Some Results on *Adjusted Winner*[1]

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Abstract

We study the *Adjusted Winner* procedure of Brams and Taylor for dividing goods fairly between two individuals, and prove several results. In particular we show rigorously that as the differences between the two individuals become more acute they both benefit. We introduce a geometric approach which allows us to give alternate proofs of some of the Brams-Taylor results and which gives some hope for understanding the many-agent case also. We also point out that while honesty may not always be the best policy, it *is* as Parikh and Pacuit [4] point out in the context of voting, the only *safe* one. Finally, we show that provided that the assignments of valuation points are allowed to be real numbers, the final result is a continuous function of the valuations given by the two agents and suggest a generalization of the adjusted winner function to take into account nonlinear utility functions.

1 Introduction

In this paper we study one particular algorithm, or procedure, for settling a dispute between two players over a finite set of goods. The algorithm we are interested in is called *Adjusted Winner* (*AW*) and due to Steven Brams and Alan Taylor [2]. See also [1] for a relevant discussion. Suppose there are two players, called Ann (A) and Bob (B), and n (divisible[2]) goods ($G_1, \ldots, G_n$) which must be distributed to Ann and Bob. The goal of the Adjusted Winner algorithm is to *fairly* distribute the n goods between Ann and Bob. We begin by discussing an example which illustrates the Adjusted Winner algorithm.

Suppose Ann and Bob are dividing three goods: G_1, G_2, and G_3. *Adjusted Winner* begins by giving both Ann and Bob 100 points to divide among the three goods. Suppose that Ann and Bob assign these points according to the following table.

Item	Ann	Bob
G_1	<u>10</u>	7
G_2	<u>65</u>	43
G_3	25	<u>50</u>
Total	100	100

[1]Working paper which has been presented at the Stony Brook International Game Theory Conference, June 2005 and Multiagent Resource Allocation Workshop (MARA), September 2005.

[2]Actually all we need to assume is that *one* good is divisible. However, since we do not know before the algorithm begins *which* good will be divided, we assume all goods are divisible. See [2, 3] for a discussion of this fact.

The first step of the procedure is to give G_1 and G_2 to Ann since she assigned more points to those items, and item G_3 to Bob. However this is not an equitable outcome since Ann has received 75 points while Bob only received 50 points (each according to their personal valuation). We must now transfer some of Ann's goods to Bob. In order to determine which goods should be transfered from Ann to Bob, we look at the ratios of Ann's valuations to Bob's valuations. For G_1 the ratio is $10/7 \approx 1.43$ and for G_2 the ratio is $65/43 \approx 1.51$. Since 1.43 is less than 1.51, we transfer as much of G_1 as needed from Ann to Bob[3] to achieve equitability.

However, even giving all of item G_1 to Bob will not create an equitable division since Ann still has 65 points, while Bob has only 57 points. In order to create equitability, we must transfer part of item G_2 from Ann to Bob. Let p be the proportion of item G_2 that Ann will keep. p should then satisfy

$$65p = 100 - 43p$$

yielding $p = 100/108 = 0.9259$, so Ann will keep 92.59% of item G_2 and Bob will get 7.41% of item G_2. Thus both Ann and Bob receive 60.185 points. It turns out that this allocation (Ann receives 92.59% of item G_2 and Bob receives all of item G_1 and item G_3 plus 7.41% of item G_2) is *envy-free*, *equitable* and *efficient*, or *Pareto optimal*. In fact, Brams and Taylor show that Adjusted Winner *always* produces such an allocation [2]. We will discuss these properties in more detail below.

2 The Adjusted Winner Procedure

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods, or items. A **valuation** of these goods is a vector of natural numbers $\langle a_1, \ldots, a_n \rangle$ whose sum is 100. Let $\alpha, \alpha', \alpha'', \ldots$ denote possible valuations for Ann and $\beta, \beta', \beta'', \ldots$ denote possible valuations for Bob. An **allocation** is a vector of n real numbers where each component is between 0 and 1 (inclusive). An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is interpreted as follows. For each $i = 1, \ldots, n$, s_i is the proportion of G_i given to Ann. Thus if there are three goods, then $\langle 1, 0.5, 0 \rangle$ means, "Give all of item 1 and half of item 2 to Ann and all of item 3 and half of item 2 to Bob." Thus AW can be viewed as a function that accepts Ann's valuation α and Bob's valuation β and returns an allocation σ. It is not hard to see that every allocation produced by AW will have a special form: all components except one will be either 1 or 0.

We now give the details of the procedure. Suppose that Ann and Bob are each given 100 points to distribute among n goods as he/she sees fit. In other words, Ann and Bob each select a valuation, $\alpha = \langle a_1, \ldots, a_n \rangle$ and $\beta = \langle b_1, \ldots, b_n \rangle$ respectively. For convenience rename the goods so that

$$a_1/b_1 \geq a_2/b_2 \geq \cdots a_r/b_r \geq 1 > a_{r+1}/b_{r+1} \geq \cdots a_n/b_n$$

[3]When the ratio is closer to 1, a unit gain for Bob costs a smaller loss for Ann.

Let α/β be the above vector of real numbers (after renaming of the goods). Notice that this renaming of the goods ensures that Ann, based on her valuation α, values the goods $G_1, \ldots, G_r$ at least as much as Bob; and Bob, based on his valuation β, values the goods $G_{r+1}, \ldots, G_n$ more than Ann does. Then the AW algorithm proceeds as follows:

1. Give all the goods $G_1, \ldots, G_r$ to Ann and $G_{r+1}, \ldots, G_n$ to Bob. Let X, Y be the number of points received by Ann and Bob respectively. Assume for simplicity that $X \geq Y$.
2. If $X = Y$, then stop. Otherwise, transfer a portion of G_r from Ann to Bob which makes $X = Y$. If equitability is not achieved even with all of G_r going to Bob, transfer $G_{r-1}, G_{r-2}, \ldots, G_1$ in that order to Bob until equitability is achieved.

Thus the AW procedure is a function from pairs of valuations to allocations. Let $\mathsf{AW}(\alpha, \beta) = \sigma$ mean that σ is the allocation given by the procedure AW when Ann announces valuation α and Bob announces valuation $\underline{e}ta$. In [2, 3], it is argued that AW is a "fair" procedure, where fairness is judged according to the following properties.

Let $\alpha = \langle a_1, \ldots, a_n \rangle$ and $\beta = \langle b_1, \ldots b_n \rangle$ be valuations for Ann and Bob respectively. An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is

- **Proportional** if both Ann and Bob receive at least 50% of their valuation. That is, $\sum_{i=1}^{n} s_i a_i \geq 50$ and $\sum_{i=1}^{n} (1 - s_i) b_i \geq 50$
- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation. That is, $\sum_{i=1}^{n} s_1 a_i \geq \sum_{i=1}^{n} (1 - s_i) a_i$ and $\sum_{i=1}^{n} (1 - s_i) b_i \geq \sum_{i=1}^{n} s_i b_i$.
- **Equitable** if both players receive the same total number of points. That is $\sum_{i=1}^{n} s_i a_i = \sum_{i=1}^{n} (1 - s_i) b_i$
- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party. That is for each allocation $\sigma' = \langle s_1', \ldots, s_n' \rangle$ if $\sum_{i=1}^{n} a_i s_i' > \sum_{i=1}^{n} a_i s_i$, then $\sum_{i=1}^{n} (1 - s_i') b_i < \sum_{i=1}^{n} (1 - s_i) b_i$. (Similarly for Bob).

In order to simplify notation, let $V_A(\alpha, \sigma)$ be the total number of points Ann receives according to valuation α and allocation σ and $V_B(\beta, \sigma)$ the total number of points Bob receives according to valuation β and allocation σ.

It is not hard to see that for two-party disputes, proportionality and envy-freeness are equivalent. For a proof, notice that

$$\sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} a_i (1 - s_i) = \sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i s_i = 100$$

Then if σ is envy free for Ann, then $\sum_{i=1}^n a_i s_i \geq \sum_{i=1}^n a_i(1-s_i)$. Hence, $2\sum_{i=1}^n a_i s_i \geq \sum_{i=1}^n a_i = 100$. And so, $\sum_{i=1}^n a_i s_i \geq 50$. The argument is similar for Bob. Conversely, suppose that σ is proportional. Then since $\sum_{i=1}^n a_i s_i \geq 50$, $\sum_{i=1}^n a_i s_i + \sum_{i=1}^n a_i s_i \geq 100 = \sum_{i=1}^n a_i$. Then $\sum_{i=1}^n a_i s_i + \sum_{i=1}^n a_i s_i - \sum_{i=1}^n a_i \geq 0$. Hence, $\sum_{i=1}^n a_i s_i - \sum_{i=1}^n a_i(1-s_i) \geq 0$. And so, $\sum_{i=1}^n a_i s_i \geq \sum_{i=1}^n a_i(1-s_i)$. The proof is similar for Bob.

Returning to AW, it is easy to see the AW only produces equitable allocations (equitability is essentially built in to the procedure). Brams and Taylor go on to show that AW, in fact, satisfies all of the above properties.

Theorem 1 (Brams and Taylor [2]) *AW produces an allocation of the goods based on the announced valuations that is efficient, equitable and envy-free.*

A formal proof of this Theorem is provided in [2]. For completeness, we include here a proof that AW is proportional (and hence envy-free). Effiency is discussed in the next section.

Lemma 2 *For all α, β, $V_{AW}(\alpha, \beta) \geq 50$.*

Proof Suppose not. That is suppose that $V_{AW}(\alpha,\beta) < 50$. Then the goods can be reordered so that

$$a_1 + \cdots + pa_r = (1-p)b_r + \cdots + b_n < 50$$

Hence $a_1 + \cdots + pa_r + (1-p)b_r + \cdots + b_n < 100$. Now since for each $j = 1, \ldots, r$, $a_j \geq b_j$, we have

$$\begin{aligned} 100 &> a_1 + \cdots + pa_r + (1-p)b_r + \cdots + b_n + pa_r + (1-p)b_r + \cdots + b_n \\ &\geq b_1 + \cdots + pb_r + (1-p)b_r + \cdots + b_n \end{aligned}$$

This is a contradiction since $b_1 + \cdots + pb_r + (1-p)b_r + \cdots + b_n = 100$. □

In fact, we can show something more — AW gives each agent 50 points precisely when the agents input the same valuations.

Lemma 3 *For all α, β, $\alpha = \beta$ iff $V_{AW}(\alpha, \beta) = 50$*

Proof ($\Rightarrow$) Suppose that $\alpha = \beta$. Let $G_1, G_2, \ldots$ be the order of goods induced by the AW procedure. Now the AW procedure will distribute the goods so that

$$a_1 + a_2 + \cdots + pa_r = (1-p)b_r + b_{r+1} + \cdots b_n$$

Since $\alpha = \beta$, for each $j = r, \ldots, n$, $b_j = a_j$. Hence, we have

$$a_1 + a_2 + \cdots + pa_r = (1-p)a_r + a_{r+1} + \cdots a_n$$

Now, since $\sum_{i=1}^{n} a_i = 100$,

$$a_1 + a_2 + \cdots + pa_r = (1-p)a_r + 100 - (a_1 + \cdots + a_r)$$

Thus $2(a_1+a_2+\cdots+pa_r) = 100$ and so $a_1+\cdots+pa_r = 50$. Hence, $V_{AW}(\alpha,\beta) = 50$.

($\Leftarrow$) Suppose that $V_{AW}(\alpha,\beta) = 50$. Suppose that $\alpha \neq \beta$. Then there exist i and j such that $a_i > b_i$ and $a_j < b_j$. The AW procedure produces an allocation where (after renaming the goods)

$$a_1 + \cdots + pa_r = (1-p)b_r + \cdots + b_n = 50$$

Furthermore, the procedure ensures that $i \leq r$. WLOG we can assume $i = 1$ by simply choosing the i that maximizes the ratio a_i/b_i. Using basic algebra, we have

$$a_1 + a_2 + \cdots + a_{r-1} + b_{r+1} + b_{r+2} + \cdots b_n = 100 - pa_r - (1-p)b_r$$

Since $a_1 > b_1$ and for each $k = 2, \ldots, r-1$, $a_k \geq b_k$, we have

$$\begin{array}{rcl} 100 - pa_r - (1-p)b_r & = & a_1 + a_2 + \cdots + a_{r-1} + b_{r+1} + b_{r+2} + \cdots b_n \\ & > & b_1 + b_2 + \cdots + b_{r-1} + b_{r+1} + \cdots + b_n \end{array}$$

Hence,

$$100 - pa_r - pb_r > b_1 + b_2 + \cdots b_n = 100$$

This is a contradiction since $p, a_r, b_r > 0$. $\square$

3 A Geometrical Interpretation of *AW*

In this section and the one on continuity, it will be useful to think of both valuations and allocations as vectors in n-space, and to use vector notation where such notation will assist our geometric intuition.

Notice that the AW procedure only produces allocations in which all components, except possibly one, are either 1 or 0. In this section, we show that this is not an accident. We will be working in $\mathbb{R}^k$ for $k \geq 1$. An allocation is a vector $\vec{x} \in \mathbb{R}^k$ where each component is a non-negative real less than or equal to 1. Thus the set of all possible allocations is a hypercube in $\mathbb{R}^k$. Let $\mathcal{C}_k = \{\vec{x} \mid \forall i \;\; 0 \leq x_i \leq 1\}$ be this hypercube of dimension k (we will leave out the k when possible).

A **valuation** is a vector $\vec{P} \in \mathbb{R}^k$ where $\sum_{i=1}^{k} P_i = 100$. Let $\cdot$ denote the dot product, that is $\vec{x} \cdot \vec{P} = \sum_{i=1}^{k} x_i P_i$. Now, let $\vec{P}$ and $\vec{Q}$ be two fixed vectors (Ann's valuation and Bob's valuation). As we want to ensure that Ann and Bob both receive the same valuation, we are interested in the hyperplane $\mathcal{H}_{\vec{P},\vec{Q}}$ generated by the following equation

$$\vec{x} \cdot \vec{P} = (\vec{1} - \vec{x}) \cdot \vec{Q}$$

Since $\vec{1} \cdot \vec{Q} = 100$, we have

$$\vec{x} \cdot (\vec{P} + \vec{Q}) = \vec{x} \cdot (\vec{Q} + \vec{P}) = \vec{x} \cdot \vec{Q} + (\vec{1} - \vec{x}) \cdot \vec{Q} = \vec{1} \cdot \vec{Q} = 100$$

Thus $\mathcal{H}_{\vec{P},\vec{Q}} = \{\vec{x} \mid \vec{x} \cdot (\vec{P} + \vec{Q}) = 100\}$. Again we will leave out the subscripts when possible.

For a fixed $\vec{P}$ and $\vec{Q}$, wanting efficency, we can ask for the allocations $\vec{x}$ that maximize $\vec{x} \cdot \vec{P}$ (subject to the above constraints): Let $\mathcal{I} = \mathcal{C}_k \cap \mathcal{H}_{\vec{P},\vec{Q}}$. Define the function $f : \mathcal{I} \to \mathbb{R}$ by $f(\vec{x}) = \vec{x} \cdot \vec{P}$. Then, since $\mathcal{I}$ is a closed and bounded subset of $\mathbb{R}^k$ (hence compact by the Heine-Borel Theorem), f has a maximum value on $\mathcal{I} = \mathcal{C}_k \cap \mathcal{H}_{\vec{P},\vec{Q}}$. Let m be this maximum value, so that for each $\vec{x} \in \mathcal{I}$, $f(\vec{x}) \leq m$ and the set $\mathcal{M} = \{\vec{x} \mid f(\vec{x}) = m\} \neq \emptyset$.

We claim that there is a point of $\mathcal{M}$ which lies on an edge of the hypercube $\mathcal{C}_k$. More formally,

Theorem 4 *There is a point $\vec{x} \in \mathcal{M}$ with all components either 1 or 0 except possibly one. I.e., $\exists j$ such that $\forall i$, if $i \neq j$ then $x_i = 1$ or $x_i = 0$.*

Proof We will show that
$(*)$ if $\vec{x} \in \mathcal{M}$ with $0 < x_i < 1$ and $0 < x_j < 1$ for $i \neq j$, then there is a point $\vec{x}' \in \mathcal{M}$ with $x_l = x'_l$ for all $l \neq i, j$ and either $x'_i = 1$ or $x'_j = 1$.

To see that this statement implies the theorem, take an arbitrary element $\vec{x} \in \mathcal{M}$ (such an element exists since $\mathcal{M}$ is nonempty). Now, each time that $(*)$ is used, the number of strictly fractional components (not 0 or 1) decreases by one. Thus when we are finished there will be at most one fractional component left.

To prove $(*)$ WLOG we may assume that $i = 1$ and $j = 2$. Thus we have

$$x_1 P_1 + x_2 P_2 + \sum_{i=3}^{k} x_i P_i = m$$

where m is the maximum of the function f. Now we must show that either there is $0 \leq x'_1 \leq 1$

$$x'_1 P_1 + P_2 + \sum_{i=3}^{k} x_i P_i = m$$

or there is $0 \leq x'_2 \leq 1$ such that

$$P_1 + x'_2 P_2 + \sum_{i=3}^{k} x_i P_i = m$$

Now if we set $x'_1 = \frac{x_1 P_1 + x_2 P_2 - P_2}{P_1}$, and $x'_2 = 1$ then it is not hard to see that $x'_1 P_1 + P_2 + \sum_{i=3}^{k} x_i P_i = m$. Similarly, if we set $x''_2 = \frac{x_1 P_1 + x_2 P_2 - P_1}{P_2}$ and $x''_1 = 1$. But to show that one of the other of these assignments work, we still need to show that either $0 \leq x'_1 \leq 1$ or $0 \leq x''_2 \leq 1$.

Since x_1 and x_2 are both between 0 and 1, $x_1P_1 + x_2P_2 < P_1 + P_2$. Thus using basic algebra, $x_1' < 1$ and $x_2'' < 1$.

Suppose that $x_1' < 0$ and $x_2'' < 0$. Then since P_1 and P_2 are both positive real numbers, $x_1P_1 + x_2P_2 - P_2 < 0$ and $x_1P_1 + x_2P_2 - P_1 < 0$. Therefore, $x_1P_1 + x_2P_2 < P_2$ and $x_1P_1 + x_2P_x < P_1$ and so $x_1P_1 + x_2P_2 < \frac{1}{2}P_1 + \frac{1}{2}P_2$. Thus

$$\frac{1}{2}P_1 + \frac{1}{2}P_2 + \sum_{i=3}^{k} x_iP_i > x_1P_1 + x_2P_2 + \sum_{i=3}^{k} x_iP_i = m$$

which is a contradiction since we could clearly have used $\frac{1}{2}, \frac{1}{2}$ as our values, and m is the maximum. □

This proof shows that *there exists* an efficient and equitable allocation that only splits one item. Of course, this is not the same as proving that the algorithm *adjusted winner* actually produces such an outcome. This is what Brams and Taylor show in [2].

4 Continuity

Intuitively, as a function from pairs of vectors to real numbers, AW is continuous. That is, "minor changes" in the valuations produces small changes in the points assigned to the agents by AW. In this section we make this statement precise.

For this section assume that there are k goods. We will think of AW as a function that takes two vectors of *real* numbers and returns a real number, i.e., $\mathsf{AW} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ where $\mathsf{AW}(\alpha, \beta) = V_A(\alpha, AW(\alpha, \beta))$. Of course, stated this way AW is only a partial function on $\mathbb{R}^k \times \mathbb{R}^k$ (only defined on pairs of vectors whose components add up to 100).

Two notions of continuity relevant for our study. The first is the standard notion of continuity and it amounts to AW being continuout in the number of *points* received. Given $v \in \mathbb{R}^k$, the Euclidean norm of v is $||v|| = \sqrt{\sum_{i=1}^{k} v_i^2}$. We say that $F : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ is **continuous** in its frist argument provided for a fixed $v \in \mathbb{R}^k$, for all $\epsilon > 0$ there exists a $\delta > 0$ such that $||x - y|| < \delta$ implies $|F(x, v) - F(y, v)| < \epsilon$. Similarly for "continuity in its second argument". As we will see below, AW is continuous in both of its arguments. The second notion of continuity involves the set of items received by each agent. Thus we think of AW as a funciton from pairs of vectors of real number to *allocations*.

Definition 1 *A function F from $\mathbb{R}^k \times \mathbb{R}^k$ to allocations is said to be* **item continuous in the first argument** *if for a fixed $v \in \mathbb{R}^k$, for all $\epsilon > 0$ there exists $v_1, v_2 \in \mathbb{R}^k \times \mathbb{R}^k$ with $F(v_1, v) = \sigma$, $F(v_2, v) = \sigma'$ and $||v_1 - v_2|| < \epsilon$, then for all $i = 1, \ldots, k$, $\sigma_i = 1$ iff $\sigma_i' = 1$ and $\sigma_i = 0$ iff $\sigma_i' = 0$.*

In other words, small changes in valuations allocates the same set of items to the agents. As we see below, AW is *not* item continuous. We now show that AW is continuous in both arguments. The result follows from the next Lemma.

Suppose that α is Ann's valuation, β is Bob's valuation and σ is the allocation produced by AW (that is $\mathsf{AW}(\alpha, \beta) = \sigma$). Let r be the ratio a_i/b_i where G_i is the item that is divided by the procedure. Define $I = \{l \mid a_l/b_l = r\}$, i.e., I is the set of indices of the goods that have the same ratio as the item which is divided by the procedure.

Lemma 5 *Suppose that α, β, σ and I are defined as above. Suppose that y_1, y_2, y_3 where y_2 is Ann's value of the item being split and y_1, y_3 is Ann's value of all other items in I. Suppose that we choose another item from I to split, call this allocaiton σ'. Say z_1, z_2, z_3 are integers where z_2 is Ann's value of the (new) item being split and z_1, z_3 are Ann's values for all other items in I. Then $V_A(\alpha, \sigma) = V_A(\alpha, \sigma')$, i.e., Ann (and hence Bob) receives the same number of points.*

Proof Let X be the value of allocation out side I that will be allocated to Bob by his valuation. Let Y be the value of allocation out side I that will be allocated to Ann by her valuation. Then

$$V_A(\alpha, \sigma) = X + ry_1 + pry_2 = Y + y_3 + (1 - p)y_2$$

where p is the percentage that Bob will get from the item that correspond to y_2. On the other hand

$$V_A(\alpha, \sigma') = X + rz_1 + qrz_2 = Y + z_3 + (1 - q)z_2$$

where q is the percentage that Bob will get from the item that correspond to z_2. Also note that $y_1 + y_2 + y_3 = z_1 + z_2 + z_3$. Let $S = y_1 + y_2 + y_3$.

Let $A = ry_1 + pry_2$ and let $B = y_3 + (1 - p)y_2$ then $A/r + B = S$ and that gives us $A = r(S - B)$. Substitute in the above equation we get $V_A(\alpha, \sigma) = X + r(S - B) = Y + B$ then $(Y + B)(1 + r) = X + rS + rY$ and that give us $V_A(\alpha, \sigma) = Y + B = (X + rS + rY)/(1 + r)$.

In a similar argument, Let $A' = ry_1 + pry_2$ and let $B' = y_3 + (1 - p)y_2$ then $A'/r + B' = S$ and that gives us $A' = r(S - B')$. Substitute in the above equation we get $V_A(\alpha, \sigma') = X + r(S - B') = Y + B'$ then $(Y + B')(1 + r) = X + rS + rY$ and that give us $V_A(\alpha, \sigma) = Y + B' = (X + rS + rY)/(1 + r)$. Thus we $V_A(\alpha, \sigma) = V_A(\alpha, \sigma')$.

□

5 Discontinuity on the Set of Items

For the rest of this section, assume we have k goods. Let α, β be Ann's and Bob's valuations respectively. Define $V_\Sigma(\alpha, \beta, \sigma) = V_A(\alpha, \sigma) + V_B(\beta, \sigma) = \Sigma s_i a_i + \Sigma(1 - s_i)b_i$. For simplicity we will write $V_\Sigma(\sigma)$ instead of $V_\Sigma(\alpha, \beta, \sigma)$ when α, β are clear in the context. Consider the following example.

Assume we have four items and given this valuation v_1 by both player to be:

	Ann	Bob
G_1	25+$\varepsilon/2$	25-$\varepsilon/2$
G_2	25+$\varepsilon/2$	25-$\varepsilon/2$
G_3	25-$\varepsilon/2$	25+$\varepsilon/2$
G_4	25-$\varepsilon/2$	25+$\varepsilon/2$

Clearly, Ann will get the first two items and Bob will get the last two items. Let us consider this valuation v_2 by both player to be:

	Ann	Bob
G_1	25-$\varepsilon/2$	25+$\varepsilon/2$
G_2	25-$\varepsilon/2$	25+$\varepsilon/2$
G_3	25+$\varepsilon/2$	25-$\varepsilon/2$
G_4	25+$\varepsilon/2$	25-$\varepsilon/2$

According to AW, Ann will get the last two items and Bob will get the first two instead. Note that $||v_1 - v_2|| = \varepsilon$. In fact, we have the following straightforward proposition.

Proposition 6 *Assume we have k goods. For any $\varepsilon > 0$ there are valuations v_1 and v_2 such that:*

- $||v_1 - v_2|| = \varepsilon$
- $\forall i$ *we have* $\sigma_1(i) = 1$ *iff* $\sigma_2(i) = 0$
- $\forall i$ *we have* $\sigma_1(i) = 0$ *iff* $\sigma_2(i) = 1$

Proof We have two cases. First, assume that k is even. Then define v_1 as following: Ann's and Bob's valuation are $a_i = 100/k + \varepsilon/2, b_i = 100/k - \varepsilon/2$ for $i \leq k/2$, i.e. for the first half of the goods, and $a_i = 100/k - \varepsilon/2, b_i = 100/k + \varepsilon/2$ for $i > k/2$. Define v_2 as following: Ann's and Bob's valuation are $a_i = 100/k - \varepsilon/2, b_i = 100/k + \varepsilon/2$ for $i \leq k/2$, i.e. for the first half of the goods, and $a_i = 100/k + \varepsilon/2, b_i = 100/k - \varepsilon/2$ for $i > k/2$. Then these v_1, v_2 satisfies all the three properties. The case when k is odd is similar. □

6 The Distance Between Announced Allocations

In this section we formalize the intuition that the more the valuations differ, the more points each agent will receive. Since *AW* only produces equitable allocations, we can think of the function AW as a function from pairs of valuations to real numbers. Let $V_{AW}(\alpha, \beta)$ denote the total points that *AW* allocates to each agent – say Ann, (according to the announced valuations α and β). Formally, $V_{AW}(\alpha, \beta)$ is defined to be $V_A(\alpha, \mathsf{AW}(\alpha, \beta))$. Of course, we coild define it in terms of Bob's valuation, but they are equal so it does not matter which definition is used.

Given an allocation α for Ann, if Ann increases any component then she must decrease another component as the sum of the components must be 100. Now if Ann wants to accentuate the difference between her allocation and Bob's allocation, then she will only increase points on goods that she values more than Bob. Let α, α' and β, β' be two valuations for Ann and Bob, respectively. We say that $(\alpha, \beta) \prec_{ij}^{A} (\alpha', \beta')$ if

1. $\beta = \beta'$
2. $\alpha_i > \beta_i$, $\alpha_j < \beta_j$, $\alpha'_i = \alpha_i + 1$ and $\alpha'_j = \alpha_j - 1$.
3. for all $k \neq i, j$, $\alpha'_k = \alpha_k$

Similarly, we define $\prec_{ij}^{B}$ with respect to Bob's valuation. The intuition is that if $(\alpha, \beta) \prec_{ij}^{A} (\alpha', \beta')$, then the pair (α', β') represents a situation in which Ann has "increased" by 1 unit the difference between α and β. We say $(\alpha, \beta) \prec (\alpha', \beta')$ if there is a sequence of pairs of valuations linearly ordered by the $\prec_{ij}^{A}, \prec_{ij}^{B}$ relations (with varying i, j) that begins with (α, β) and ends with (α', β'). Thus $\prec$ is the transitive closure of the **union** of the relations $\prec_{ij}^{A}$ and $\prec_{ij}^{B}$. It is not hard to see that $\prec$ is a (non-reflexive) partial order. The main theorem of this section is

Theorem 7 *If $(\alpha, \beta) \prec (\alpha', \beta')$, then $V_{AW}(\alpha, \beta) < V_{AW}(\alpha', \beta')$.*

We return to the proof of the main theorem of this section (Theorem 7). The proof of the theorem is an easy consequence of the following fact.

Lemma 8 *Suppose that $(\alpha, \beta) \prec_{ij}^{A} (\alpha', \beta')$, then $V_A(\alpha, \mathsf{AW}(\alpha, \beta)) < V_A(\alpha', \mathsf{AW}(\alpha', \beta'))$.*

Proof To see this, note that when Ann increased some valuations by 1, where it already exceeded Bob's valuation for that item, then she gets that item in the initial allocation both before this change and after the change. Hence Ann receives more points in her first allocation, and Bob must be compensated for this fact in the final allocation. Thus Bob's final score will increase. But since both Ann and Bob receive the same final score, they will both benefit. We postpone the details and the arithmetic to the final version of the paper. □

7 NonLinear Utility Functions

There are two assumptions about the agent's utility functions that are needed for the previous discussions. First of all, the agents utilities are assumed to be **additive**. That is the utility of a set of goods is the sum of the utilities assigned to each individual good. Second, the utility function for each individual good is assumed to be linear. In this section we consider situations in which this second assumption is dropped.

The intuition for dropping the linearity assumption is that there are many situations in which agents mare share a good but each may get *more* utility than can be described by a linear utility function. For example, suppose that Ann and Bob both assign 100 points to a car. The *AW* procedure would force Ann and Bob to split the car in half. Thus both receive 50 points. Suppose that both Ann and Bob only want to use the car on weekends. Some weekends Bob uses the car and some weekends Ann uses the car. If It is not always the case that they both need to use the car at the same time, then it is possible that each agent can actually receive *more* than 50 points. Suppose that on only half of the weekends is there a conflict between Ann and Bob over the use of the car. Thus both Ann and Bob get to use the car 75% of the time they need to. This can be interpreted as both Ann and Bob receiving 75 points.

Another good example is roommates. When two roommates share an apartment, they are not getting half of the value of that apartment. They are still both are getting to use the Kitchen, the bathroom and the living as if they are living by themselves. It is not the case the roommates always need to use the same resources at the same time.

The following example illustrates the type of situations we have in mind. Suppose that Ann and Bob have the following valuation:

Item	Ann	Bob
G_1	30	20
G_2	30	20
G_3	20	30
G_4	20	30
Total	100	100

In this case, *AW* will give the first two items to Ann and the last two items to Bob and they both receive 60 points. Now assume that both agents' partial utility function of each item is given by the equation $2x - x^2$ (x is the percentage of the good that the agent receives). Thus for good G_1, the total number of points that Ann receives from $(x \times 100)\%$ of G_1 is $60x - 30x^2$ for Ann and $40x - 20x^2$ for Bob. If Ann gets 60% of the first two items and 40% of the last two items, and Bob gets 40% of the first two items and 60% of the last two items, then they both end up with 76 points.

We propose a generalization of the adjusted winner procedure that takes into account the fact that agents' may have nonlinear utilities.

Formulation: Each player will supply two numbers for each items: his/her valuation of getting 100% of the item and his/her valuation of getting 50%. They we can compute the function that will represent each player. For example see the valuations below:

Item	Ann 100%	Ann 50%	Bob 100%	Ann 50%
G_1	30	20	20	15
G_2	30	20	50	30
G_3	40	30	30	20
Total:	100	70	100	65

Using these values, we can approximate the agents' (quadratic) valuation function. Then, using standard techniques, find the maximal total utility subject to the constraint that the agents' total valuations are the same. The details are left for the full version of the paper.

More formally, let $\Gamma = \{G_1, \ldots, G_k\}$ be a set of goods. It is assumed that goods are divisible, as such it is possible for an agent to receive a portion of a good. If $p \in [0,1]$, let (p, G) represent the situation where the agent receives $(p \times 100)\%$ of G. Since agents may receive portions of goods, we define utility functions as

$$u : [0,1] \times \Gamma \to [0,100]$$

where $u(p, G) = r$ means that the agent assigns utility r to receiving $(p \times 100)\%$ of G. Assuming linearity implies that $u(p, G) = pu(1, G)$. Of course any closed interval would work here since we can always normalize. Since Γ is finite, we can think of a utility function u as a tuple $\langle u_{G_1}, u_{G_2}, \ldots, u_{G_k} \rangle$ where $u_{G_i} : [0,1] \to [0, u(1, G_i)]$.

Assuming additivity, given a utility function u, we define the function $\overline{u}$ on the set of subsets of Γ as follows. Let $\Delta \subseteq \Gamma$, then

$$\overline{u}(\Delta) = \sum_{G \in \Delta} u(G)$$

In order to simplify notation, we will write $u(\Delta)$ instead of $\overline{u}(\Delta)$.

Let u be the utility function of Ann and v the utility function of Bob. The *AW* procedure asks Ann and Bob to represent their utility functions as vectors whose sum of the components is 100. Given Ann's valuation $\alpha = \langle a_1, \ldots, a_k \rangle$, *AW* approximates Ann's utility function as follows: each u_{G_i} is the straight line going through (0,0) and $(1, a_i)$. Given Bob's valuation $\beta = \langle b_1, \ldots, b_k \rangle$, *AW* approximates his utility function as follows: each v_{G_i} is the straight line from $(0, b_i)$ to $(1, 0)$. Viewed in this light, *AW* is a function that accepts two linear utility functions and returns an allocation which is equitable, envy-free and efficient with respect to its two arguments.

More generally, let F be any function from pairs of utility functions to the set of allocations. We will show that under suitable conditions, there is a function F such that $F(u, v)$ is envy-free, equitable and efficient. Furthermore, $F(u, v)$ will produce an allocation which is more efficient than the allocation produced by *AW*.

Definition 1 *Suppose that u is a utility functions, G a good and $x, y \in [0,1]$.*

- *u is* **strictly monotonic with respect to** G_i *if $x < y$ implies $u_G(x) < u_G(y)$*

- u *is* **strictly anti-monotonic with respect to** G *if* $x < y$ *implies* $u_G(x) > u_G(y)$
- u *is* **strictly concave with respect to** G *if for all* $\lambda \in [0,1]$, $u_G \lambda x + (1-\lambda)y) > \lambda u_G(x) + (1-\lambda)u_G(y)$
- u *is* **strictly convex with respect to** G *if for all* $\lambda \in [0,1]$, $u_G(\lambda x + (1-\lambda)t) < \lambda u(x) + (1-\lambda)u_G(x)$

We say that u is strictly monotonic if u is strictly monotonic with respect to G for each good G. Similarly for the other properties. The following fact is straightforward.

Fact: If u is strictly monotonic with respect to G, v is anti-monotonic with respect to G and u_G and v_G intersect, then they interesect at a unique point.

Definition 2 *Let* u *and* v *be two utility functions. We say that* u *and* v *are* **complementary with respect to** G *if*

1. u_G *is monotonic;*
2. v_G *is anti-monotonic; and*
3. $u_G(0) = v_G(1)$ *and* $u_G(1) = v_G(0)$.

We say u and v are complementary utility functions of u and v are complementary with respect to G for each good G. Finally, we say that a utility function is continuous if u_G is continuous for each good G. The following lemma shows that for one good, if we assume the agents' utility functions are complementary, continuous and concave then we can find an allocation which is better for both agents than the allocation produced by AW.

Lemma 9 *Suppose that* u *and* v *are continuous and complementary utility functions with respect to* G*. Then if* u_G *and* v_G *are concave, there exists a unique point* x_0 *such that* $u_G(x_0) = v_G(x_0)$ *and* $u_G(x_0) \geq (u_G(0) + u_G(1))/2$ *(*$v_G(x_0) \geq (u_G(0) + u_G(1))/2$*).*

Proof By assumption u_G is strictly monotonic, continuous and concave; v_G is continuous, strictly anti-monotonic and concave; and $u_G(0) = v_G(1)$ and $u_G(1) = v_G(0)$. It is easy to see that there must be a unique point x_0 such that $u_G(x_0) = v_G(x_0)$. We must show $u_G(x_0) \geq (u_G(0) + u_G(1))/2$. Suppose $u_G(x_0) < (u_G(0) + u_G(1))/2$. Then since u_G is concave,

$$(*) \qquad v_G(x_0) = u_G(x_0) < (u_G(0) + u_G(1))/2 \leq u_G(1/2)$$

Furthermore since, $u_G(0) = v_G(1)$ and $u_G(1) = v_G(0)$ and v_G is concave.

$$(**) \qquad v_G(x_0) = u_G(x_0) < (u_G(0) + u_G(1))/2 = (v_G(1) + v_G(0))/2 \leq v_G(1/2)$$

There are three cases to consider:

1. $x_0 < 1/2$. Then since v_G is anti-monotonic, $v_G(x_0) > v_G(1/2)$. But this contradicts $(**)$
2. $x_0 > 1/2$. Then since u_G is monotonic, $u_G(x_0) > u_G(1/2)$. But this contradicts $(*)$.
3. $x_0 = 1/2$. This contradicts both $(*)$ and $(**)$.

□

With one good, the *AW* procedure splits the good in half giving each agent 50 points. Thus the above theorem shows that under suitable assumptions about the utility function, there exists an envy-free, equitable and efficient allocation which is better for both parties than the one produced by *AW*. Can a similar argument be constructed for any number of goods?

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