Special functions and Lie theory

Exercises, week 11

Exercise 1 Let the Bessel function $\mathcal{J}_\alpha$ be defined as in Exercise 1 of Week 10. Show that

$$\Big(\frac{d^2}{dr^2}+\frac{2\alpha+1}{r}\,\frac{d}{dr}\Big)\mathcal{J}_\alpha(\lambda r)=-\lambda^2\,\mathcal{J}_\alpha(\lambda r)\qquad(\lambda,r>0),$$

first for general α by using the defining power series, and second for $\alpha=\frac12 d-1$ ($d=1,2,\ldots$) by using that the function $x\mapsto\mathcal{J}_{\frac12 d-1}(\lambda|x|)$ on $\mathbb{R}^d$ is obtained by averaging $x\mapsto e^{i\lambda x_1}$ with respect to the group $SO(d)$.

Exercise 2 Show that the group $G:=SL(2,\mathbb{R})$ acts transitively on the upper half plane $X:=\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$ by

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}.z:=\frac{az+b}{cz+d},$$

and that the stabilizer of i in G equals $K:=SO(2)$. Determine the orbits of the subgroups $A:=\left\{\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}\right\}$ and $N:=\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix}\right\}$ for this action.

Make X into a Riemannian manifold such that the line element is G-invariant. Find the corresponding volume element and Laplace-Beltrami operator.