Computational Social Choice: Spring 2017

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Plan for Today

So far we saw three voting rules: plurality, plurality with runoff, Borda.

Today we are going to see many more *voting rules*. Most importantly:

- Positional scoring rules
- Condorcet extensions

We are going to *compare* rules and consider how to *classify* them.

Much (not all) of this (and more) is also covered by Zwicker (2016).

W.S. Zwicker. Introduction to the Theory of Voting. In F. Brandt et al. (eds.), *Handbook of Computational Social Choice*. CUP, 2016.

Formal Framework

Need to choose from a finite set $X = \{x_1, \ldots, x_m\}$ of *alternatives*.

Let $\mathcal{L}(X)$ denote the set of all strict linear orders $\succ$ on X. We use elements of $\mathcal{L}(X)$ to model (true) *preferences* and (declared) *ballots*.

Each member of a finite set $N = \{1, \ldots, n\}$ of *voters* supplies us with a ballot, giving rise to a *profile* $\boldsymbol{\succ} = (\succ_1, \ldots, \succ_n) \in \mathcal{L}(X)^n$.

A *voting rule* (or *social choice function*) for N and X selects one or more winners for every such profile:

$$F : \mathcal{L}(X)^n \to 2^X \setminus \{\emptyset\}$$

If $|F(\boldsymbol{\succ})| = 1$ for all profiles $\boldsymbol{\succ}$, then F is called *resolute*.

Most natural voting rules are *irresolute* and have to be paired with a *tie-breaking rule* to always select a unique election winner.

Examples: random tie-breaking, lexicographic tie-breaking

Preview: Some Axioms

Our focus today is going to be on concrete voting rules. Still, we are going to use these *axioms* to highlight some issues they suffer from:

- *Participation Principle:* It should be in the best interest of voters to participate: voting truthfully should be no worse than abstaining.
- *Pareto Principle:* There should be no alternative that every voters strictly prefers to the alternative selected by the voting rule.
- *Condorcet Principle:* If there is an alternative that is preferred to every other alternative by a majority of voters, then it should win.

Majority Rule and Condorcet Paradox

Suppose there are only *two alternatives* and an odd number of voters.
Then we can use the *majority rule:*

$$F(\succ) \quad = \quad \begin{cases} x_1 & \text{if } |\{i \in N \mid x_1 \succ_i x_2\}| > \frac{n}{2} \\ x_2 & \text{otherwise} \end{cases}$$

This is the perfect rule (we'll prove this formally next time, *twice*).
But it only is well-defined for two alternatives ...

For three or more alternatives, sometimes none of them beats all others in pairwise majority contests. This is the famous *Condorcet Paradox:*

$$A \succ_1 B \succ_1 C$$
$$B \succ_2 C \succ_2 A$$
$$C \succ_3 A \succ_3 B$$

M.J.A.N. de Caritat (Marquis de Condorcet). *Essai sur l'application de l'analyse à la probabilté des décisions rendues a la pluralité des voix*. Paris, 1785.

Single Transferable Vote (STV)

STV (used, e.g., in Australia) works in stages:

- If some alternative is top for an *absolute majority*, then it wins.
- Otherwise, the alternative ranked at the top by the fewest voters (the plurality loser) gets *eliminated* from the race.
- Votes for eliminated alternatives get *transferred:* delete removed alternatives from ballots and "shift" rankings (i.e., if your 1st choice got eliminated, then your 2nd choice becomes 1st).

Various options for how to deal with *ties* during elimination.

In practice, voters need not be required to rank all alternatives (non-ranked alternatives are assumed to be ranked lowest).

STV (suitably generalised) is often used to elect committees.

For three alternatives, STV and *plurality with runoff* coincide.

<u>Variants:</u> *Coombs*, *Baldwin*, *Nanson* (different elimination criteria)

The No-Show Paradox

Under plurality with runoff (and thus under STV), it may be better to abstain than to vote for your favourite alternative!

25 voters: $A \succ B \succ C$
46 voters: $C \succ A \succ B$
24 voters: $B \succ C \succ A$

Given these voter preferences, B gets eliminated in the first round, and C beats A 70:25 in the runoff.

Now suppose two voters from the first group abstain:

23 voters: $A \succ B \succ C$
46 voters: $C \succ A \succ B$
24 voters: $B \succ C \succ A$

A gets eliminated, and B beats C 47:46 in the runoff.

P.C. Fishburn and S.J Brams. Paradoxes of Preferential Voting. *Mathematics Magazine*, 56(4):207–214, 1983.

Cup Rules via Voting Trees

We can define a voting rule via a *binary tree*, with the alternatives labelling the leaves, and an alternative progressing to a parent node if it beats its sibling in a *majority contest*.

Two examples for such *cup rules* and a possible profile of ballots:

(1)

```
        o
       / \
      o   C
     / \
    A   B
```

(2)

```
        o
       / \
      /   \
     o     o
    / \   / \
   A   B B   C
```

$A \succ B \succ C$

$B \succ C \succ A$

$C \succ A \succ B$

Rule (1): C wins

Rule (2): A wins

Cup Rules and the Pareto Principle

The (weak) *Pareto Principle* requires that we should never elect an alternative that is strictly dominated in *every* voter's ballot.

Cup rules do not always satisfy this most basic principle!

```
        o
       / \
      o   D
     / \
    o   A
   / \
  B   C
```

Consider this profile with three voters:

Ann:	$A \succ B \succ C \succ D$
Bob:	$B \succ C \succ D \succ A$
Cindy:	$C \succ D \succ A \succ B$

D *wins!* (despite being dominated by C)

What happened? Note how this "embeds" the Condorcet Paradox, with every occurrence of C being replaced by $C \succ D$ ...

The Condorcet Principle

An alternative that beats every other alternative in pairwise majority contests is called a *Condorcet winner*. Sometimes there is no CW.

The *Condorcet Principle* says that, if it exists, only the CW should win. Voting rules that satisfy this principle are called *Condorcet extensions*.

Exercise: *Show that every cup rule is a Condorcet extension.*

But some other rules, such as the *Borda rule*, don't. Example:

3 voters: $C \succ B \succ A$
2 voters: $B \succ A \succ C$

Positional Scoring Rules

We can generalise the idea underlying the Borda rule as follows:

A *positional scoring rule* (PSR) is defined by a so-called *scoring vector* $\boldsymbol{s} = (s_1, \ldots, s_m) \in \mathbb{R}^m$ with $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$ and $s_1 > s_m$.

Each voter submits a ranking of the m alternatives. Each alternative receives s_i points for every voter putting it at the ith position.

The alternative(s) with the highest score (sum of points) win(s).

Examples:

- *Borda rule* = PSR with scoring vector $(m-1, m-2, \ldots, 0)$
- *Plurality rule* = PSR with scoring vector $(1, 0, \ldots, 0)$
- *Antiplurality* (or *veto*) *rule* = PSR with scoring vector $(1, \ldots, 1, 0)$
- For any $k < m$, *k-approval* = PSR with $(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$

Positional Scoring Rules and the Condorcet Principle

Consider this example with three alternatives and seven voters:

$$\begin{array}{ll} \text{3 voters:} & A \succ B \succ C \\ \text{2 voters:} & B \succ C \succ A \\ \text{1 voter:} & B \succ A \succ C \\ \text{1 voter:} & C \succ A \succ B \end{array}$$

A is the *Condorcet winner:* she beats both B and C $4:3$. But any *positional scoring rule* makes B win (because $s_1 \geqslant s_2 \geqslant s_3$):

$$\begin{array}{ll} A: & 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3 \\ B: & 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3 \\ C: & 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3 \end{array}$$

Thus, *no positional scoring rule* for three (or more) alternatives can possibly satisfy the *Condorcet Principle*.

Copeland Rule and Majority Graph

Under the *Copeland rule* an alternative gets $+1$ point for every pairwise majority contest won and -1 point for every such contest lost.

Exercise: *Show that the Copeland rule is a Condorcet extension.*

Remark: We only need to look at the *majority graph* (with an edge from A to B whenever A beats B in a pairwise majority contest).

Exercise: *How can you characterise the Condorcet winner (if it exists) in graph-theoretical terms in a given majority graph?*

A.H. Copeland. *A "Reasonable" Social Welfare Function*. Seminar on Mathematics in Social Sciences, University of Michigan, 1951.

F. Brandt, M. Brill, and P. Harrenstein. Tournament Solutions. In F. Brandt et al. (eds.), *Handbook of Computational Social Choice*. CUP, 2016.

Aside: McGarvey's Theorem

Let $\langle X, \succ_M \rangle$ denote the *majority graph*. For odd n, $\langle X, \succ_M \rangle$ always is a complete directed graph (a "*tournament*"). Surprisingly:

Theorem 1 (McGarvey, 1953) *For any given tournament, there exists a profile that induces that tournament as its majority graph.*

Proof: Given tournament $\langle X, \twoheadrightarrow \rangle$ with $|X| = m$, introduce two voters i_{xy} and i'_{xy} for every $x, y \in X$ with $x \twoheadrightarrow y$ with these preferences:

$$x \succ_{i_{xy}} y \succ_{i_{xy}} x_1 \succ_{i_{xy}} x_2 \succ_{i_{xy}} \cdots \succ_{i_{xy}} x_{m-2}$$

$$x_{m-2} \succ_{i'_{xy}} \cdots \succ_{i'_{xy}} x_2 \succ_{i'_{xy}} x_1 \succ_{i'_{xy}} x \succ_{i'_{xy}} y$$

Here $\{x_1, \ldots, x_{m-2}\} = X \setminus \{x, y\}$.
We get $\langle X, \twoheadrightarrow \rangle = \langle X, \succ_M \rangle$ for this profile of $m \cdot (m-1)$ voters. ✓

D.C. McGarvey. A Theorem on the Construction of Voting Paradoxes. *Econometrica*, 21(4):608–610, 1953.

Kemeny Rule and Weighted Majority Graph

Under the *Kemeny rule* an alternative wins if it is maximal in a ranking that minimises the sum of pairwise disagreements with the individual ballots. That is:

(1) For every possible ranking $\succ$, count the number of triples (i, x, y) s.t. $\succ$ disagrees with voter i on the ranking of alternatives x and y.

(2) Find all rankings $\succ$ that have a minimal score in the above sense.

(3) Elect any alternative that is maximal in such a "closest" ranking.

Exercise: *Show that the Kemeny rule is a Condorcet extension.*

The Kemeny rule needs more information than just the majority graph. But it can be computed from the *weighted majority graph*.

J. Kemeny. Mathematics without Numbers. *Daedalus*, 88:571–591, 1959.

F. Fischer, O. Hudry, and R. Niedermeier. Weighted Tournament Solutions. In F. Brandt et al. (eds.), *Handbook of Computational Social Choice*. CUP, 2016.

More Voting Rules

Here are a few more voting rules (still not all there is!):

- *Slater:* Find ranking that minimises number of edges in majority graph we'd have to switch. Elect top alternative in that ranking.
- *Ranked-Pairs:* Build a full ranking by "locking in" ordered pairs in order of majority strength (but avoid cycles). Elect top alternative.
- *Young:* Elect alternative x that minimises the number of voters we need to remove before x becomes the Condorcet winner.
- *Dodgson:* Elect alternative x that minimises the number of swaps of adjacent alternatives in the profile we need to perform before x becomes the Condorcet winner. (Note difference to Kemeny!)

Trivia: Dodgson also went by Lewis Carroll ("*Alice in Wonderland*").

Fishburn's Classification

Can classify voting rules on the basis of the *information* they require. The best known such classification is due to Fishburn (1977):

- *C1:* Winners can be computed from the *majority graph* alone.

  Examples: Copeland, Slater

- *C2:* Winners can be computed from the *weighted majority graph* (but not from the majority graph alone).

  Examples: Kemeny, Ranked-Pairs, Borda (*think about it!*)

- *C3:* All other voting rules. Examples: Young, Dodgson, STV

Remark: Fishburn originally intended this for Condorcet extensions only, but the concept also applies to all other voting rules.

P.C. Fishburn. Condorcet Social Choice Functions. *SIAM Journal on Applied Mathematics*, 33(3):469–489, 1977.

Computational Complexity

We can also classify voting rules according to the computational complexity of the *winner determination problem*. Omitting details:

Proposition 2 *For any positional scoring rule, the election winners can be computed in polynomial time.*

Theorem 3 (Brill and Fischer, 2012) *Deciding whether a given alternative is a winner under the ranked-pairs rule is NP-complete.*

Theorem 4 (Hemaspaandra et al., 1997) *Deciding whether a given alternative is a Dodgson winner is complete for parallel access to NP.*

M. Brill and F. Fischer. The Price of Neutrality for the Ranked Pairs Method. Proc. 26th AAAI Conference on Artificial Intelligence, 2012.

E. Hemaspaandra, L.A. Hemaspaandra, and J. Rothe. Exact Analysis of Dodgson Elections. *Journal of the ACM*, 44(6):806–825, 1997.

Nonstandard Ballots

We defined voting rules over profiles of strict linear orders (even if some rules, e.g., plurality, don't use all information). Other options:

- *Approval voting:* You can approve of any subset of the alternatives. The alternative with the most approvals wins.
- *Even-and-equal cumulative voting:* You vote as for AV, but 1 point gets split evenly amongst the alternatives you approve.
- *Range voting:* You vote by dividing 100 points amongst the alternatives as you see fit (as long every share is an integer).
- *Majority judgment* ($\neq$ JA): You award a grade to each alternative ("excellent", "good", etc.). Highest median grade wins.

The most important of these is approval voting.

Remark: *k-approval* and *approval voting* are very different rules!

Summary

We have introduced a large number of voting rules:

- *Staged procedures:* plurality with runoff, STV (and variants)
- *Positional scoring rules:* Borda, plurality, antiplurality, k-approval
- *Condorcet extensions:* cup rules, Copeland, Slater, Kemeny, ranked-pairs rule, Dodgson, Young
- *Nonstandard rules:* approval voting, even-and-equal cumulative voting, range voting, majority judgment

We have analysed these rules from a variety of angles:

- *Axioms:* Condorcet, Pareto, Participation (no no-show paradox)
- *Informational basis:* majority graph, weighted majority graph
- *Computational complexity:* polynomial, NP, beyond NP

What next? Characterising voting rules, to help choose the right one.