Tutorial on Fairness and Uncertainty
TFG-MARA in Budapest

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CNRS-EUREQua

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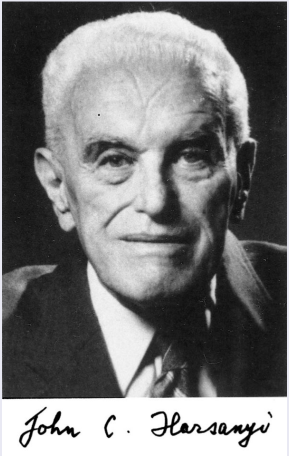
Once upon a time in Budapest...

Computer Science & Decision Theory

John von Neumann

Once upon a time in Budapest...

Decision Theory & Ethics



John Harsanyi

Decision Theory & Ethics

Decision Theory

normative theory, that tries to figure out what a rational behavior (i.e., a goal-directed and consistent behavior) should be.

Social Choice

normative theory, that tries to figure out what a moral behavior should be.

Indeed, most philosophers also regard *moral behavior* as a special form of rational behavior. If we accept this view (as I think we should) then the theory of morality, i.e, moral philosophy or ethics, becomes another normative discipline dealing with rational behavior.

J. Harsanyi

Uncertainty & Ethics

- Problem: allocating an indivisible item between two persons
- Conventional wisdom : let a fair coin decide who will get the good.

Uncertainty plays a fundamental role in our intuitive perception of fairness.

Uncertainty as Fairness

Uncertainty & Ethics

- most of the Social Choice literature : what is actually relevant in collective decisions is individuals' preferences
- Social Choice: attempt of conciliate individuals' preferences into a collective one.

Most of real alternatives involve Risk or Uncertainty.

FAIRNESS UNDER UNCERTAINTY

Road Map

1. Uncertainty and Fairness: Objects
2. Uncertainty as Fairness
3. Fairness under Uncertainty

Part I

Uncertainty and Fairness: Objects

Lotteries are Income Distributions

Lotteries

- $\mathcal{X}$: outcome space (e.g. $\mathcal{X} = \mathbb{R}$)
- $L : \mathcal{X} \to [0,1]$: lottery ($\mathcal{L}$: set of lotteries)
- $L(x) = p$: you get $x \in \mathcal{X}$ with probability p

Income Distribution

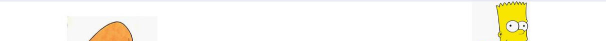
- $\mathcal{Y}$: incomes (e.g. $\mathcal{Y} = \mathbb{R}$)
- $X : \mathcal{Y} \to [0,1]$: income distribution
- $X(y) = p$: a fraction p of the population gets income y

LOTTERY = INCOME DISTRIBUTION

Hidden Assumptions

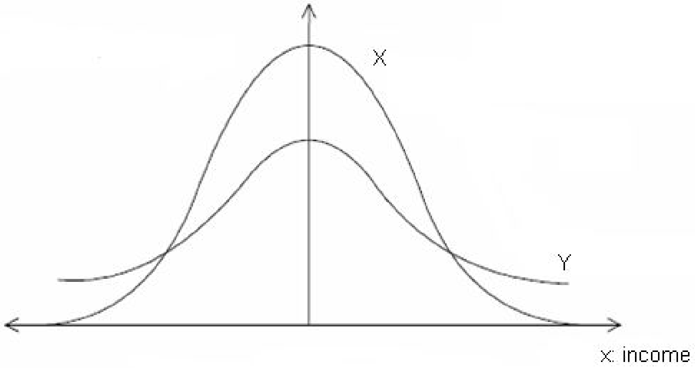
Anonymity

Population Principle



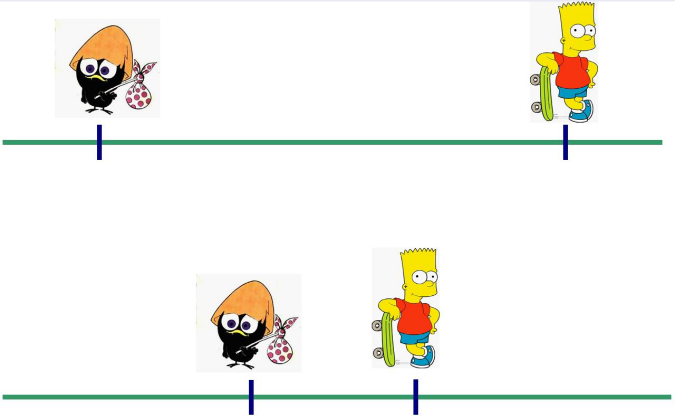
Risk and Inequality

Risk: Mean Preserving Spread



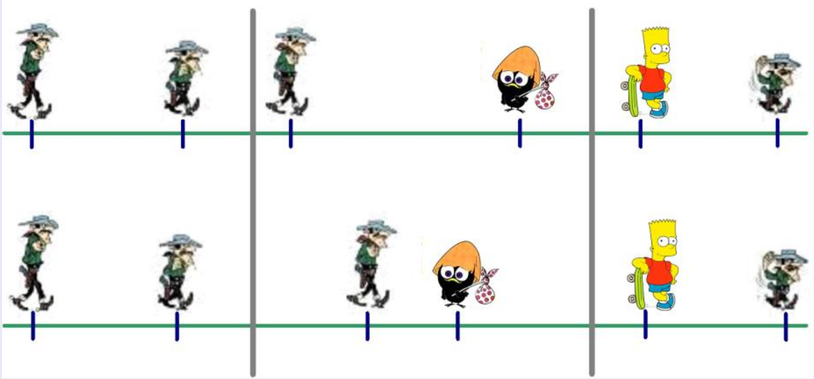
Risk and Inequality

Inequality: Pigou Transfer Principle



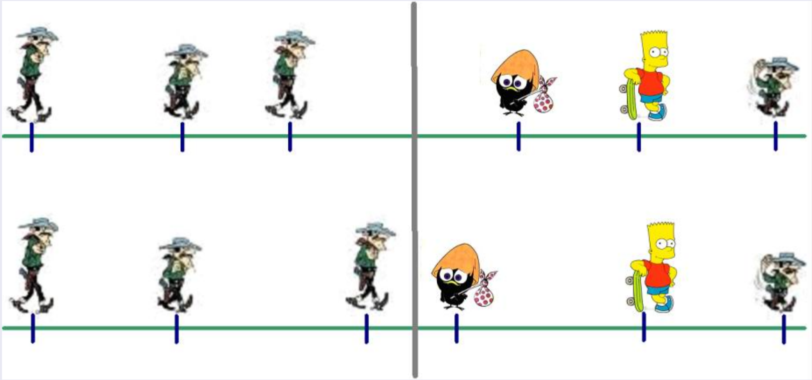
Risk and Inequality

Inequality: Pigou-Dalton Transfer Principle



Risk and Inequality

Inequality: Pigou-Dalton Transfer Principle



Risk and inequality aversion

Risk aversion

A decision maker is *risk averse* if $X \succeq Y$ whenever Y is obtained from X by a sequence of Mean Preserving Spreads.

Inequality aversion

A society is *inequality averse* if $X \succeq Y$ whenever X is obtained from Y by a sequence of Pigou-Dalton transfers

The connection

Y is obtained from X by a sequence of Mean Preserving spreads iff X is obtained from Y by a sequence of Pigou-Dalton Transfers

RISK AVERSION = INEQUALITY AVERSION

Expected Utility

Axiom (Order)

$\succeq$ *is a complete, continuous, transitive, binary relation on* $\mathcal{L}$.

Axiom (Independence)

For all $L_1, L_2, L_3 \in \mathcal{L}$, *all* $\alpha \in (0,1)$,

$$L_1 \succeq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$$

Theorem (von Neumann - Morgenstern)

$\succeq$ *satisfies Axioms [Order] and [Independence] iff there exists a* $u : \mathcal{X} \to \mathbb{R}$ *such that* $(x_1, p_1; \cdots, x_n, p_n) \succeq (x_1', p_1'; \cdots; x_n', p_n')$ *iff:*

$$\sum_i p_i u(x_i) \geq \sum_i p_i' u(x_i').$$

Preferences on Income Distributions

Mixture of income distributions

- Four countries: A, B, C and D.
- A and B : same size (n), income distributions X and Y
- C and D : same size (m), income distribution Z
- Merging A and C : $\frac{n}{n+m}X + (1 - \frac{n}{n+m})Z$
- Merging B and D : $\frac{n}{n+m}Y + (1 - \frac{n}{n+m})Z$

Independence for Income Distributions

If you prefer society A to society B, you also prefer society (A, C) to society (B, D)

Extend vNM Theorem to income distributions

Preferences on Income Distributions

Axiom (homogeneity)

The ranking of two income distributions is not affected if all incomes are multiplied by the same strictly positive factor

Inequality averse social evaluation function

$\succeq$ satisfies axioms [Order], [Independence], [Homogeneity] and is inequality averse iff it can be represented by:

$$\begin{cases} W(X) = \sum_i p_i \frac{x_i^{1-\sigma}}{1-\sigma}, \ \sigma \neq 1 \\ W(X) = \sum_i p_i \ln(x_i) \end{cases}$$

Furthermore, the degree of inequality aversion increase with σ.

USED TO BUILD INEQUALITY INDICES

Inequality and Risk: Conclusion

- formal analogy between lotteries and income distributions
- formal analogy between risk and inequality aversion
- Decision under risk can be used to perform inequality analysis
- Many results are available
- e.g.: the well known Gini index corresponds to the Rank Dependent Expected Utility model

Uncertainty

Savage Acts

- S : state space
- $\mathcal{X}$: set of consequences
- $f : S \to \mathcal{X}$: act

- Lottery: known probabilities $=$ RISK
- Savage Acts : probabilities are unknown $=$ UNCERTAINTY

Problem

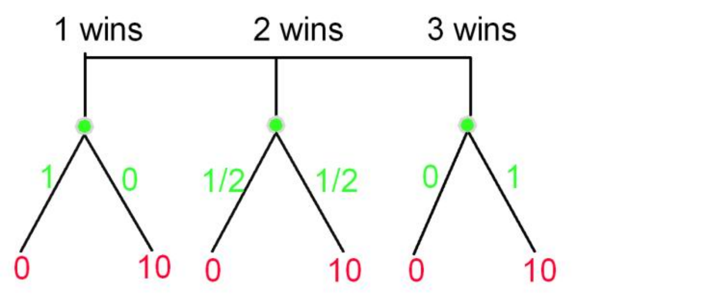
- The set of Savage acts has almost no structure
- In particular: it's not a mixture space

Anscombe-Aumann acts

Definition

- $\mathbb{X}$ set of consequences
- $\mathbb{Y}$ set of distributions over $\mathbb{X}$ (roulette lottery)
- Act: $f : S \to \mathbb{Y}$ (set of AA acts : $\mathcal{A}$) (horse lottery)

Example



Uncertain Income Distributions

Example

	1	2	3
s_1	10	0	5
s_2	20	100	20

$$\begin{cases} f(1) = (0, \frac{1}{3}; 5, \frac{1}{3}; 10, \frac{1}{3}) \\ f(2) = (20, \frac{2}{3}; 100, \frac{1}{3}) \end{cases}$$

UNCERTAIN INCOME DISTRIBUTIONS = ANSCOMBE-AUMANN ACTS

Subjective Expected Utility

Theorem (Anscombe-Aumann's Theorem)

Axioms [Order], [Continuity], [Independence], [Monotonicity], and [Non-degeneracy] hold iff $\succeq$ can be represented by:

$$V(f) = \sum_s p_s u(f(s)),$$

where $p \in \Delta(S)$ is unique, and $u : \mathbb{Y} \to \mathbb{R}$, is a linear function, unique up to a positive affine transformation.

Evaluating uncertain income distributions with SEU?

f	a	b
s	1	0
t	0	1

g	a	b
s	1	0
t	1	0

- $V(f) = p_s(\frac{1}{2} \times 1 + \frac{1}{2} \times 0) + p_t(\frac{1}{2} \times 0 + \frac{1}{2} \times 1) = \frac{1}{2}p_s + \frac{1}{2}p_t$
- $V(g) = p_s(\frac{1}{2} \times 1 + \frac{1}{2} \times 0) + p_t(\frac{1}{2} \times 1 + \frac{1}{2} \times 0) = \frac{1}{2}p_s + \frac{1}{2}p_t$
- $\Rightarrow f \sim g$
- f and g are indeed equivalent ex post
- But ex ante, f seems more equal than g...

Key issue

EX ANTE AND EX POST EGALITARIANISM: DIAMOND'S CRITICS

Two steps aggregation

f	a	b
s	1	0
t	0	1

g	a	b
s	1	0
t	1	0

h	a	b
s	0	0
t	1	1

- "natural order": $h \succ f \succ g$
- f and g are equivalent *ex post*
- f and h are equivalent *ex ante*
- $\Rightarrow$ two steps aggregation cannot generate $h \succ f \succ g$

Solution?

$$\begin{array}{c|cc} f & a & b \\ \hline s & \alpha & \beta \\ t & \gamma & \delta \end{array} \rightarrow \left(I_a \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, I_a \begin{pmatrix} \beta \\ \delta \end{pmatrix} \right) \rightarrow I_p \left(I_a \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, I_a \begin{pmatrix} \beta \\ \delta \end{pmatrix} \right)$$

$$\begin{array}{c|cc} f & a & b \\ \hline s & \alpha & \beta \\ t & \gamma & \delta \end{array} \rightarrow \begin{pmatrix} I_p(\alpha, \beta) \\ I_p(\gamma, \delta) \end{pmatrix} \rightarrow I_a \begin{pmatrix} I_p(\alpha, \beta) \\ I_p(\gamma, \delta) \end{pmatrix}$$

$$\rightarrow \Psi \left(I_p \left(I_a \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, I_a \begin{pmatrix} \beta \\ \delta \end{pmatrix} \right), I_a \begin{pmatrix} I_p(\alpha, \beta) \\ I_p(\gamma, \delta) \end{pmatrix} \right)$$

Can be generalized and axiomatized, using decision theoretic techniques

Conclusion

SOCIAL CHOICE IS JUST DECISION THEORY

Part II

Uncertainty as Fairness

Overview

From Impartiality to ignorance

- Principle of justice are those a rational decision maker would chose under appropriate conditions of impartiality
- A decision is impartial if the decision maker is in a situation of complete ignorance of what his own position, and the position of those near to his heart, would be within the system chosen." (Harsanyi)

IMPARTIALITY VIEWED AS IGNORANCE

Harsanyi and Rawls

- they agree on IMPARTIALITY=IGNORANCE
- they *disagree* on what "ignorance" means...

Harsanyi & Rawls

What "ignorance" means

- Harsanyi : ignorance = equal probability of being any individual = Impartial Observer
- Rawls : ignorance = no information at all about who you'll be = Veil of Ignorance

Consequences

- Harsanyi: Utilitarianism. $W = \sum_i U_i$
- Rawls : MaxMin. $W = \min_i U_i$

Setup

Individuals

- $N = \{1, \cdots, n\}$: society
- $\succeq_i$: individual i's preferences, over lotteries $\mathbb{Y}$ (complete description of the society)
- Assumption: $\succeq_i$ are of vNM type

Extended preferences

- Observer should be able to say: "I prefer being Mr. i and getting x_i than being Mr. j and getting x_j
- Preferences on extended lotteries
- Formally: preferences on $\mathcal{E} = \Delta(\mathbb{Y} \times N)$

Extended Lotteries

Extended Lotteries

- $\rho : \mathbb{Y} \times N \to [0, 1]$
- $\rho(x, i) =$ probability of being in i's shoes, and getting x

Personal identity lottery & Allocations

- $p \in \Delta(N) =$ personal identity lottery
- $f : N \to \mathbb{Y} \in \mathcal{A} =$ allocation
- One may identify ρ and some (f, p)

Extended Lotteries

Extended lottery

	1	2	3
a	$3/8$	$1/12$	$1/8$
b	$1/4$	$1/12$	$1/12$

Associated Personal Identity Lottery

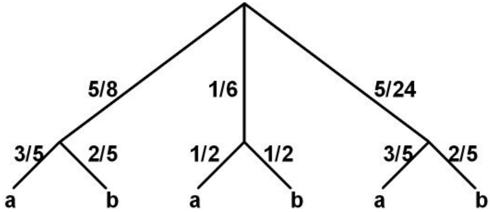
	1	2	2
$p(\rho(i))$	$5/8$	$1/6$	$5/24$

Associated Allocation

	1	2	3
a	$3/5$	$1/2$	$3/5$
b	$2/5$	$1/2$	$2/5$

Extended Lotteries

	1	2	3
a	$3/5$	$1/2$	$3/5$
b	$2/5$	$1/2$	$2/5$
$p(\rho(i))$	$5/8$	$1/6$	$5/24$



The Impartial Observer Theorem

Assumptions

- $\succeq_i$ on $\mathbb{Y}$ of vNM type
- $\succeq$ on $\mathcal{E}$ of vNM type
- $(f, \delta_i) \succeq (g, \delta_i) \Leftrightarrow f \succeq_i g$ (Acceptance Principle)
- Equal Chance : $\forall y, z \in \mathbb{Y}$, $y \succeq^* z \Leftrightarrow (k_y, \mu) \succeq (k_z, \mu)$

Result

Under these assumptions,

$$y \succeq^* z \Leftrightarrow \sum_i \frac{1}{n} V_i(y) \geq \sum_i \frac{1}{n} V_i(z)$$

where V_i are vNM representations of individuals' preferences

Critics of Harsanyi's theorem

Diamond's critics

- The Independence assumption is unacceptable for the social preferences (because of *ex ante* inequality)
- Impartial Observer without Independence: Epstein & Segal

Rawls' critics

- Ignorance shouldn't be reduced to equiprobability
- Only fact-based (direct or indirect evidence) probabilities are allowed
- There is no such information under the Veil of Ignorance
- The bayesian model is irrelevant
- A rational model should be of MaxMin type

Revisiting Rawls-Harsanyi debate

Questions

- Harsanyi's claim against Rawls: Utilitarianism follows from epistemic axioms
- Rawls: epistemic arguments should lead to MaxMin
- Difficult to say, since Rawls doesn't provide any formal model

Aim

- Build a model that can accommodate both Rawls' and Harsanyi's views
- Discuss on the epistemic foundation of Utilitarianism and MaxMin

Modeling Ignorance

Revisiting extended lotteries

- Extended lottery (f, p): the personal identity lottery is known
- Rawls: this is not true. Replace p by, say, $\Delta(N)$
- More generally: $\mathbb{P}$ = set of closed subsets of $\Delta(N)$
- $(f, \mathcal{P})$: you just know that p belongs to $\mathcal{P} \in \mathbb{P}$

Comments

- In decision theory: Jaffray (1989) takes $\mathbb{P}$ = set of cores of beliefs. Not compatible with EU.
- Recent models (in particular GTV) consider the general case
- Problem: these models are state-independent, and would force all individuals' preferences to be identical
- One should modify a bit these models...

The Observer's preferences: Main Theorem

Main Theorem

A reasonable set of ▸ Axioms hold iff $\succeq$ can be represented by

$$V(f,\mathcal{P}) = \min_{p \in \mathcal{F}(\mathcal{P})} \sum_i p(i) V_i(f(i))$$

1. V_i are affine functions representing $\hat{\succeq}_i$
2. $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$

1. $\mathcal{F}(\mathcal{P}) \subseteq co(\mathcal{P})$
2. $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) = \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$

- $\mathcal{F}$ is unique
- The V_i are unique up to common positive affine trans.

A more precise representation

Theorem

Under additional ▸ Axioms*, the restriction of* $\succeq$ *to* $\mathcal{A} \times \mathbb{B}$ *can be represented by:*

$$V(f, \mathcal{P}) = \theta \min_{p \in \mathcal{P}} \sum_i p(i) V_i(f(i)) + (1 - \theta) \sum_i c_{\mathcal{P}}(i) V_i(f(i))$$

where $c_{\mathcal{P}}$ *is the Shapley value of* $\mathcal{P}$*. Furthermore* θ *is unique and the* V_i *are* $\propto$ *unique.*

The Ignorant Observer Model, Rawls and Harsanyi

Acceptance Principe

$$(f, \{\delta_i\}) \succeq (g, \{\delta_i\}) \Leftrightarrow f(i) \succeq_i g(i)$$

Ignorant Observer Theorem

$$W(f, \mathcal{P}) = \min_{p \in \mathcal{F}(\mathcal{P})} \sum_i p(i) V_i(f(i)),$$

where V_i are vNM representations of $\succeq_i$

Ignorant Observer Theorem: special case

$$W(f, \mathcal{P}) = \theta \min_{p \in \mathcal{F}(\mathcal{P})} p_i V_i(f_i) + (1-\theta) \sum_i c_{\mathcal{P}}(i) V_i(f(i)),$$

where V_i are vNM representations of $\succeq_i$

Complete Ignorance

Epistemic foundations of Rawls' and Harsanyi's rules

The problem

- We found a plurality of rules
- Harsanyi's Utilitarianism and Rawls' maxmin are particular cases
- can any of these rules be justified on an epistemic basis?

In search for an epistemic justification

Axiom (Neutrality towards Uncertainty)

$$(f, \mathcal{P}) \sim (g, \mathcal{P}) \Rightarrow (\alpha f + (1-\alpha) g, \mathcal{P}) \sim (f, \mathcal{P})$$

- Neutrality towards uncertainty $\Leftrightarrow$ utilitarianism
- In contradiction with ▸ Ellsberg Paradox

Axiom (Extreme Aversion towards Uncertainty)

$$\forall p \in \mathcal{P},\ (f, \{p\}) \succeq (f, \mathcal{P})$$

- Extreme aversion towards uncertainty $\Leftrightarrow$ Rawls' rule
- Very unlikely

Conclusion

Maybe, after all, social choice is a bit more than just decision theory...

Part III

Fairness under Uncertainty

The Aggregation Problem

Aggregating n preferences into one that:

1. satisfies the same "rationality" requirements as individuals' preferences
2. is non dictatorial
3. does not provoke unanimous opposition

Harsanyi's Theorem

Assumptions

- $N' = \{1, \cdots, n\}$ agents, $N = \{0\} \cup N'$ where $0 =$ "society"
- $\mathbb{Y} =$ set of alternatives (lotteries)
- All agents and the society are expected utility maximizers
- Agents preferences are Independent
- $y \succeq_i z, \forall i \in N' \Rightarrow y \succeq_0 z$ (Pareto)

Result

There exit unique weights $\lambda_i \geq 0$, and a unique number μ, such that:

$$V_0 = \sum_i \lambda_i V_i + \mu$$

Subjective Expected Utility: Bad News

Assumption

- $\mathcal{A}$ = alternatives (Anscombe-Aumann acts)
- Individuals and Society are SEU
- Not necessarily agreement on probabilities anymore
- Preferences are independent

Result

- If all individuals and the society have the same priors: back to Harsanyi's Theorem
- Otherwise: impossibility result

Subjective Expected Utility: Good News?

Assumption

Individuals and Society are SEU, with state dependent preferences

Result

- Harsanyi's Theorem again
- But this is trivial (re-normalization of utilities: Mongin)
- Fixing priors $\Rightarrow$ Impossibility again

Subjective Expected Utility: Good News?

Gilboa, Samet & Schmeidler's Assumption

- Individuals and Society are SEU, with state dependent preferences
- Pareto restricted to cases where individuals agree on probabilities

Result

Linear aggregation of beliefs and tastes (separated):

- $u_0 = \sum_i \lambda_i u_i$
- $p_0 = \sum_i \theta_i p_i$

An Example

2 individuals MMEU with $\mathcal{P}_1 = \mathcal{P}_2 = \Delta$

	E	E^c
f	$(0, 0)$	$(0, 0)$
g	$(1, 0)$	$(0, 1)$

$V_0 = \lambda V_1 + (1 - \lambda) V_2$

$V_1(f) = 0$ $\quad V_2(f) = 0$ $\quad V_0(f) = 0$

$V_1(g) = 0$ $\quad V_2(g) = 0$ $\quad V_0(g) = 0$

$$f \sim_0 g$$

$u_0(f(E)) = 0$ $\quad u_0(f(E^c)) = 0$

$u_0(g(E)) = \lambda$ $\quad u_0(g(E^c)) = 1 - \lambda$

A General Impossibility Theorem

Theorem

If:

- $\succeq_i$ $(i \in N)$ are complete, transitive, continuous and regular
- $\succeq_i$ $(i \in N')$ are independent

then Pareto holds iff

1. there exist $\mathcal{A}^c-$affine representations of $\succeq_i$ (V_i), $(\lambda_1, \cdots, \lambda_n) > 0$, $\mu \in \mathbb{R}$ (unique) s.t.:

$$V_0(f) = \sum_i \lambda_i V_i(f) + \mu, \; \forall f \in \mathcal{A}$$

2. $\lambda_i \lambda_j \neq 0$ iff i and j are neutral towards uncertainty ...

Interpretation

In words...

- Either social preferences are a linear aggregation of uncertainty neutral individual preferences;
- Or there is a dictator.

Consequences:

1. If social preferences are not neutral towards uncertainty, then there is a dictator;
2. It is in some sense stronger than Harsanyi's Theorem, since neutrality towards uncertainty is a consequence, not an assumption.

Conclusion: Individual and Collective Rationality

Restoring the possibility

- Relaxing Pareto?
  - GSS proposal would not work
  - Paternalism?
- Relaxing the "rationality" requirement at the collective level.

What "Collective Rationality" means?

- Buchanan critics: "Who" are we talking about?
- Monotonicity: with respect to what?
  - Individuals' utilities (V_i)
  - Outcome ($f(s)$)

Towards a theory of group decision making?

Introduction

"The timing-effect is often an issue in moral debate, as when people argue about whether a social system should be judged with respects to its actual income distribution or with respect to its distribution of economic opportunities."

Myerson

Questions

- Definition(s) of envy-freeness under uncertainty?
- Existence of envy-free and efficient allocations?

Setup

- Two-period economy
  1. no consumption in period 1
  2. S states of nature in period 2
  3. C commodities
- H Agents SEU, with priors π_h, and concave utilities u_h
- $e(s) \in \mathbb{R}_+^C$: total endowment in state s
- $(x_1, \cdots, x_H) \in \mathbb{R}_+^{HSC}$

An allocation x is feasible if for all s, $\sum_h x_h(s) \leq e(s)$

Efficiency

Ex ante efficiency

$x \in P_a$ if there is no feasible allocation y such that:

$$\sum_s \pi_h(s) u_h(y_h(s)) \geq \sum_s \pi_h(s) u_h(x_h(s))$$

for all h, with a strict inequality for at least one h

Ex post efficiency

$x \in P_p$ if there is no feasible allocation y such that:

$$u_h(y(s)) \geq u_h(x(s))$$

for all h and all s, with a strict inequality for at least one h and s

$$P_a \subset P_p$$

Envy

Ex ante envy-freeness

$x \in E_a$ if:

$$\sum_s \pi_h(s) u_h(x_h(s)) \geq \sum_s \pi_h(s) u_h(x_k(s)), \forall h, k$$

Ex post envy-freeness

$x \in E_p$ if :

$$u_h(x_h(s)) \geq u_h(x_k(s)), \forall h, kj, s$$

$$E_p \subset E_a$$

$P_a \cap E_p$: INTERTEMPORALLY FAIR ALLOCATIONS

Individual Risk

Idea

- As a whole, society does not face any risk
- Agents have different exposure to risk

Assumptions

- No aggregate risk: $e_s = e, \forall s$
- Each agent separately bears some individual risk:
  - Interpret h to be type: N_h agents of type h
  - $\sum_h N_h = N$
  - Each individual of type h correctly believes that its probability of being in individual state s is $\pi_h(s)$
  - In fact, exactly $\pi_h(s)N_h$ agents of type h will be in state s

Result

Under Individual Risk: $P_a \cap E_p \neq \emptyset$

No aggregate risk and same beliefs

Assumptions

- No aggregate risk: $e_s = e$, $\forall s$
- All agents have same beliefs: $\pi_h = \pi_k$, $\forall h, k$

Result

If there is no aggregate risk and all agents have the same beliefs, then:

$$P_a \cap E_p \neq \emptyset$$

Open Issues

- In general, intertemporally fair allocation might exist or not...
- Beliefs seems to play a crucial role
- Conjecture: the "closer the beliefs", the closer we can approach an intertemporally fair allocation

The Observer's preferences

Axiom (Order)

$\succeq$ is a complete, continuous, and non-degenerated binary relation on $\mathcal{A} \times \mathbb{P}$

Axiom (Set-Mixture Independence)

$$\left.\begin{array}{l}(f,\mathcal{P}_1) \succeq (\succ)(g,\mathcal{Q}_1)\\ (f,\mathcal{P}_2) \succeq (g,\mathcal{Q}_2)\end{array}\right\} \Rightarrow \begin{array}{r}(f,\alpha\mathcal{P}_1 + (1-\alpha)\mathcal{P}_2)\\ \succeq (\succ)(g,\alpha\mathcal{Q}_1 + (1-\alpha)\mathcal{Q}_2)\end{array}$$

Comment

- Implies the Independence Axiom when one considers sets of information reduced to singletons
- $\Rightarrow$ vNM when information is reduced to singletons

The Observer's preferences

Constant-valued acts

$$\mathcal{A}^{cv} = \{f \in \mathcal{A} \,|\, (f, \{\delta_i\}) \sim (f, \{\delta_j\}),\ \forall i, j \in N\}$$

Axiom (Boundedness)

For all $\mathcal{P} \in \mathbb{P}$, $f \in \mathcal{A}$, there exist $\bar{f}, \underline{f} \in \mathcal{A}^{cv}$ such that:

$$(\bar{f}, \mathcal{P}) \succeq (f, \mathcal{P}) \succeq (\underline{f}, \mathcal{P})$$

Axiom ($\mathcal{A}^{cv}$-Independence)

For all $f, g \in \mathcal{A}$, $h \in \mathcal{A}^{cv}$, $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$, and $\alpha \in (0,1)$,

$$(f, \mathcal{P}) \succeq (g, \mathcal{Q}) \Leftrightarrow (\alpha f + (1-\alpha)h, \mathcal{P}) \succeq (\alpha g + (1-\alpha)h, \mathcal{Q})$$

The Observer's preferences

Axiom (Equivalence)

- $\forall h \in \mathcal{A}^{cv}, \mathcal{P}, \mathcal{Q} \in \mathbb{P}, (h, \mathcal{P}) \sim (h, \mathcal{Q})$
- $\forall f, g \in \mathcal{A}, \mathcal{P} \in \mathbb{P}, (f, \mathcal{P}) \sim (f_{S(\mathcal{P})} g, \mathcal{P})$

Axiom (Uncertainty Aversion)

$$(f, \mathcal{P}) \sim (g, \mathcal{P}) \Rightarrow (\alpha f + (1-\alpha) g, \mathcal{P}) \succeq (f, \mathcal{P})$$

Axiom (Pareto)

If for all $p \in \mathcal{P}$, $(f, \{p\}) \succeq (g, \{p\})$, *then* $(f, \mathcal{P}) \succeq (g, \mathcal{P})$

Conditional Preferences

$$f(i) \hat{\succeq}_i g(i) \Leftrightarrow (f, \{\delta_i\}) \succeq (g, \{\delta_i\})$$

Theorem

A more precise representation

Permuting utilities

For all $f \in \mathcal{A}$, and all permutation $\varphi : N \to N$:

$$\mathcal{A}(f^{\varphi}) = \left\{ g \in \mathcal{A} \,\middle|\, (g, \delta_i) \sim (f, \delta_{\varphi_{-1}}(i)) \right\}$$

Axiom (Anonymity)

For all $(f, \mathcal{P})$, all permutation $\varphi : N \to N$, and all $g \in \mathcal{A}(f^{\varphi})$, $(f, \mathcal{P}) \sim (g, \mathcal{P}^{\varphi})$

Axiom (Mixture Neutrality Under Same Worst Case)

If there exists $p^* \in \mathcal{P}$ such that $(f, \{p\}) \succeq (f, \{p^*\})$ and $(g, \{p\}) \succeq (g, \{p^*\})$ for all $p \in \mathcal{P}$, then:

$$(f, \mathcal{P}) \sim (g, \mathcal{P}) \Leftrightarrow (\alpha f + (1 - \alpha) g, \mathcal{P}) \sim (f, \mathcal{P})$$

Theorem

Ellsberg Paradox

	E	E^c
f	1	0
g	0	1
h	α	$1-\alpha$

Neutrality towards uncertainty: $f \sim g \Rightarrow \alpha f + (1-\alpha)g \sim f$

- SEU: $f \sim g \Rightarrow p(E) = p(E^c) = \frac{1}{2} \Rightarrow \alpha f + (1-\alpha)g \sim f$
  EU: uncertainty neutral
- MaxMin EU:
  $V(f) = \min_{p\in\Delta} p(E) = \min_{p\in\Delta} p(E^c) = V(g) = 0$
  $V(\alpha f + (1-\alpha)g) = \min_{p\in\Delta} [\alpha p(E) + (1-\alpha)p(E^c)] = \min\{\alpha, 1-\alpha\} > 0$
  MaxMin EU: uncertainty aversion

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Independent Preferences

Definition

$\{\succeq_i\}$ are independent if for all $i \in N'$, there exist $\bar{y}_i, \underline{y}_i \in \mathbb{Y}$ s.t.

$$\bar{y}_i \succ_i \underline{y}_i \text{ and } \bar{y}_i \sim_j \underline{y}_i \forall j \neq i$$

Basic Result

Assume that all individuals are EU maximizers. Then their preferences are independent iff their utility functions are affinely independent, ie.,

$$\sum a_i V_i(y) + b = 0 \Rightarrow a_1 = \cdots = a_n = b = 0$$

INDEPENDENCE $\leftrightarrows$ DIVERSITY

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Regular Preferences

Aim: Define a class of preferences under uncertainty as general as possible, that encompass most of existing models

Constant acts do not reduce uncertainty

$\forall f \in \mathcal{A}^c$, $g, h \in \mathcal{A}$, $\alpha \in (0,1]$

$$g \succeq h \Leftrightarrow \alpha g + (1-\alpha) f \succeq \alpha h + (1-\alpha) f$$

Sure thing principle for binary acts

For all $f, g, h, \ell \in \mathcal{A}^c$, all event E

$$f_E h \succ g_E h \Rightarrow f_E h' \succeq g_E h'$$

A preference is regular if it satisfies these two conditions
Most of state-independent models are regulars: SEU, CEU, MMEU...

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Example

	E	E^c
f	1	0
g	0	1
h	α	$1-\alpha$

Neutrality towards uncertainty

$f \sim g \Rightarrow \alpha f + (1-\alpha)g \sim f$

- SEU: $f \sim g \Rightarrow p(E) = p(E^c) = \frac{1}{2} \Rightarrow \alpha f + (1-\alpha)g \sim f$
  EU: uncertainty neutral
- MaxMin EU:
  $V(f) = \min_{p\in\Delta} p(E) = \min_{p\in\Delta} p(E^c) = V(g) = 0$
  $V(\alpha f + (1-\alpha)g) = \min_{p\in\Delta} [\alpha p(E) + (1-\alpha)p(E^c)] =$
  $\min\{\alpha, 1-\alpha\} > 0$
  MaxMin EU: uncertainty aversion

Definition

Notation

- $f(s) = f(s'), \forall s, s' \ (\mathcal{A}^c)$
- $f_E g(s) = f(s)$ if $s \in E$, $g(s)$ otherwise

Neutrality towards uncertainty

for all event E, all constant acts f, g, h, ℓ s.t.:

$$f_E g \sim h_E \ell,$$

$$\alpha f_E g + (1-\alpha) h_E \ell \sim f_E g, \ \forall \alpha \in (0,1)$$

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