Some New Results on *Adjusted Winner*

Eric Pacuit

September 16, 2005

ILLC, University of Amsterdam
staff.science.uva.nl/∼epacuit
epacuit@science.uva.nl

Joint work with Rohit Parikh and Samer Salame, CUNY

Introduction

Adjusted winner (AW) is an algorithm for dividing n divisible goods among two people (invented by Steven Brams and Alan Taylor).

For more information see

- *Fair Division: From cake-cutting to dispute resolution* by Brams and Taylor
- *The Win-Win Solution* by Brams and Taylor
- www.nyu.edu/projects/adjustedwinner

Introduction: Example

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 1. Both Ann and Bob divide 100 points among the three goods.

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 1. Both Ann and Bob divide 100 points among the three goods.

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 2. The agent who assigns the most points receives the item.

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 2. The agent who assigns the most points receives the item.

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 2. The agent who assigns the most points receives the item.

Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Notice that $65/46 \geq 5/4 \geq 1 \geq 30/50$

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Give A to Bob (the item whose ratio is closest to 1)

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Give A to Bob (the item whose ratio is closest to 1)

Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Give A to Bob (the item whose ratio is closest to 1)

Item	Ann	Bob
A	0	4
B	65	0
C	0	50
Total	65	54

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Still not equal, so give (some of) B to Bob: $65p = 100 - 46p$.

Item	Ann	Bob
A	0	4
B	65	0
C	0	50
Total	65	54

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

yielding $p = 100/111 = 0.9009$

Item	Ann	Bob
A	0	4
B	65	0
C	0	50
Total	65	54

Introduction: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

yielding $p = 100/111 = 0.9009$

Item	Ann	Bob
A	0	4
B	58.559	4.559
C	0	50
Total	58.559	58.559

Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

A **valuation** of these goods is a vector of natural numbers $\langle a_1, \ldots, a_n \rangle$ whose sum is 100.

Let $\alpha, \alpha', \alpha'', \ldots$ denote possible valuations for Ann and $\beta, \beta', \beta'', \ldots$ denote possible valuations for Bob.

Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

An **allocation** is a vector of n real numbers where each component is between 0 and 1 (inclusive). An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is interpreted as follows.

For each $i = 1, \ldots, n$, s_i is the proportion of G_i given to Ann.

Thus if there are three goods, then $\langle 1, 0.5, 0 \rangle$ means, "Give all of item 1 and half of item 2 to Ann and all of item 3 and half of item 2 to Bob."

Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

$V_A(\alpha, \sigma) = \sum_{i=1}^{n} a_i s_i$ is the total number of points that Ann receives.

$V_B(\beta, \sigma) = \sum_{i=1}^{n} b_i (1 - s_i)$ is the total number of points that Bob receives.

Thus AW can be viewed as a function from pairs of valuations to allocations: $\mathsf{AW}(\alpha, \beta) = \sigma$ if σ is the allocation produced by the AW algorithm.

Adjusted Winner: Formal Definition

$\mathsf{AW}(\alpha, \beta)$ is defined as follows:

Adjusted Winner: Formal Definition

AW(α, β) is defined as follows:

Rename the goods so that

$$a_1/b_1 \geq a_2/b_2 \geq \cdots a_r/b_r \geq 1 > a_{r+1}/b_{r+1} \geq \cdots a_n/b_n$$

Adjusted Winner: Formal Definition

AW(α, β) is defined as follows:

Rename the goods so that

$$a_1/b_1 \geq a_2/b_2 \geq \cdots a_r/b_r \geq 1 > a_{r+1}/b_{r+1} \geq \cdots a_n/b_n$$

1. Give all the goods $G_1, \ldots, G_r$ to Ann and $G_{r+1}, \ldots, G_n$ to Bob. Let X, Y be the number of points received by Ann and Bob respectively. Assume for simplicity that $X \geq Y$.

Adjusted Winner: Formal Definition

AW(α, β) is defined as follows:

Rename the goods so that

$$a_1/b_1 \geq a_2/b_2 \geq \cdots a_r/b_r \geq 1 > a_{r+1}/b_{r+1} \geq \cdots a_n/b_n$$

1. Give all the goods $G_1, \ldots, G_r$ to Ann and $G_{r+1}, \ldots, G_n$ to Bob. Let X, Y be the number of points received by Ann and Bob respectively. Assume for simplicity that $X \geq Y$.

2. If $X = Y$, then stop. Otherwise, transfer a portion of G_r from Ann to Bob which makes $X = Y$. If equitability is not achieved even with all of G_r going to Bob, transfer $G_{r-1}, G_{r-2}, \ldots, G_1$ in that order to Bob until equitability is achieved.

Fairness Conditions

- **Proportional** if both Ann and Bob receive at least 50% of their valuation. That is, $\sum_{i=1}^{n} s_i a_i \geq 50$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq 50$
- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation. That is, $\sum_{i=1}^{n} s_1 a_i \geq \sum_{i=1}^{n}(1 - s_i)a_i$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq \sum_{i=1}^{n} s_i b_i$.
- **Equitable** if both players receive the same total number of points. That is $\sum_{i=1}^{n} s_i a_i = \sum_{i=1}^{n}(1 - s_i)b_i$
- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party. That is for each allocation $\sigma' = \langle s'_1, \ldots, s'_n \rangle$ if $\sum_{i=1}^{n} a_i s'_i > \sum_{i=1}^{n} a_i s_i$, then $\sum_{i=1}^{n}(1 - s'_i)b_i < \sum_{i=1}^{n}(1 - s_i)b_i$. (Similarly for Bob).

Fairness Conditions

- **Proportional** if both Ann and Bob receive at least 50% of their valuation. That is, $\sum_{i=1}^{n} s_i a_i \geq 50$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq 50$

- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation. That is, $\sum_{i=1}^{n} s_1 a_i \geq \sum_{i=1}^{n}(1 - s_i)a_i$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq \sum_{i=1}^{n} s_i b_i$.

- **Equitable** if both players receive the same total number of points. That is $\sum_{i=1}^{n} s_i a_i = \sum_{i=1}^{n}(1 - s_i)b_i$

- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party. That is for each allocation $\sigma' = \langle s'_1, \ldots, s'_n \rangle$ if $\sum_{i=1}^{n} a_i s'_i > \sum_{i=1}^{n} a_i s_i$, then $\sum_{i=1}^{n}(1 - s'_i)b_i < \sum_{i=1}^{n}(1 - s_i)b_i$. (Similarly for Bob).

Fairness Conditions

- **Proportional** if both Ann and Bob receive at least 50% of their valuation. That is, $\sum_{i=1}^{n} s_i a_i \geq 50$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq 50$
- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation. That is, $\sum_{i=1}^{n} s_1 a_i \geq \sum_{i=1}^{n}(1 - s_i)a_i$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq \sum_{i=1}^{n} s_i b_i$.
- **Equitable** if both players receive the same total number of points. That is $\sum_{i=1}^{n} s_i a_i = \sum_{i=1}^{n}(1 - s_i)b_i$
- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party. That is for each allocation $\sigma' = \langle s'_1, \ldots, s'_n \rangle$ if $\sum_{i=1}^{n} a_i s'_i > \sum_{i=1}^{n} a_i s_i$, then $\sum_{i=1}^{n}(1 - s'_i)b_i < \sum_{i=1}^{n}(1 - s_i)b_i$. (Similarly for Bob).

Fairness Conditions

- **Proportional** if both Ann and Bob receive at least 50% of their valuation. That is, $\sum_{i=1}^{n} s_i a_i \geq 50$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq 50$
- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation. That is, $\sum_{i=1}^{n} s_1 a_i \geq \sum_{i=1}^{n}(1 - s_i)a_i$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq \sum_{i=1}^{n} s_i b_i$.
- **Equitable** if both players receive the same total number of points. That is $\sum_{i=1}^{n} s_i a_i = \sum_{i=1}^{n}(1 - s_i)b_i$
- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party. That is for each allocation $\sigma' = \langle s'_1, \ldots, s'_n \rangle$ if $\sum_{i=1}^{n} a_i s'_i > \sum_{i=1}^{n} a_i s_i$, then $\sum_{i=1}^{n}(1 - s'_i)b_i < \sum_{i=1}^{n}(1 - s_i)b_i$. (Similarly for Bob).

Easy Observations

- For two-party disputes, proportionality and envy-freeness are equivalent.

- *AW* only produces equitable allocations (equitability is essentially built in to the procedure).

- *AW* produces allocations σ that have the following property: there is at most one i such that $0 \leq \sigma_i \leq 1$ and for all $j \neq i$, $\sigma_j \in \{0, 1\}$.

***Adjusted Winner* is Fair**

Theorem (Brams and Taylor) *AW produces allocations that are efficient, equitable and envy-free (with respect to the announced valuations)*

Some Questions

- Can we extend to more than two agents?

Some Questions

- Can we extend to more than two agents?
- Can we make use of geometric intuitions?
  - Is AW a "continuous" function?

Some Questions

- Can we extend to more than two agents?

- Can we make use of geometric intuitions?
  - Is AW a "continuous" function?

- Can an agent benefit by making use of information about the other agent's valuation?

Some Questions

- Can we extend to more than two agents?
- Can we make use of geometric intuitions?
  - Is AW a "continuous" function?
- Can an agent benefit by making use of information about the other agent's valuation?
- The agents' utility functions are assumed to be linear, what about non-linear utility functions?

More Than Two Agents

This example was given by two Dutch mathematicians J. H. Reijnierse and J. A. M. Potters.

Item	Ann	Bob	Carol
X	40	30	30
Y	50	40	30
Z	10	30	40

The only efficient and equitable allocation turns out to be give X to Ann, Y to Bob, and Z to Carol. It is not envy-free.

Geometric Intuitions

We will be working in $\mathbb{R}^k$ for $k \geq 1$.

An allocation is a vector $\vec{x} \in \mathbb{R}^k$ where each component is a non-negative real less than or equal to 1.

Thus the set of all possible allocations is a hypercube in $\mathbb{R}^k$. Let $\mathbb{C}_k = \{\vec{x} \mid \forall i \ \ 0 \leq x_i \leq 1\}$ be this hypercube of dimension k.

A **valuation** is a vector $\vec{P} \in \mathbb{R}^k$ where $\sum_{i=1}^{k} P_i = 100$.

Let $\cdot$ denote the dot product, that is $\vec{x} \cdot \vec{P} = \sum_{i=1}^{k} x_i P_i$ (the total points received by Ann).

Geometric Intuitions

Let $\vec{P}$ and $\vec{Q}$ be two fixed vectors (Ann's valuation and Bob's valuation).

We are interested in the hyperplane $\mathcal{H}_{\vec{P},\vec{Q}}$ generated by the following equation

$$\vec{x} \cdot \vec{P} = (\vec{1} - \vec{x}) \cdot \vec{Q}$$

For a fixed $\vec{P}$ and $\vec{Q}$, wanting efficency, we can ask for the allocations $\vec{x}$ that maximize $\vec{x} \cdot \vec{P}$ (subject to the above constraints)

Geometric Intuitions

Let $\mathcal{I} = \mathbb{C}_k \cap \mathcal{H}_{\vec{P},\vec{Q}}$. Define the function $f : \mathcal{I} \to \mathbb{R}$ by $f(\vec{x}) = \vec{x} \cdot \vec{P}$.

It is not hard to see that f has a maximum value on $\mathcal{I} = \mathbb{C}_k \cap \mathcal{H}_{\vec{P},\vec{Q}}$ (Heine-Borel).

Let $\mathcal{M} = \{\vec{x} \mid f(\vec{x}) = m\} \neq \emptyset$.

Theorem *There is a point of $\mathcal{M}$ which lies on an edge of the hypercube.*

Thus, the fact that AW produces allocations in which all components, except possibly one, are either 1 or 0 is no accident.

Is AW Continuous?

Is AW Continuous?

Yes and No (depends what is meant by continuous).

Is AW Continuous?

Yes and No (depends what is meant by continuous).

Yes: Small fluctuations in the agents' valuations produces small fluctuations in the total number of points the agent receives.

Is AW Continuous?

Yes and No (depends what is meant by continuous).

Yes: Small fluctuations in the agents' valuations produces small fluctuations in the total number of points the agent receives.

No: Small fluctuations in the agents' valuations can drastically change the set of items assigned to an agent by AW.

Is AW Continuous?

Yes and No (depends what is meant by continuous).

Yes: Small fluctuations in the agents' valuations produces small fluctuations in the total number of points the agent receives.

No: Small fluctuations in the agents' valuations can drastically change the set of items assigned to an agent by AW.

Item	Ann	Bob
G_1	$25+\varepsilon/2$	$25-\varepsilon/2$
G_2	$25+\varepsilon/2$	$25-\varepsilon/2$
G_3	$25-\varepsilon/2$	$25+\varepsilon/2$
G_4	$25-\varepsilon/2$	$25+\varepsilon/2$

Item	Ann	Bob
G_1	$25-\varepsilon/2$	$25+\varepsilon/2$
G_2	$25-\varepsilon/2$	$25+\varepsilon/2$
G_3	$25+\varepsilon/2$	$25-\varepsilon/2$
G_4	$25+\varepsilon/2$	$25-\varepsilon/2$

Strategizing

Can the agents improve their allocation by misrepresenting their preferences?

Strategizing

Can the agents improve their allocation by misrepresenting their preferences?

Yes

Strategizing

Can the agents improve their allocation by misrepresenting their preferences?

Yes

However, while honesty may not always be the best policy it is the only **safe** *one, i.e., the only one which will guarantee 50%.*

See *Safe votes, sincere votes, and strategizing* [PP] for more on the notion of a “safe” strategy.

Strategizing

Item	Ann	Bob
Matisse	75	25
Picasso	25	75

Ann will get the Matisse and Bob will get the Picasso and each gets 75 of his or her points.

Strategizing: Example

Suppose Ann knows Bob's preferences, but Bob does not know Ann's.

Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	25
P	74	75

So Ann will get M plus a portion of P.

According to Ann's announced allocation, she receives 50 points

According to Ann's actual allocation, she receives $75 + 0.33 * 25 = 83.33$ points.

Strategizing: A Theorem

Theorem (Brams and Taylor) *Assume there are two goods,* G_1 *and* G_2*, all true and announced values are restricted to integers, and suppose Bob's announced valuation of* G_1 *is* x*, where* $x \geq 50$*. Assume Ann's true valuation of* G_1 *is* b*. Then her optimal announced valuation of* G_1 *is:*

$$\begin{cases} x+1 & \text{if } b > x \\ x & \text{if } b = x \\ x-1 & \text{if } b < x \end{cases}$$

Strategizing: Example

Suppose *both* players know each other's preferences but neither knows that the other knows their own preference.

Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	74
P	74	26

Each will get 74 of his or her announced points, but each one is really getting only 25 of his or her *true* points.

Strategizing: Example

Suppose *both* players know each other's preferences. Moreover, Ann knows that Bob knows her preference and Bob doesn't know that Ann knows.

Item	Ann	Bob
M	26	74
P	74	26

Item	Ann	Bob
M	73	74
P	27	26

What happens as the level of knowledge increases?

Non-linear Utility Functions

It is assumed that both Ann and Bob have linear utility functions.

If the agents have, say concave or convex, utility functions can be make use of this information and produce an outcome which is better for both parties than the one produced by AW?

Non-linear Utility Functions

Two functions u_1 and u_2 are called **complementary benefit functions** if

1. u_1 and u_2 are continuous
2. u_1 is monotonic ($x < y$ implies $u_1(x) < u_1(y)$)
3. u_2 is anti-monotonic ($x < y$ implies $u_2(y) < u_2(x)$)
4. $u_1(0) = u_2(1)$ and $u_1(1) = u_2(0)$

Non-linear Utility Functions

Two functions u_1 and u_2 are called **complementary benefit functions** if

1. u_1 and u_2 are continuous
2. u_1 is monotonic ($x < y$ implies $u_1(x) < u_1(y)$)
3. u_2 is anti-monotonic ($x < y$ implies $u_2(y) < u_2(x)$)
4. $u_1(0) = u_2(1)$ and $u_1(1) = u_2(0)$

Lemma *Suppose that u_1 and u_2 are complementary benefit functions. Then if u_1 and u_2 are concave, there exists a unique point x_0 such that $u_1(x_0) = u_2(x_0)$ and $u_1(x_0) \geq (u_1(0) + u_1(1))/2$ ($u_2(x_0) \geq (u_1(0) + u_1(1))/2$).*

Non-linear Utility Functions

So, with one good we can find a better (equitable, efficient and envy-free) outcome.

Conjecture: *Let AW^+ be the procedures which accepts two complementary utility functions and finds a point where they are equal and maximal. If the utility functions are concave, then AW^+ will yield better allocations than AW.*

Conclusions

- AW fairly divides n goods among two agents. The procedure is envy-free, efficient and equitable.

- Although strategizing is a possibility, it is not a safe strategy.

- Extending to more than two people requires dropping one of the fairness properties. Thinking of AW as a function from pairs of vectors in $\mathbb{R}^k$ to $\mathbb{R}$ may help us understand the multi-agent case.

- If we know the agents' utility functions are non-linear, we may be able to find a more efficient outcome.

Thank you.