Computational Social Choice: Autumn 2011

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Plan for Today

Fair division is the problem of dividing one or several goods amongst two or more agents in a way that satisfies a suitable fairness criterion.

This can be considered a problem of *social choice:*

- A group of agents each have individual preferences over a collective agreement (the allocation of goods to be found).
- But: in fair division preferences are often assumed to be cardinal (*utility functions*) rather than ordinal (as in voting)
- And: fair division problems come with some *internal structure* often absent from other social choice problems (e.g., I will be indifferent between allocations giving me the same set of goods)

This will be a very brief introduction to fair division. For more details and references, see my lecture notes on the topic.

U. Endriss. *Lecture Notes on Fair Division*. ILLC, University of Amsterdam, 2010.

Fair Division: Formal Framework

- Let $\mathcal{N} = \{1, \ldots, n\}$ be a set of *agents* (or *players*, or *individuals*) who need to share several *goods* (or *resources*, *items*, *objects*).
- An *allocation* A is a mapping of agents to *bundles* of goods.
- Each agent $i \in \mathcal{N}$ has a *utility function* u_i (or *valuation function*), mapping allocations to the reals, to model their preferences.
  - Typically, u_i first defined on bundles, so: $u_i(A) = u_i(A(i))$.
  - Discussion: preference intensity, interpersonal comparison
- An allocation A gives rise to a *utility vector* $\langle u_1(A), \ldots, u_n(A)\rangle$.
- Sometimes, we are going to define social preference structures directly over utility vectors $u = \langle u_1, \ldots, u_n\rangle$ (elements of $\mathbb{R}^n$), rather than speaking about the allocations generating them.

Pareto Efficiency

Allocation A is *Pareto dominated* by allocation A' if $u_i(A) \leqslant u_i(A')$ for all agents $i \in \mathcal{N}$ and this inequality is strict in at least one case.

An allocation A is *Pareto efficient* if there is no other feasible allocation A' such that A is Pareto dominated by A'.

The idea goes back to Vilfredo Pareto (Italian economist, 1848–1923).

Discussion:

- Pareto efficiency is very often considered a minimum requirement for any reasonable allocation. It is a very weak criterion.
- Only the ordinal content of preferences is needed to check Pareto efficiency (no preference intensity, no interpersonal comparison).

Collective Utility Functions

A *collective utility function* (CUF) is a function $\mathrm{SW} : \mathbb{R}^n \to \mathbb{R}$ mapping utility vectors to the reals. Three important examples:

- the *utilitarian* CUF ranking allocations by total utility

$$\mathrm{SW}_{\mathrm{util}}(u) \quad = \quad \sum_{i \in \mathcal{N}} u_i$$

- the *egalitarian* CUF that ties social welfare to the poorest agent

$$\mathrm{SW}_{\mathrm{egal}}(u) \quad = \quad min\{u_i \mid i \in \mathcal{N}\}$$

- the *Nash* CUF that tries to balance fairness and efficiency

$$\mathrm{SW}_{\mathrm{nash}}(u) \quad = \quad \prod_{i \in \mathcal{N}} u_i$$

H. Moulin. *Axioms of Cooperative Decision Making*. Econometric Society Monographs, Cambridge University Press, 1988.

Social Welfare Orderings

Each CUF defines a *social welfare ordering* (SWO) $\succeq$, a binary relation on $\mathbb{R}^n$ that is reflexive, transitive and complete.

We can also define SWOs directly. The most important example is the *leximin ordering* $\succeq_\ell$, extending the egalitarian principle:

- $u \succ_\ell v$ if, for some k, the k lowest elements of u and v coincide and the $k{+}1$st element of u is greater than the $k{+}1$st element of v.

Axiomatic Method

Similarly to Arrovian social choice theory, there are a lot of interesting results regarding axiomatisations of certain social welfare orderings.

Some of the axioms used in the literature:

- An SWO $\succeq$ is *zero independent* if $u \succeq v$ entails $(u+w) \succeq (v+w)$ for all $u, v, w \in \mathbb{R}^n$.
- An SWO $\succeq$ is *independent of the common utility pace* if $u \succeq v$ entails $f(u) \succeq f(v)$ for all $u, v \in \mathbb{R}^n$ and for every increasing bijection $f : \mathbb{R} \to \mathbb{R}$. (Example: taxes)
- An SWO satisfies the *Pigou-Dalton Principle* if $u \succeq v$ whenever u can be reached from v by means of an inequality-reducing yet mean-preserving transfer of utility between two agents.

H. Moulin. *Axioms of Cooperative Decision Making*. Econometric Society Monographs, Cambridge University Press, 1988.

Envy-Freeness

An allocation is called *envy-free* if no agent would rather have one of the bundles allocated to any of the other agents:

$$u_i(A(i)) \quad \geqslant \quad u_i(A(j))$$

Recall that $A(i)$ is the bundle allocated to agent i in allocation A.

Remark: Envy-free allocations do not always *exist* (at least not if we require either complete or Pareto efficient allocations).

Degrees of Envy

As we cannot always ensure envy-free allocations, another approach would be to try to reduce the *degree of envy* as much as possible.

- Envy between two agents:
  $\max\{u_i(A(j)) - u_i(A(i)), 0\}$ or
  1 if $u_i(A(j)) > u_i(A(i))$ and 0 otherwise
- Degree of envy of a single agent:
  $\max$, sum
- Degree of envy of a society:
  $\max$, sum [or indeed any CUF]

Y. Chevaleyre, U. Endriss, S. Estivie, and N. Maudet. *Reaching Envy-free States in Distributed Negotiation Settings*. Proc. IJCAI-2007.

Indivisible Goods: Formal Framework

Fixing goods to be *indivisible* results in a more concrete framework:

- Set of *agents* $\mathcal{N} = \{1, \ldots, n\}$ and finite set of indivisible *goods* $\mathcal{G}$.
- An *allocation* A is a partitioning of $\mathcal{G}$ amongst the agents in $\mathcal{N}$.
  Example: $A(i) = \{a, b\}$ — agent i owns items a and b
- Each agent $i \in \mathcal{N}$ has got a *valuation function* $v_i : 2^{\mathcal{G}} \to \mathbb{R}$.
  Example: $v_i(A) = v_i(A(i)) = 577.8$ — agent i is pretty happy
- If agent i receives bundle B and the sum of her payments is x, then her *utility* is $u_i(B, x) = v_i(B) - x$ ("quasi-linear utility").

For fair division of indivisible goods *without money*, assume that payment balances are always equal to 0 (and utility = valuation).

Preference Representation Languages

<u>Example:</u> Allocating 10 goods to 5 agents means $5^{10} = 9765625$ allocations and $2^{10} = 1024$ bundles for each agent to think about.

So we need to choose a good *language* to compactly represent preferences over such large numbers of alternative bundles, e.g.:

- Logic-based languages (weighted goals)
- Bidding languages for combinatorial auctions (OR/XOR)
- Program-based preference representation (straight-line programs)
- CP-nets and CI-nets (for ordinal preferences)

The choice of language affects both *algorithm design* and *complexity*.

See Chevaleyre et al. (2008) for references.

Y. Chevaleyre, U. Endriss, J. Lang, and N. Maudet. Preference Handling in Combinatorial Domains: From AI to Social Choice. *AI Magazine*, 29(4):37–46, 2008.

Complexity Results

Before we look into the "how", here are some complexity results:

- Checking whether an allocation is *Pareto efficient* is coNP-complete.
- Finding an allocation with maximal *utilitarian* social welfare is NP-hard. If all valuations are *modular* (additive) then it is polynomial.
- Finding an allocation with maximal *egalitarian* social welfare is also NP-hard, even when all valuations are modular.
- Checking whether an *envy-free* allocation exists is NP-complete; checking whether an allocation that is both Pareto efficient and envy-free exists is even Σ_2^p-complete.

References to these results (and details on the preference representation languages for which they apply) may be found in the "MARA Survey".

Y. Chevaleyre, P.E. Dunne, U. Endriss, J. Lang, M. Lemaître, N. Maudet, J. Padget, S. Phelps, J.A. Rodríguez-Aguilar and P. Sousa. Issues in Multiagent Resource Allocation. *Informatica*, 30:3–31, 2006.

Negotiating Socially Optimal Allocations

In AI, one approach has been to design simple *negotiation protocols* allowing agents to make deals regarding the exchange of goods and to analyse the dynamics of those systems.

We can distinguish two perspectives:

- The *local/individual* view: what deals will agents make, given their preferences? Example: myopic rationality
- The *global/social* view: how will the overall allocation evolve in terms of social welfare? Example: utilitarian collective utility

One of the main questions is whether negotiation will *converge* to a social optimum, or under what circumstances it will.

U. Endriss, N. Maudet, F. Sadri and F. Toni. Negotiating Socially Optimal Allocations of Resources. *Journal of AI Research*, 25:315–348, 2006.

Cake-Cutting: Formal Framework

"Cake-cutting" is the problem of fair division of a single *divisible* (and heterogeneous) good between n *agents* (or *players*).

The *cake* is represented by the unit interval $[0, 1]$:

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|----------------------|
0                      1
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Each agent i has a *utility function* u_i (or *valuation*, *measure*) mapping finite unions of subintervals of $[0, 1]$ to the reals, satisfying:

- Non-negativity: $u_i(B) \geqslant 0$ for all $B \subseteq [0, 1]$
- Normalisation: $u_i(\emptyset) = 0$ and $u_i([0, 1]) = 1$
- Additivity: $u_i(B \cup B') = u_i(B) + u_i(B')$ for disjoint $B, B' \subseteq [0, 1]$
- u_i is continuous: the Intermediate-Value Theorem applies and single points do not have any value.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A.K. Peters, 1998.

Cut-and-Choose

The classical approach for dividing a cake between *two agents:*

- *One agent cuts the cake in two pieces (she considers to be of equal value), and the other one chooses one of the pieces (the piece she prefers).*

The cut-and-choose procedure is *proportional:*

- Each agent is guaranteed at least one half (general: $1/n$) according to her own valuation.

Discussion: In fact, the first agent (if she is risk-averse) will receive exactly $1/2$, while the second will usually get more.

What if there are more than two agents?

The Dubins-Spanier Procedure

Dubins and Spanier (1961) proposed a *proportional* procedure for *arbitrary n*, produceing *contiguous* slices.

(1) A referee moves a knife slowly across the cake, from left to right. Any player may shout "stop" at any time. Whoever does so receives the piece to the left of the knife.

(2) When a piece has been cut off, we continue with the remaining $n-1$ players, until just one player is left (who takes the rest). ✓

Observe that this is also *not envy-free*. The last chooser is best off (she is the only one who can get more than $1/n$).

Note that this is not a discrete "protocol" (i.e., an algorithm) in the narrow sense of the word (you cannot actually implement this!).

L.E. Dubins and E.H. Spanier. How to Cut a Cake Fairly. *American Mathematical Monthly*, 68(1):1–17, 1961.

The Banach-Knaster Last-Diminisher Procedure

In the first ever paper on fair division, Steinhaus (1948) reports on a solution for *arbitrary n* proposed by Banach and Knaster.

(1) Agent 1 cuts off a piece (that she considers to represent $1/n$).

(2) That piece is passed around the agents. Each agent either lets it pass (if she considers it too small) or trims it down further (to what she considers $1/n$).

(3) After the piece has made the full round, the last agent to cut something off (the "last diminisher") is obliged to take it.

(4) The rest (including the trimmings) is then divided amongst the remaining $n-1$ agents. Play cut-and-choose once $n = 2$. ✓

Each agent is guaranteed a *proportional* piece. Takes $O(n^2)$ steps.

H. Steinhaus. The Problem of Fair Division. *Econometrica*, 16:101–104, 1948.

The Even-Paz Divide-and-Conquer Procedure

Even and Paz (1984) investigated *upper bounds* for the number of *queries* (cuts or marks) required for n *agents*.

They introduced the following *divide-and-conquer* protocol:

(1) Ask each agent to cut the cake at her $\lfloor \frac{n}{2} \rfloor : \lceil \frac{n}{2} \rceil$ mark.

(2) Associate the union of the leftmost $\lfloor \frac{n}{2} \rfloor$ pieces with the agents who made the leftmost $\lfloor \frac{n}{2} \rfloor$ cuts, and the rest with the others.

(3) Recursively apply the same procedure to each of the two groups, until only a single agent is left. ✓

Each agent is guaranteed a *proportional* piece. Takes $O(n \log n)$ steps.

S. Even and A. Paz. A Note on Cake Cutting. *Discrete Applied Mathematics*, 7:285–296, 1984.

Envy-Free Procedures

Achieving *envy-freeness* is much harder than achieving proportionality:

- For $n = 2$ the problem is easy: cut-and-choose does the job.
- For $n = 3$ we will see two solutions. They are already quite complicated: either the number of cuts is *not minimal* (but > 2), or *several simultaneously moving knives* are required.
- For $n = 4$, to date, no procedure producing *contiguous pieces* is known. Barbanel and Brams (2004), for example, give a moving-knife procedure requiring up to 5 cuts.
- For $n \geqslant 6$, to date, only procedures requiring an *unbounded* number of cuts are known (see e.g. Brams and Taylor, 1995).

J.B. Barbanel and S.J. Brams. Cake Division with Minimal Cuts. *Mathematical Social Sciences*, 48(3):251–269, 2004.

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

The Selfridge-Conway Procedure

The first discrete protocol achieving envy-freeness for $n = 3$ has been discovered independently by Selfridge and Conway (around 1960). Our exposition follows Brams and Taylor (1995).

(1) Player 1 cuts the cake in three pieces (she considers equal).

(2) Player 2 either "passes" (if she thinks at least two pieces are tied for largest) or trims one piece (to get two tied for largest pieces). — If she passed, then let players 3, 2, 1 pick (in that order). ✓

(3) If player 2 did trim, then let 3, 2, 1 pick (in that order), but require 2 to take the trimmed piece (unless 3 did). Keep the trimmings unallocated for now (note: the partial allocation is envy-free).

(4) Now divide the trimmings. Whoever of 2 and 3 received the *un*trimmed piece does the cutting. Let players choose in this order: non-cutter, player 1, cutter. ✓

S.J. Brams and A.D. Taylor. An Envy-free Cake Division Protocol. *American Mathematical Monthly*, 102(1):9–18, 1995.

The Stromquist Procedure

Stromquist (1980) found an envy-free procedure for $n = 3$ producing *contiguous* pieces, though requiring four simultaneously *moving knifes:*

- A referee slowly moves a knife across the cake, from left to right (supposed to cut somewhere around the $1/3$ mark).
- At the same time, each agent is moving her own knife so that it would cut the righthand piece in half (wrt. her own valuation).
- The first agent to call "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three agent knifes. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which her knife is pointing. ✓

W. Stromquist. How to Cut a Cake Fairly. *American Mathematical Monthly*, 87(8):640–644, 1980.

Summary

We have briefly touched upon three topics in fair division:

- Criteria for assessing fairness of an allocation of resources (or more generally: utility vectors)
- Combinatorial optimisation and negotiation for the fair allocation of indivisible goods
- Cake-cutting algorithms (single divisible good)

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