Limitations on Fixed n-Tone Equal Tempered Divisions

Aline Honingh

Institute for Logic, Language and Computation,
Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands
tel:+31 20 525 6508 fax:+31 20 5255206
e-mail: ahoningh@science.uva.nl

Abstract

To construct an n-tone equal tempered division other than $n = 12$, one usually uses a goodness-of-fit approach to find the n-tone system that best approximates a number of intervals from just intonation. This method leads to equal tempered divisions of size $12, 19, 31, 41$, and 53. However, to be able to use an n-tone equal tempered system for a keyboard application, there are some restrictions to be taken into consideration. We demand a surjective mapping from the note-names to the units of the n-tone equal division, in order to have a suitable mapping from a score to e.g. the keys of a piano. We will show that this demand translates into two mathematical conditions which lead to the following values for n: 5,7,12,19,26,31,43,45,50,55,69,74,81,88. Combining this result with the general results for a goodness-of-fit approach, we conclude that good divisions of the octave are $12, 19$ and 31, which have indeed been used in musical practice. Since there is a limit to the closeness just intonation can be approximated by equal temperament in the method used, we expand our note-name system so as to differentiate between different frequency ratios having the same note-name. Certain conditions apply to this new system as well and the resulting values for n can serve as an explanation as to why certain n-tone temperaments have been used in the past.

1 Introduction

Nowadays all classical Western pianos are tuned to 12-tone equal temperament. This temperament makes a compromise between just tuned thirds and just tuned fifths. A commonly accepted idea to construct an equal temperament is to approximate the most prominent intervals of just intonation. Considering an octave based equal temperament, this can be done in the following way:

$$R \approx 2^{m/n}, \tag{1}$$

where R is an interval from just intonation, n represents the n-tone temperament that is used, and m the number of steps within the n-tone equal temperament that approximates the chosen interval. Using a goodness-of-fit approach one can find the n-tone equal tempered system that best approaches a number of intervals from just intonation [6, 11, 14, 3, 6, 7, 10]. Another method to determine which n-tone equal temperament is preferred is proposed by Balzano [1]. Balzano argues to prefer n-tone equal tempered systems with $n = k(k + 1)$ for $k \in \mathbb{Z}$. Systems that satisfy this condition can not only be expressed in a semitone group where the generator is the smallest element, but is also isomorphic to two other groups: a single-generator cycle of 'keys', and a product group of 'triads'. However, the goodness-of-fit method only checks just intonation intervals without considering if the resulting n-tone temperament is realistic to use, and Balzano's method results in $k(k+1)$-tone temperaments that contain a diatonic scale of $2k + 1$ notes, so clearly unusable for Western music. Therefore, in this paper we address another than the two methods mentioned to find a suitable division of the octave for n-tone equal temperament. Historically, it has not always been clear for what reasons certain choices for n-tone equal tempered systems have been made. Yunik and Swift [15] write "Through often convoluted, difficult-to-follow logic, various other values for n have been proposed”. In this paper, a possible explanation is given for the choice of these equal tempered systems.

In the next section we will explain that a specific note name should only be attached to one unit of the octave division, in order to have a suitable mapping from a score to for example the keys of a piano. This results in a restriction on the possible octave divisions. Then, in section 2.1.1 we discuss the consequences of this condition to the addition of intervals, and in section 2.2 we impose another restriction on the division of the octave to make sure all possible note names can be used. The result of both restrictions is visually explained in section 3, and in section 4 this visual representation is extended to differentiate between all frequency ratios. We finish with conclusions in section 5.

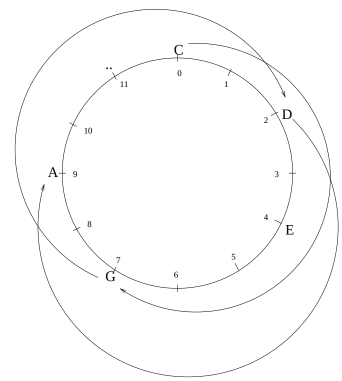

Figure 1: Start of attaching note-names to the 12-tone equal tempered division. The fifth is approximated by 7 steps in the 12-tone division.

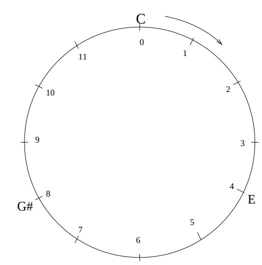

Figure 2: Start of attaching note-names to the 12-tone equal tempered division using the major third that is approximated by 4 steps in this division.

2 Attaching note-names to an octave division

All possible note-names for musical tones can be formulated in the following way

$$\ldots A\flat - E\flat - B\flat - F - C - G - D - A - E - B - F\sharp \ldots \quad (2)$$

which is an infinite series in both directions. One possibility to attach note-names to an equal tempered division is to calculate the number of (smallest) steps m in the n-tone temperament that approximates the fifth $R = 3/2$ and give adjacent note-names from (2) to unit number

$$k \cdot m \bmod n, \qquad k \in \mathbb{Z} \quad (3)$$

of the equal tempered division. The example in figure 1 is given for 12-tone temperament. For 12-tone equal temperament $m = 7$ since $3/2 \approx 2^{7/12}$. In this way all note-names are attached to a certain position.

Another way to attach note-names to this division is to use another interval (than the fifth) as ’generator of note-names’. Figure 2 gives an example of the major third $5/4 \approx 2^{4/12}$ dividing the notes. Using this method of dividing, only one fourth of all possible note-names are used as one can understand, comparing the line of thirds

$$\ldots - F\flat - A\flat - C - E - G\sharp - B\sharp - D\sharp\sharp - \ldots \quad (4)$$

to the line of fifths (2).

In the example of the 12-tone temperament, the note-divisions are in agreement with each other; the notes selected by the major third are in the same position as those notes selected by the perfect fifth (compare the two E’s, both attached to position ’4’). Not for all n-tone divisions this is the case. For example for the 15-tone temperament, the third C-E measures 6 (out of 15) units if calculated from the fifth, and 5 units if calculated from the major third (see figure 3). If an equal temperament is to be used for a keyboard application, this is not preferable since there is no consensus about which keys to press when a score is read.

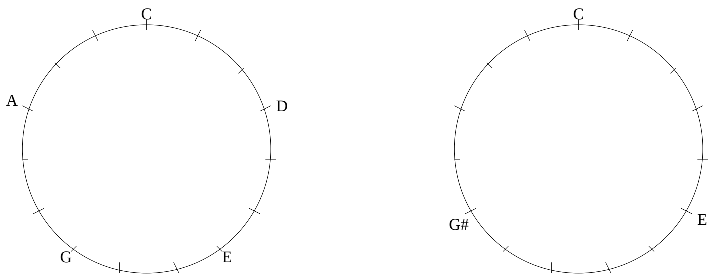

Figure 3: Example of 15-tone equal temperament. Left: attaching note-names using fifth, right: attaching note-names using major thirds

2.1 Enharmonicity condition

What we want is to make a surjective mapping from the note-names to the units of the n-tone equal division. This means, two different note-names can refer to the same unit (and are therefore enharmonically equivalent), but one note name cannot refer to two different units. Furthermore, all units should have a note-name. We come back later to this last condition.

In this way, reading a score, it is clear which keys on a piano to press. To gain this result, the fifths have to match with the major thirds as illustrated in the previous section. Also, other intervals (besides the fifth and the major third) can be chosen to generate the note-names, and demand that they match with each other.

To determine whether an n-tone temperament has a good match of thirds and fifths, one has to check if the major third (5/4) is approximated by the same number of steps as the Pythagorean third (81/64, third constructed from four perfect fifths). The difference in cents[1] between the two intervals measures 21.51 cents. Therefore, the better these intervals are approximated, the less chance they both map on the same part of the tempered system. And because there is always a better fit if n is chosen big enough, there will be a certain maximum to n for which the major thirds and the fifths still match. The number of units m that approximates an interval R in an n-tone equal temperament is given by (see eq. 1):

$$m_R(n) = \lfloor n \log_2 R + 1/2 \rfloor, \tag{5}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Since four fifths up and using octave equivalence gives a major third (C-G-D-A-E), the following equivalence relation can be demanded[2]:

$$4 \cdot m_{\frac{3}{2}} \bmod n = m_{\frac{5}{4}} \tag{6}$$

Furthermore, three fifths give a major sixth (C-G-D-A), and demanding that fifths should match with major sixths gives the following condition

$$3 \cdot m_{\frac{3}{2}} \bmod n = m_{\frac{5}{3}} \tag{7}$$

It is possible to go on writing conditions like this, but first we take a look at what is covered by conditions (6) and (7).

When we look at the line of fifths:

$$\ldots A\flat - E\flat - B\flat - F - C - G - D - A - E - B - F\sharp \ldots \tag{8}$$

and we realize that an equal tempered approximation to an interval R is just as good as the approximation to the inverse of the interval $2/R$, we understand (and it can be shown) that the following relations hold for the same values of n:

$$x \cdot m_{\frac{3}{2}} \bmod n = m_R \tag{9}$$

$$x \cdot m_{\frac{4}{3}} \bmod n = m_{\frac{2}{R}} \tag{10}$$

for $x \in \mathbb{Z}^+$ and $R \in [1, 2)$ ($R \in \mathbb{Q}$). When m steps approximate the fifth, $n - m$ steps approximate the fourth:

$$m_{\frac{4}{3}} = n - m_{\frac{3}{2}} \tag{11}$$

Substituting (11) in (9) gives

$$-x \cdot m_{\frac{4}{3}} \bmod n = m_R, \tag{12}$$

and comparing (9) and (12) shows us that a number of x fifths up gives the same interval as x fourths down (or $-x$ fourths up). Therefore, equations (6) and (7) make sure that all consonant intervals $3/2, 4/3, 5/4, 6/5, 8/5, 5/3$ match with each other. Conditions (6) and (7) together we call the enharmonicity conditions. The possible values for n so that

[1] An interval $\frac{x}{y}$ has a width of $1200 \cdot \log_2(\frac{x}{y})$ cents.

[2] Recall the following intervals and their frequency ratios: perfect fifth $3/2$, perfect fourth $4/3$, major third $5/4$, minor third $6/5$, major sixth $5/3$, minor sixth $8/5$.

(6) and (7) are true can be obtained by running a simple perl script:

$$n = \quad 5, 7, 12, 19, 24, 26, 31, 36, 38, 43, 45, 50, 55, 57, 62, 69, 74, 76, 81, 88, 93, 100 \tag{13}$$

It is not realistic to demand that more intervals should match in the n-tone system, because there is no consensus about the frequency ratios. For example, a major second is denoted by 9/8 but also by 10/9.

2.1.1 Adding intervals

Playing a major or minor triad in an n-tone equal tempered system, it is desirable to have the all containing intervals: major third, minor third and perfect fifth represented by the number of units that approximate the intervals best. This is not automatically satisfied when the number of units m for each interval is calculated. For this an extra condition is required: the number of units that approximates an interval added to the number of units that approximates another interval should equal the number of units that approximates the sum-interval. For example, adding a major third to a minor third gives a perfect fifth.

$$5/4 \cdot 6/5 = 3/2. \tag{14}$$

Translated to the approximations in 12-tone equal temperament this reads:

$$2^{4/12} \cdot 2^{3/12} = 2^{7/12} \tag{15}$$

where the $'='$ sign is still valid. Not in every n-tone equal temperament this is the case; in 14-tone temperament (14) would be translated to

$$2^{5/14} \cdot 2^{4/14} \neq 2^{8/14}. \tag{16}$$

For the correct way of adding intervals R_1 and R_2, the condition is:

$$m_{R_1} + m_{R_2} = m_{R_3}, \tag{17}$$

where $R_3 = R_1 \cdot R_2$. Of course this condition cannot be satisfied for all values of R_1 and R_2 for a specific n.

We wonder for which intervals condition (17) is true for the values of n that satisfy the conditions (6, 7). Using (9), equations (6, 7) translate into the following six equations:

$$x * m_{\frac{3}{2}} \bmod n = m_R, \tag{18}$$

$$\begin{aligned} \text{for} \quad & R = 3/2\,(x = 1), \\ & R = 4/3\,(x = -1), \\ & R = 5/4\,(x = 4), \\ & R = 8/5\,(x = -4), \\ & R = 5/3\,(x = 3), \\ & R = 6/5\,(x = -3). \end{aligned}$$

Substituting this in (17) gives

$$x_1 \cdot m_{\frac{3}{2}} + x_2 \cdot m_{\frac{3}{2}} = x_3 \cdot m_{\frac{3}{2}} \tag{19}$$

which is true for combinations of x_1, x_2 and x_3 from 18 satisfying

$$x_1 + x_2 = x_3. \tag{20}$$

This means that the example of adding major and minor thirds (14,15) is always correct for all n from (13). Furthermore the additions $6/5 \cdot 4/3 = 8/5$, $5/3 \cdot 3/2 = 5/4$ and $5/3 \cdot 8/5 = 4/3$ are also translated correctly into equal temperament.

2.2 Generating fifth

With the conditions (6,7) we have now established that in these n-tone equal tempered systems all consonant intervals are indicated by the number of units that best approximates these intervals. Also, these conditions make sure that enharmonic equivalence of notes is possible, but no note can refer to two different units in the equal tempered system. Furthermore, the enharmonicity conditions make sure that the condition for correctly adding intervals is automatically satisfied for the major and minor third adding up to a perfect fifth.

The conditions (6,7) do not yet make sure that all units in the n-tone division get a note-name. When n is divisible by m such that $n/m = t$, $t \in \mathbb{Z}$, only t units of the n-tone division get a note-name. See for example figure 2 where all notes are mapped on units 0, 4 and 8. In this particular case (where the tones are only generated by the major third) the

12-tone system can be reduced to a 3-tone system, without changing the deviation of the equal tempered notes to the frequency ratios from just intonation (the notes are equally well approximated as in the 12-tone system).

In our method we have been using the approximation to three intervals: $3/2, 5/4, 6/5$ (or their inverses), to divide the note-names over the n units of the equal tempered system. If one of these numbers $m_{\frac{3}{2}}, m_{\frac{5}{4}}$ or $m_{\frac{6}{5}}$ makes sure all units get a note-name, this m is a so called generator of n. The number m is a generator of n if

$$\mathrm{GCD}[m, n] = 1, \tag{21}$$

that is, the greatest common divisor (GCD) of m and n is 1.

When the n-tone equal tempered system does not have generators among $m_{\frac{3}{2}}, m_{\frac{5}{4}}$ or $m_{\frac{6}{5}}$ (but still does follow the enharmonicity condition), the n-tone temperament can be simplified to an n'-tone temperament, such that

$$n = k \cdot n' \qquad k \in \mathbb{Z}. \tag{22}$$

If $m_{\frac{3}{2}}$ is not a generator of the system but the enharmonicity condition is true, then the notes generated by $m_{\frac{5}{4}}$ and $m_{\frac{6}{5}}$ will be located in the same positions as the same notes generated by $m_{\frac{3}{2}}$ (since the notes generated by the major and minor third are subgroups of the notes generated by the fifth). So, then $m_{\frac{5}{4}}$ or $m_{\frac{6}{5}}$ (or their inverses) can never be a generator of the system. Thus, when $m_{\frac{3}{2}}$ is not a generator of the system, it can always be reduced to a simplified system following (22). Since those n-tone systems that can be reduced to n'-tone systems are not interesting here (because they do not approximate the consonant intervals better than the reduced systems), we want to find n-tone equal tempered systems of which $m_{\frac{3}{2}}$, the number of steps approximating the fifth, is a generator of n:

$$\mathrm{GCD}[m_{\frac{3}{2}}, n] = 1. \tag{23}$$

Combining this condition (23) with the enharmonicity condition, the following values for n are left:

$$n = 5, 7, 12, 19, 26, 31, 43, 45, 50, 55, 69, 74, 81, 88 \tag{24}$$

Combining this with general results for a goodness-of-fit approach, which leads to systems of size:$12, 19, 31, 41$, and 53, we see that divisions of the octave in $12, 19$ or 31 parts would be a good choice. Indeed, keyboard applications for these temperaments have been constructed, like for example the 19-tone harmonium in 1854 [13] and the 31-tone organ by Fokker [5].

3 Thirds space

So far, we have seen that the choice of using the Western note name system has given two restrictions (equations (6, 7) and 23) on the possible divisions of the octave. To better understand what these restrictions mean for the resulting n-tone temperaments, it is useful to present them visually in a tonal space. It is known that tone-spaces can be represented as 2-dimensional lattices $\mathbb{Z}^2$ [1, 9]. Analogous to Balzano [1] we use here the major and minor third as unit intervals (or unit vectors) to build the space, and therefore call this space 'thirds'-space. A thirds-space can be made from frequency ratios, note-names or equal tempered unit numbers. The first two are shown in figure 4. Both are infinite spaces and can be mapped onto each other to see which note corresponds to which frequency ratio (having fixed the prime interval '1' to 'C'). Different versions of a thirds-space for equal tempered unit numbers are possible, depending on the number of parts the octave is divided in. A mapping of an n-tone equal temperament with n from (24) can be made with this space. Figure (5) displays the situation for $n = 12$ and $n = 19$. The tone-space displayed left in figure 5 can be rolled up along the sides of the square (indicated by boldface numbers) to become a torus. Every lattice point with the same number is actually the same point, so clearly these are not infinite spaces. We see from figures 5 and 6 that the n-tone temperaments we obtained can be represented in parallelograms. The number of elements n is exactly the area spanned up by the parallelogram. We notice that in each n-tone system the length of one diagonal is kept constant. Comparing this space to the note-name-space in figure 4 we see that this makes sure that all notes that have the same name are identified with each other, which is a consequence of the enharmonicity conditions. The other diagonal of the parallelogram indicates which other notes are identified with each other; in other words, which notes are enharmonically equivalent. In 12-tone equal temperament the length of this

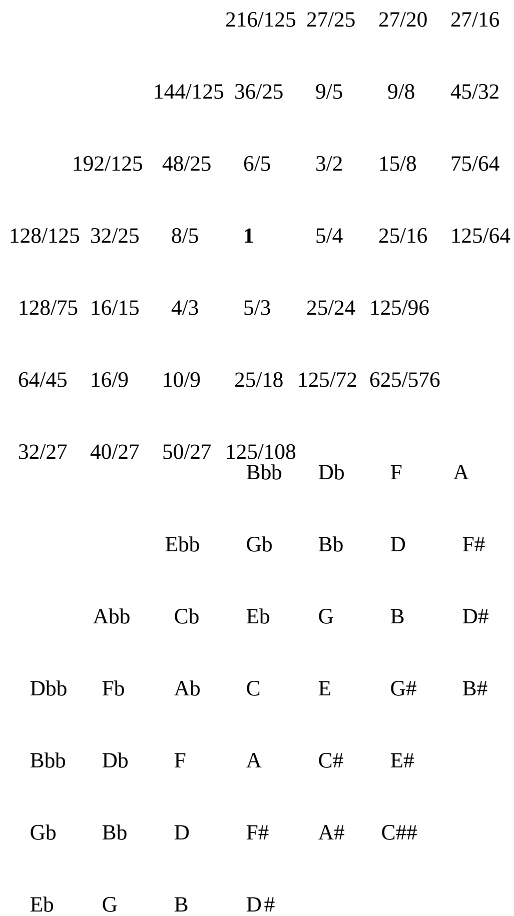

Figure 4: Projection of Just intonation intervals generated by major and minor third on the note-names

diagonal is “3 major thirds up, 4 minor thirds down”, meaning that a note is enharmonically equivalent to the note which is 3 major thirds up and 4 minor thirds down. For the 19, 26 and 31 tone case the distances can be read from figures 5 and 6. Coming back to the reasons for the enharmonicity condition, we can now better understand these. In figure 4 we can see the path between two notes with the same name. This path can be read as ’3 major thirds and 4 minor thirds up’, or ’3 perfect fifths and 1 minor third up’, or any other path can be chosen to go from one to another with the same name. Infinitely many choices can be made to describe the path between these two notes in terms of combinations of

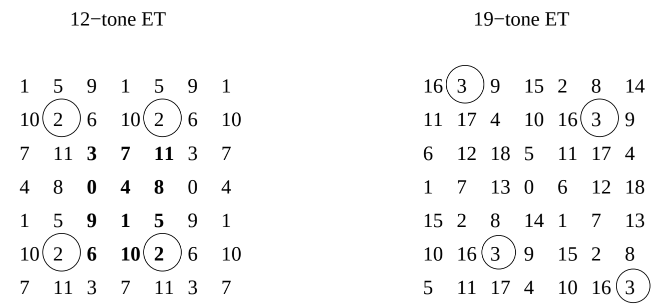

Figure 5: Two examples of n-tone equal tempered mappings on Just intonation ratio’s and note-names

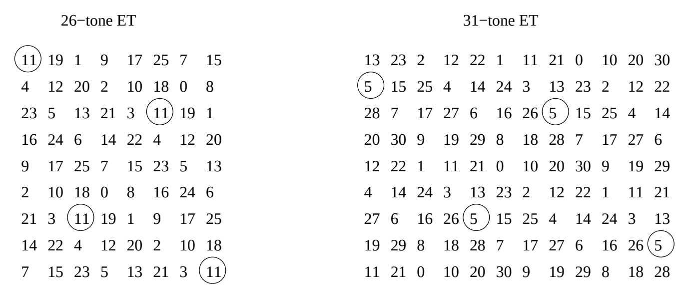

Figure 6: Thirds space represented in a 26 and a 31 tone equal tempered system

other intervals. What we did in constructing equation (6) is in fact saying that: 4 perfect fifths minus 1 major third should return to the same note. Equation (7) translates into: 3 perfect fifths plus 1 minor third (or minus 1 major sixth) returns the same note.

4 Extended note systems

Now that we have given a visual representation for the n-tone systems resulting from (24), we wonder how we can extend this in such a way that every frequency ratio is represented by a separate note-name. This would be very useful since then we can distinguish between two different ratios with the same note-name. If this can be done, the restrictions on the equal division of the octave can probably be changed such that more divisions can be used. Eitz [4] created a note-name system that distinguishes between different ratios from just intonation. Eitz departs from Pythagorian ratios and gives them 'normal' note-names with the superscript 0. Ratios differing t times a factor $81/80$ he has given the note-name with a $\pm t$ superscript, see figure 7 for a representation. With this new note-name system, what conditions are prefer-

			Bbb^{+3}	Db^{+2}	F^{+1}	A^{0}
		Ebb^{+3}	Gb^{+2}	Bb^{+1}	D^{0}	F#$^{-1}$
	Abb^{+3}	Cb^{+2}	Eb^{+1}	G^{0}	B^{-1}	D#$^{-2}$
Dbb^{+3}	Fb^{+2}	Ab^{+1}	C^{0}	E^{-1}	G#$^{-2}$	B#$^{-3}$
Bb^{+2}	Db^{+1}	F^{0}	A^{-1}	C#$^{-2}$	E#$^{-3}$	
Gb^{+1}	Bb^{0}	D^{-1}	F#$^{-2}$	A#$^{-3}$	C##$^{-4}$	
Eb^{0}	G^{-1}	B^{-2}	D#$^{-3}$			

Figure 7: Eitz' notation for the names of the intervals

able for constructing an n-tone equal tempered system? Are there some notes which we would like to be enharmonically equivalent? We focus for a moment on the well-known problem which goes with playing in just intonation. The major chords I, IV and V form the basis of the just major scale, namely $C^0 - E^{-1} - G^0$, $F^0 - A^{-1} - C^0$ and $G^0 - B^{-1} - D^0$ in the key of C major. The triads $A^{-1} - C^0 - E^{-1}$ and $E^{-1} - G^0 - B^{-1}$ form the minor triads VI and III. The problems come by spelling the II triad. Should it be spelled as $D^{-1} - F^0 - A^{-1}$ or as $D^0 - F^{+1} - A^0$ or even something else? Playing in just intonation, problems like this will always arise. To give an example, how to spell the chord sequence: $I - VI - II - V - I$, keeping notes that appear in adjacent triads in the same spelling. Following this last rule, the spelling would be: $C^0E^{-1}G^0$-$A^{-1}C^0E^{-1}$-$D^{-1}F^0A^{-1}$-$G^{-1}B^{-2}D^{-1}$-$C^{-1}E^{-2}G^{-1}$. Comparing the first I triad to the last I triad, we see that C^0 has changed to C^{-1} which is called a downward drift. To avoid this drift, we have to identify C^0 with C^{-1}, and in general identify a note-name X^0 to X^t ($r \in \mathbb{Z}$). We have seen that infinitely many paths can be chosen to go from X^{-1} to X^0, and besides the enharmonicity condition (6,7) infinitely many other conditions can be stated, all making sure X^0 is being identified with X^t. And when X^0 is identified with X^t, the familiar note-name system as in figure 4 can again be used. Thus, a choice has to be made between avoiding downward drift and approximating just intonation as well as possible.

Then what should be taken into account when trying to derive a suitable equal octave division that maps on figure 7? Ideally, in a mapping from an equal tempered unit number thirds-space to the frequency ratio thirds-space, every ratio is approximated by the closest unit number in that n-tone equal tempered system. But if that is achieved, intervals (like major third, minor seventh etc.) do not have a fixed unit number representing them any more, and that would mean that a piece of music can not be transposed.

For all n-tone systems where the $m_{\frac{3}{2}}$ is a generator of n, a representation like figure 5 and 6 can be made. That is: to every point of the lattice a number can be assigned. The enharmonicity conditions made sure that notes with identical names are identified and that addition of consonant intervals is correct. In our new tone-space equal note-names do not need to be identified to each other anymore. However, the addition of consonant intervals is still required. From the origin of the space (indicated by 1 in the ratio space and by 0 in the number space) the intervals major and minor third (or major and minor sixth) can be seen as the unit vectors building the space. Then, the thirds space of equal tempered numbers can be constructed by knowing the values of $m_{\frac{5}{4}}$ and $m_{\frac{6}{5}}$. The fifth should be connected to the number that approximates this interval best. Since a major third plus a minor third should equal a perfect fifth, this is established if

$$m_{\frac{5}{4}} + m_{\frac{6}{5}} = m_{\frac{3}{2}} \qquad (25)$$

And when the major and minor third are approximated correctly, their inverses minor and major sixth are too.

When using the Western note name system, there was also the generating fifth condition to make sure the resulting equal temperaments could not be reduced to a lower number system. Using Eitz note name system, this condition is changed a bit. If for a certain n-tone equal temperament not all numbers are used when they are divided by the fifth, major third and minor third (or their inverses), the system can be reduced to an n'-tone system ($n = k \cdot n', k \in \mathbb{Z}$) in the same way as before. To establish that every number $(1 \ldots n)$ can be reached by a linear combination of $m_{\frac{3}{2}}$, $m_{\frac{5}{4}}$ and $m_{\frac{6}{5}}$, representing the number of units that approximate the fifth, major third and minor third in the n-tone system respectively, we have to demand that

$$\mathrm{GCD}[a \cdot m_{\frac{3}{2}} + b \cdot m_{\frac{5}{4}} + c \cdot m_{\frac{6}{5}}, n] = 1, \tag{26}$$

where $a, b, c \in \mathbb{Z}$. Using the Euclidean algorithm (see for example [12]) this condition can be rewritten as:

$$\mathrm{GCD}[\mathrm{GCD}[\mathrm{GCD}[m_{\frac{3}{2}}, m_{\frac{5}{4}}], m_{\frac{6}{5}}], n] = 1, \tag{27}$$

which in turn can be written shorter as:

$$\mathrm{GCD}[m_{\frac{3}{2}}, m_{\frac{5}{4}}, m_{\frac{6}{5}}, n] = 1. \tag{28}$$

This condition together with the adding interval condition (25) result in the following values for n:

$$\begin{aligned} n \;=\; & 3, 4, 5, 7, 8, 9, 10, 12, 15, 16, 18, 19, 22, 23, 25, 26, \\ & 27, 28, 29, 31, 34, 35, 37, 39, 41, 42, 43, 45, 46, 47, \\ & 48, 49, 50, 53, 55, 56, 58, 59, 60, 61, 63, 65, 69, 70, \\ & 71, 72, 73, 74, 75, 77, 78, 79, 80, 81, 83, 84, 87, 88, \\ & 89, 90, 91, 94, 95, 96, 97, 99, \ldots \end{aligned} \tag{29}$$

Here are only the values for n until 100 given, but there is no limit to the value for n that satisfy these conditions. Comparing these results to (24), we see that a lot more divions of the octave can be used if the restriction to use Western note-names is abandoned. However, condition (25) was constructed especially for Western music, since it is important to have triads (the building blocks of Western music) that are in tune. So, the n-tone equal tempered systems with n from (29) are to be used for Western music, but music written in the normal Western note-name system has to be translated to a system like Eitz' before it is possible to play it.

5 Conclusions

We argued that for an n-tone equal tempered system suitable for keyboard application two mathematical conditions should be satisfied, leading to a number of values for n. Remarkably (but intuitively understandably) there is a certain maximum to the value of n. As a consequence of this, there is a maximum bound to the closeness we can approximate just intonation with equal temperament, when using the familiar note-name system. Good divisions of the octave to approximate just intonation that do not satisfy the two required mathematical conditions cannot be used in combination with the Western note-name system. If one still wishes to use such tone systems this has to be in combination with a note-name system like Eitz' system. Combining our results for good equal divisions of the octave with the general results using a goodness of fit approach, we conclude that possible divisions for a keyboard are $12, 19, 31$ notes per octave. Indeed, keyboard systems with these octave divisions have been constructed [13, 5].

In finding a suitable equal tempered system that maps on Eitz' note-name system, there are still certain conditions to satisfy. The sequence of possible values of the octave division is an infinite sequence now. We did not find values like $n = 41, 53$, which result from a goodness of fit approach in our results using the traditional note-names, but we do find them in our results using Eitz notation. The in goodness-of-fit outstanding 53-tone system has been translated in 53-tone harmonium built in the 19th century [2, 8]. For finding good n-tone systems to play Western music the first demand may be that it should approximate the ratios from just intonation well, but in this paper we gave some insight in the possibilities for actually using these temperaments and thus a possible explanation for the historical choices of certain tone-systems.

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