (=81/80). In the tables 1, 2 and 3, some diatonic chords are listed with two possibilities for tuning.

Table 1: Harmonic chords consisting of 3 notes. Of each chord, the convex configuration is given, together with another possible configuration. More configurations (intonations) are possible but only one is given here. The circles represent the notes in the frequency ratio space, the black circle representing the tonic C of the chord.

3-note chords	convex	
major triad C-E-G	1 – 5/4 – 3/2 · ⊙ · · ● ⊙ · · · Γ = 9	1 – 5/4 – 40/27 ⊙ · · ⊙ · · · ● Γ = 16
minor triad C-E♭-G	1 – 6/5 – 3/2 · · · · ● ⊙ · · ⊙ Γ = 9	1 – 32/27 – 3/2 · · · · · ⊙ · · ● ⊙ · · · · · Γ = 15
diminished triad C-E♭-G♭	1 – 6/5 – 36/25 ● · · · ⊙ · · · ⊙ Γ = 15	1–32/27–64/45 ⊙ · · ● · ⊙ · · Γ = 17
augmented triad C-E-G♯	1 – 5/4 – 25/16 · ⊙ · · ⊙ · · ● · Γ = 13	1 – 5/4 – 125/81 ⊙ · · · · · · · · · · · · · ⊙ · · · · ● Γ = 23

Since the tone space is infinitely big, there are infinitely many tunings for a chord, however only some musically logical ones are listed here to give an example. In the first column of every table the name of the chords with corresponding note names is given. In the other columns different tunings and their composition in the plane are given. The tones are indicated by circles, the black circle being the root of the chord (C was chosen to be the root in all cases). For every chord, the Gradus Suaventatis is calculated and given in the tables. We can test our first hypothesis which says that the convex composition (and if there is ambiguity, the most compact convex composition) represents the preferred intonation. One can see that for almost every chord the convex composition of it in the tone space is more consonant according to Euler (= lower value for Γ) than the other. There are two exceptions in this which are the diminished seventh chord and the dominant eleventh chord.

Table 2: Harmonic chords consisting of 4 notes.

4-note chords	convex	
dominant seventh chord C-E-G-B♭	1:5/4:3/2:9/5 ⊙ · · • ⊙ · · · ⊙ Γ = 15	1 : 5/4 : 3/2 : 16/9 · · ⊙ · ⊙ · • ⊙ · · · · Γ = 17
major seventh chord C-E-G-B	1 : 5/4 : 3/2 : 15/8 · ⊙ ⊙ · • ⊙ · · · Γ = 10	1 : 5/4 : 3/2 : 50/27 ⊙ · · · · · · · ⊙ · · · · • ⊙ Γ = 18
minor seventh chord C-E♭-G-B♭	1 : 6/5 : 3/2 : 9/5 · · · • ⊙ · · ⊙ ⊙ Γ = 11	1 : 6/5 : 3/2 : 16/9 ⊙ · • ⊙ · · · ⊙ Γ = 16
half-diminished seventh chord C-E♭-G♭-B♭	1 : 6/5 : 36/25 : 9/5 • · · · ⊙ ⊙ · · ⊙ Γ = 15	1 : 6/5 : 36/25 : 16/9 ⊙ · • · · · · · ⊙ · · · · · ⊙ Γ = 19
major-minor seventh chord C-E♭-G-B	1 : 6/5 : 3/2 : 15/8 · ⊙ • ⊙ · ⊙ Γ = 15	1 : 6/5 : 3/2 : 50/27 ⊙ · · · · · · · · · · · · • ⊙ · · · · ⊙ Γ = 23
augmented seventh chord C-E-G♯-B	1:5/4:25/16:15/8 ⊙ · ⊙ ⊙ • · Γ = 15	1:5/4:125/81:50/27 ⊙ · · · · · ⊙ · · · · · · · ⊙ · · · · • Γ = 25
diminished seventh chord C-E♭-G♭-B♭♭	1:6/5:36/25:216/125 • · · · · ⊙ · · · · ⊙ · · · · ⊙ Γ = 22	1:6/5:64/45:128/75 · · • · ⊙ · · ⊙ · ⊙ · · Γ = 22
major triad with added sixth C-E-G-A	1 : 5/4 : 3/2 : 5/3 ⊙ ⊙ · · • ⊙ Γ = 11	1 : 5/4 : 3/2 : 27/16 ⊙ · · · • ⊙ · ⊙ Γ = 13
minor triad with added sixth C-E♭-G-A♭	1:6/5:3/2:8/5 · · · · • ⊙ · ⊙ ⊙ Γ = 11	1:6/5:3/2:81/50 • ⊙ · · · · ⊙ · · · · · · · ⊙ Γ = 19

The dominant eleventh chord is in a sense a reduction of the dominant thirteenth chord, only one note is missing. Filling in the missing note in the composition that is most favored, one obtains the most consonant thirteenth chord which is convex as well. In this way, we can understand why this particular composition for the eleventh chord is more consonant than the convex one. However, this second composition is more compact than the first one, supporting hypothesis number 2, which says to prefer the most compact configuration. The compactness of the chords has not been explicitly indicated in the tables, but hypothesis 2 has been validated in all cases except for the diminished seventh chord. The second configuration of the diminished seventh chord listed in table 3 is the most compact one.

Table 3: Diatonic chords consisting of 5, 6 or 7 notes.

5/6/7-note chords	convex	
dominant ninth chord C-E-G-B♭-D	1:5/4:3/2:9/5:9/8 ⊙ · · • ⊙ ⊙ · · ⊙ Γ = 16	1:5/4:3/2:16/9:10/9 ⊙ · ⊙ · ⊙ · • ⊙ Γ = 17
dominant eleventh chord C-E-G-B♭-D-F	1:5/4:3/2:9/5:9/8:27/20 ⊙ · · · • ⊙ ⊙ · · · ⊙ ⊙ Γ = 18	1:5/4:3/2:16/9:10/9:4/3 ⊙ · ⊙ · ⊙ ⊙ • ⊙ · · · · Γ = 17
dominant thirteenth chord C-E-G-B♭-D-F-A	1 : 5/4 : 3/2 : 9/5 : 9/8 : 27/20 : 27/16 ⊙ · · · • ⊙ ⊙ ⊙ · · ⊙ ⊙ Γ = 19	1 : 5/4 : 3/2 : 16/9 : 10/9 : 4/3 : 5/3 ⊙ ⊙ ⊙ · ⊙ ⊙ • ⊙ · · · · also convex! Γ = 17

Note that both listed dominant thirteenth chords are convex. The second one listed is the preferred one according to both hypotheses, since it is more compact. This is also the configuration which is preferred by Euler's function. To sum up, we proposed two hypotheses in order to present the best intonation, the first saying to prefer (the most compact) convex configuration, and the second saying to prefer the most compact configuration. The values of consonance of the chords were calculated using Euler's Gradus function. Of the 16 chords, for 14 of them hypothesis 1 was validated. For 15 of them, hypothesis 2 was validated.

We want to stress that ``preferred tuning" in this case is only based on the sound of the chord in isolation. In musical practice, there can be more than one choice for the intonation of a chord depending on its musical function in the chord sequence. However, this can still be a very useful measure because it can serve as a beginning of a full tuning theory.

CONVEXITY, COMPACTNESS AND CONSONANCE

After studying some harmonic chords with respect to its convexity, compactness and consonance according to Euler, we are now interested in the relation between these three notions when more chords are studied. Considering a lattice like fig. 1, each set of points can be interpreted as a chord. We are then interested in the following question. Which percentage of the sets of notes that have a possible convex configuration, has a convex configuration that corresponds with 1) the most compact configuration, and 2) the most consonant configuration. The total amount of sets of notes is here equivalent to the amount of sets that have a possible convex configuration. This is due to the fact that for some chords there is no possible intonation such that the notes form a convex set in the tone space. For these chords, only the compactness can say something about the preferred intonation. Figure 2 illustrates what percentages we are looking for.

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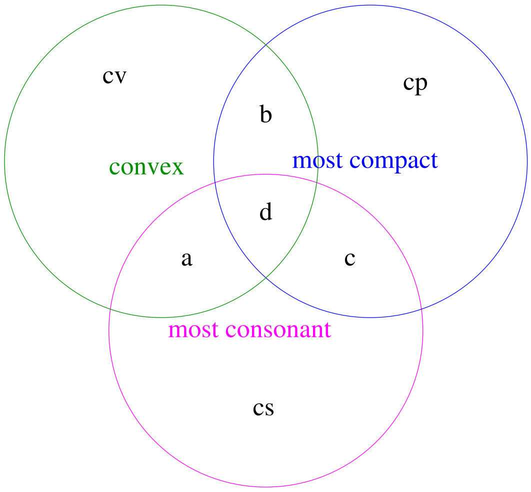

Figure 2: Illustration of overlap of convex, most compact and most consonant configuration when trying to find the preferred intonation of a chord.

The total number of sets, the sets with a possible convex configuration, is equivalent to cv+a+b+d (see figure 2), which is in turn equivalent to cp+b+c+d and to cs+a+c+d. Therefore, note that 'total' is reflected three times in the areas 'convex', 'most compact' and 'most consonant'. The percentages we are interested in are given here:

b+d = (compact & convex) / total

a+d = (convex & consonant) / total

d+c = (compact & consonant) / total

d = (compact & convex) / consonant

For example, b+d from figure 2 is given by the percentage of sets (with a convex possibility) whose configurations that are most compact are also convex.

There could be more than one convex possibility per set (as is the case with the dominant thirteenth chord in table 3). Also, it is possible that more than one configuration has the same (lowest) value for the compactness or consonance. This only means that some solutions are not unique, but since we count the number of sets and not the number of configurations, this does not change the obtained percentages.

We have written a program in Matlab that finds all possible 2,3 and 4 note sets in a 9 × 9 (coordinates run from -4 to 4) 2-dimensional lattice, and calculates of each set 1) whether it has a convex configuration and which configurations are convex, 2) the configuration that is most compact, and 3) the configuration that is most consonant. The convexity and compactness of a set is calculated from the coordinates of the elements in the set. From short test-runs was concluded that it is sufficient to work with a 9 × 9 2-dimensional lattice, a bigger lattice did not significantly change the percentages. One point is chosen in the center (0,0), so for n=2 only one point is varied, for n=3 two points, and so on. To make sure to not double count some sets, point 3 is varied over the points that point 2 has not been varied over and so on for the points thereafter. The results of the Matlab program are shown in table 4. The number of sets that are examined (last row in table 4) is the number of sets that have a possible convex configuration.

Table 4: Results of the percentages as indicated in fig. 2.

percentage	**n=2**	**n=3**	**n=4**
compact & consonant (c+d)	97.5 %	85.4 %	85.6 %
convex & consonant (b+d)	16.3 %	41.4 %	41.2 %
convex & compact (a+d)	11.3 %	40.8 %	36.0 %
compact & convex & consonant (d)	11.3 %	37.3 %	34.1 %
number of sets examined	80	1590	14810

Observing the results, one can see that the biggest correlation can be found between the most compact and consonant sets, as we expected. The correlation between convexity and the other items is very low for n=2 and is getter higher as the number of notes increases. When considering only 2 notes, the notion of convexity differs a lot from the notion of compactness, since two notes form a convex set if a line can be drawn between the two notes on which no other notes lie. Therefore it is not easier for two notes to form a convex set if the notes lie close to each other than when the notes lie far from each other, as can be seen from the low correlation between convexity and compactness for n=2. However, for increasing n, the correlation between convexity and compactness increases as well. Note that regions a and d are really small, especially for small n (for n=2, a=0). This means that when the most consonant configurations are also convex, they are most likely to be also the most compact configurations (a); and when the most compact configurations are also convex, they are most likely to be also the most consonant configurations (d).

CONCLUDING REMARKS

In this paper we have motivated to use the notions of convexity and compactness as measure of intonation for chords in isolation. As a measure of consonance, Euler's Gradus function was used for comparison. Although a strong relation was obtained between consonance and

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both convexity and compactness when tested on some diatonic chords, after better investigations it turned out that convexity is a poor indication of consonance for chords in isolation. The notion of compactness however, showed a strong relation with consonance for chords with 2, 3 or 4 notes.

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