(Complex) eigenvalues and rotational axes

You have seen in other problems that rotations are represented by orthonormal matrices. When you combine rotations, you multiply a lot of these; yet the final result should again be a rotation. You can see this both from the physics of the situation, and from the mathematics: the product of a number of orthonormal matrices is again an orthonormal matrix.

Since this result is a rotation, it is natural to ask: what is the rotation axis, and what is the rotation angle. You can solve this easily using the *eigenvalues* and *eigenvectors* of the matrix. But: some of the eigenvalues will be complex numbers!

(1) Let us take as an example the matrix:

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

You should recognize this as a rotation matrix, around the z-axis, over an angle $\frac{\pi}{6}$. (If not, do the 2D rotation problem `rotation2d.ps`, and the 3D rotation problem `rotation3d.ps`.)

(2) When the rotation matrix acts on points, it changes their coordinates so that they appear to rotate. This is true for all points. What is the special property of points on the rotation axis?

(3) How do you express that property in terms of eigenvalues?

(4) So a point on the rotation axis (which passes through the origin) is represented by one of the eigenvectors of eigenvalue 1. No other points remain the same, or change to a multiple of themselves, so would expect them to have no real eigenvalues. And you are right: the eigenvalues are complex.

(5) Compute the eigenvalues of $\mathbf{A}$. (**Answer**: $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}\sqrt{3}+\frac{1}{2}i = e^{\frac{\pi}{6}i}$, $\lambda_3 = \frac{1}{2}\sqrt{3}-\frac{1}{2}i = e^{-\frac{\pi}{6}i}$. Note that the eigenvalues are either real, or a pair of complex conjugate values, $\lambda_2 = \overline{\lambda_3}$.)

(6) Compute the corresponding eigenvectors of $\mathbf{A}$. (**Answer**: $(0,0,1)^T$, $(i,1,0)^T$, $(-i,1,0)^T$; or as an orthonormal coordinate system: $(0,0,1)^T$, $(i/\sqrt{2},1/\sqrt{2},0)^T$, $(-i/\sqrt{2},1/\sqrt{2},0)^T$)

(7) You see that the eigenvector of eigenvalue 1 is indeed along the rotation axis of $\mathbf{A}$. The complex eigenvectors are hard to interpret; but note that the complex eigenvalues $e^{\frac{\pi}{6}i}$ and $e^{-\frac{\pi}{6}i}$ contain the rotation angle $\frac{\pi}{6}$.

(8) Use the eigenvectors as a basis. What is the coordinate transformation matrix $\mathbf{B}$ that can be used to interpret a vector of that basis? What kind of a matrix is it, mathematically ?

(**Answer**: take for example the coordinate transformation:

$$\mathbf{B} = \begin{pmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix} \tag{2}$$

(which has determinant i) from the eigenvector basis. It can be used to re-represent vectors, e.g. $\mathbf{B}(0,1,0)^T = (i/\sqrt{2},1/\sqrt{2},0)^T$ is the representation of the second eigenvector in terms of our original coordinates. $\mathbf{B}$ is a *unitary* matrix, satisfying $\mathbf{B}^{-1} = \overline{\mathbf{B}}^T$, with '' denoting the complex conjugate. Check this!)

(9) Use $\mathbf{B}$ to represent $\mathbf{A}$ on the eigenvector basis. Why do you get a diagonal matrix? What is on the diagonal?

(**Answer**: The diagonal matrix $\mathbf{\Delta}$ follows from $\mathbf{A} = \mathbf{B}\mathbf{\Delta}\mathbf{B}^{-1}$ as:

$$\mathbf{\Delta} = \overline{\mathbf{B}}^T \mathbf{A}\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{\pi}{6}i} & 0 \\ 0 & 0 & e^{-\frac{\pi}{6}i} \end{pmatrix}. \tag{3}$$

The eigenvalues are on the diagonal, of course.)

(10) Thus by a complex, unitary coordinate transformation we achieve diagonalization of a rotation matrix. The real eigenvector with eigenvalue 1 is along the rotation axis. The other eigenvalues are each other's complex conjugate, and their argument is (plus or minus) the rotation angle.