Dick Askey's positive addition to Amsterdam

Tom Koornwinder

Korteweg-de Vries Institute, University of Amsterdam
T.H.Koornwinder@uva.nl

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Askey's sabbatical, Amsterdam, 1969–1970

Mathematisch Centrum, 2e Boerhaavestraat 49, Amsterdam
(earlier situation; in 1969 the top floor looked different)

`http://beeldbank.amsterdam.nl/afbeelding/010003018854`

Askey's sabbatical, Amsterdam, 1969–1970 (cntd.)

Dick Askey reading a math book in his Dutch home, Amstelveen, 1970

Askey's sabbatical (cntd.)

stichting mathematisch centrum

MC

AFDELING TOEGEPASTE WISKUNDE TC 51/70 APRIL

RICHARD ASKEY
EIGHT LECTURES ON ORTHOGONAL POLYNOMIALS

TW

2e boerhaavestraat 49 amsterdam

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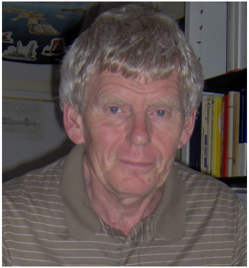
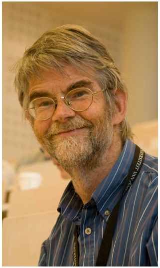
Askey's sabbatical (cntd.)

Lecturing in Amsterdam to interested PhD students



Nico Temme

Herman Bavinck



Tom K

and their boss

prof. Hans Lauwerier

Askey's sabbatical (cntd.)

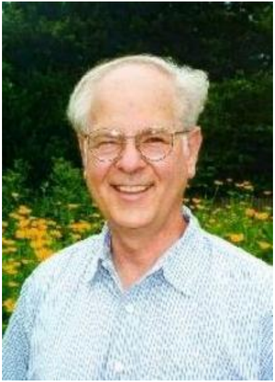
Dick liked the library of the Math. Centrum,
as did some of his students

Askey's sabbatical (cntd.)

Dick told in his lectures about his heroes,

Gábor Szegő

and about the wonder boy in the States who solved all his problems.

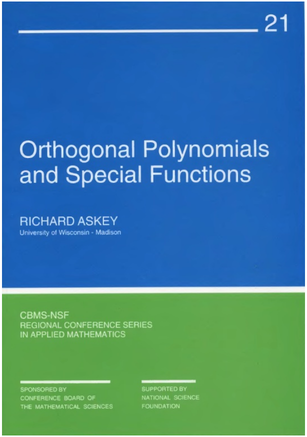


George Gasper

Dick's wise lessons

- Always try to write as hypergeometric function.
- Is your integral possibly a fractional integral?
- Depth is in positivity.
- Even more depth can be imported from group theory.
- What is the underlying addition formula?
- Study the old masters.
- Look for interactions with and applications to other fields.
- Special functions are useful functions!   (Paul Turán)

Askey’s 1975 SIAM Lecture Notes



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http://www.ec-securehost.com/SIAM/CB21.html

Four canonical problems

$\{p_n\}$ and $\{q_n\}$ systems of orthogonal polynomials;

$$\int p_m(x)p_n(x)\,d\mu(x) = h_n\,\delta_{m,n}.$$

- product formula:

$$p_n(x)p_n(y) = \int p_n(z)K(x,y,z)\,d\mu(z).$$

- transmutation:

$$q_n(x) = \int p_n(y)A(x,y)\,d\mu(y).$$

- linearization of products:

$$p_m(x)p_n(x) = \sum_{k=|m-n|}^{m+n} c_{m,n,k}\,p_k(x)/h_k.$$

- connection formula:

$$q_n(x) = \sum_{k=0}^{n} a_{n,k}\,p_k(x)/h_k.$$

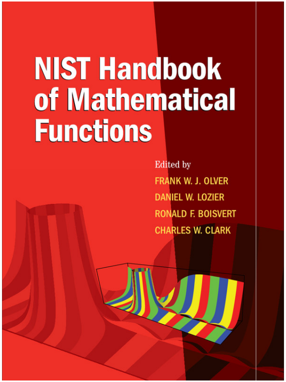
Find the integral and summation kernels explicitly and/or see when these kernels are nonnegative.

Two more topics

- **History of mathematics and mathematicians**
  See for instance R. Askey, *How can mathematicians and mathematical historians help each other?*, in: *History and philosophy*, Univ. of Minnesota Press, 1988, pp. 201–217.
- **Compendia of special functions**
  See for instance R. Askey, *Handbooks of special functions*, in: *A century of mathematics in America, Part III*, American Mathematical Society, 1989, pp. 369–391.

  Askey was very critical about most compendia that had appeared, except for *Higher transcendental functions*, which got his praise.

Digital Library of Mathematical Functions



Later Askey was an associate editor of the *Digital Library of Mathematical Functions* (DLMF) and he was coauthor of three chapters therein.

`http://dlmf.nist.gov/about/bio/RAAskey`

Product formulas

$$p_n(x)\,p_n(y) = \int p_n(z)\,K(x,y,z)\,d\mu(z)$$

$$K(x,y,z) = \sum_{n=0}^{\infty} \frac{p_n(x)\,p_n(y)\,p_n(z)}{h_n}$$

Gegenbauer product formula

Jacobi polynomials $(\alpha,\beta>-1)$:

$$R_n^{(\alpha,\beta)}(x):=\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)},\quad d\mu_{\alpha,\beta}(x):=\frac{(1-x)^\alpha(1+x)^\beta\,dx}{\int_{-1}^1(1-x)^\alpha(1+x)^\beta\,dx},$$

$$\int_{-1}^{1} R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)=h_n^{(\alpha,\beta)}\,\delta_{m,n}.$$

Gegenbauer product formula $(\alpha>-\frac12)$: $R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)$

$$=\int_{-1}^{1} R_n^{(\alpha,\alpha)}\big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\big)\,d\mu_{\alpha-\frac12,\alpha-\frac12}(t).$$

For $\alpha=-\frac12$: $\cos n\phi\cos n\psi=\frac12\big(\cos n(\phi+\psi)+\cos n(\phi-\psi)\big)$.

Generalized translation (Levitan, Bochner, Hirschman):

$$T_y[f](x):=\int_{-1}^{1} f\big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\big)\,d\mu_{\alpha-\frac12,\alpha-\frac12}(t).$$

Positivity of generalized translation:

$$f(x)=\sum_{n=0}^\infty\frac{\widehat{f}(n)R_n^{(\alpha,\alpha)}(x)}{h_n}\ge0\Leftrightarrow T_y[f](x)=\sum_{n=0}^\infty\frac{\widehat{f}(n)R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)}{h_n}\ge0$$

Gegenbauer product formula (cntd.)

Product formula and generalized translation in kernel form:

$$R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)=\int_{-1}^1 R_n^{(\alpha,\alpha)}(z)\,K_{\alpha,\alpha}(x,y,z)\,d\mu_{\alpha,\alpha}(z),$$

$$T_y[f](x)=\int_{-1}^1 f(z)\,K_{\alpha,\alpha}(x,y,z)\,d\mu_{\alpha,\alpha}(z),\quad \text{where}$$

$$K_{\alpha,\alpha}(x,y,z)=\sum_{n=0}^\infty R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)R_n^{(\alpha,\alpha)}(z)/h_n^{(\alpha,\alpha)}$$

$$=\frac{\Gamma(\alpha+1)^2}{\Gamma(\alpha+\frac12)\Gamma(\alpha+\frac32)}\,\frac{(1-x^2-y^2-z^2+2xyz)_+^{\alpha-\frac12}}{\big((1-x^2)(1-y^2)(1-z^2)\big)^\alpha}\ge 0.$$

Convolution:

$$(f*g)(x):=\int_{-1}^1 T_y[f](x)\,g(y)\,d\mu_{\alpha,\alpha}(y)=\int_{-1}^1\int_{-1}^1 f(z)\,g(y)$$

$$\times K_{\alpha,\alpha}(x,y,z)\,d\mu_{\alpha,\alpha}(y)\,d\mu_{\alpha,\alpha}(z)=\sum_{n=0}^\infty \frac{\widehat{f}(n)\widehat{g}(n)R_n^{(\alpha,\alpha)}(x)}{h_n^{(\alpha,\alpha)}}\,.$$

Convolution algebra

Put $\|f\|_1 := \int_{-1}^{1} |f(x)|\, d\mu_{\alpha,\alpha}(x)$. Then we conclude:

$$\|f{*}g\|_1 \le \|f\|_1\, \|g\|_1, \quad \|f{*}g\|_\infty \le \|f\|_\infty\, \|g\|_1, \quad f,g \ge 0 \Rightarrow f{*}g \ge 0.$$

The same machinery would work for other orthogonal systems, provided we have a product formula with positive kernel.

Gegenbauer case $\alpha = \frac{1}{2}(d-3)$ **by group theory**:
$G = O(d), \quad K = O(d-1), \quad \Omega = G/K = S^{d-1}$ (Gelfand pair).

$$(f * g)(x) = (g * f)(x) = \int_G f(y)\, g(y^{-1}x)\, dy \quad (f, g\ K\text{-biinvariant}),$$

$$(F * G)(\langle x, y\rangle) = \int_\Omega F(\langle x, z\rangle)\, G(\langle z, y\rangle)\, d\omega(z) \quad (x, z \in \Omega),$$

$$\phi(x)\,\phi(y) = \int_K \phi(xky)\, dk \quad (x, y \in G,\ \phi(x) = R_n^{(\alpha,\alpha)}(\langle xe_1, e_1\rangle)\ ).$$

ϕ is spherical function (Gelfand); immediate positivity results; works also for certain other Jacobi parameters (Gangolli).

Jacobi product formula

Askey & Wainger (1969) attacked the analogous Jacobi problem for $\alpha > \beta > -\frac{1}{2}$. Not yet positivity but boundedness:

$$\int_{-1}^{1} \left| \frac{\sum_{n=0}^{\infty} r^n R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y) R_n^{(\alpha,\beta)}(z)}{h_n^{(\alpha,\beta)}} \right| d\mu_{\alpha,\beta}(z) \le M \quad (0 \le r < 1),$$

Then $\quad \|f * g\|_1 \le M \|f\|_1 \, \|g\|_1$.

Gasper (1971) showed that

$$R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y) = \int_{-1}^{1} R_n^{(\alpha,\beta)}(z) \, K_{\alpha,\beta}(x,y,z) \, d\mu_{\alpha,\beta}(z)$$

with $K_{\alpha,\beta}(x,y,z) \ge 0$ as a definite integral of an explicit nonnegative elementary function. In fact he found this by combining two formulas in Watson's *Treatise on the theory of Bessel functions*, see there pages 411 and 413.

Jacobi product formula (cntd.)

Watson essentially has the same nonnegative kernel in the following two product formulas ($\alpha > \beta > -\frac{1}{2}$):

$$\frac{J_\alpha(x)}{x^\alpha}\,\frac{J_\beta(y)}{y^\beta} = \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^\infty \frac{J_\beta(z)}{z^\beta}\,\widetilde{K}_{\alpha,\beta}(x,y,z)\,z^{2\beta+1}\,dz,$$

$$R_n^{(\alpha,\beta)}(\cos 2\theta_1)\,R_n^{(\alpha,\beta)}(\cos 2\theta_2) = \int_0^{\pi/2} R_n^{(\alpha,\beta)}(\cos 2\theta_3)$$

$$\times\,\widetilde{K}_{\alpha,\beta}(\sin\theta_1\sin\theta_2,\cos\theta_1\cos\theta_2,\cos\theta_3)\,(\cos\theta_3)^{2\beta+1}\,\sin\theta_3\,d\theta_3.$$

Askey's question

Rewrite the Gasper-Watson Jacobi product formula as something similar to the Gegenbauer product formula

$$R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y) = \int_{-1}^{1} R_n^{(\alpha,\alpha)}\big(xy+(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}t\big)\,d\mu_{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}(t).$$

For this purpose work with addition formulas and group theory.

Gegenbauer addition formula

The Gegenbauer product formula gives the constant term in the Gegenbauer addition formula:

$$R_n^{(\alpha,\alpha)}(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t) = \sum_{k=0}^{n} \frac{(-1)^k(-n)_k\,(n+2\alpha+1)_k}{2^{2k}((\alpha+1)_k)^2\,h_k^{(\alpha-\frac12,\alpha-\frac12)}}$$

$$\times(1-x^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-y^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(y)\,R_k^{(\alpha-\frac12,\alpha-\frac12)}(t).$$

For $\alpha=\frac12(d-3)$ by group theory: $G=O(d)\supset K=O(d-1)\supset M=O(d-2)$, $A=SO(2)\subset G$ commuting with M;
ϕ spherical function for (G,K) and ψ_δ for (K,M) (Gelfand pairs):

$$\phi(a_1ka_2)=\sum_{\delta\in(K/M)^\wedge}\widehat{\phi}_{a_1,a_2}(\delta)\,d_\delta\,\psi_\delta(k)\quad(k\in K,\ a_1,a_2\in A).$$

Or as reproducing kernel for spherical harmonics of degree n:

$$R_n^{(\alpha,\alpha)}(\langle x,y\rangle)=\frac{1}{d_n}\sum_{k=1}^{d_n}Y_{n,k}(x)\,Y_{n,k}(y)\quad(x,y\in S^{d-1}=O(d)/O(d-1)).$$

disk polynomials

Gangolli: Jacobi polynomials $R_n^{(d-2,0)}$ are spherical functions on complex projective space $P^{d-1}(\mathbb{C}) = U(d)/(U(1) \times U(d-1))$ (a compact Riemannian symmetric space of rank one).

But this is the space of $U(1)$-orbits on $S^{2d-1} = U(d)/U(d-1)$ (unit sphere in $\mathbb{C}^d$). Functions on $P^{d-1}(\mathbb{C})$ are $U(1)$-invariant functions on S^{2d-1}.

Moreover $(U(d), U(d-1))$ is Gelfand pair with Zernike's disk polynomials $R_{m,n}^{\alpha}(z)$ $(\alpha = d-2)$ as spherical functions.

$$R_{m,n}^{\alpha}(re^{i\phi}) := R_{\min(m,n)}^{(\alpha,|m-n|)}(2r^2-1)\, r^{|m-n|}\, e^{i(m-n)\phi},$$

$$\int_D R_{m,n}^{\alpha}(x+iy)\,\overline{R_{k,l}^{\alpha}(x+iy)}\,(1-x^2-y^2)^{\alpha}\,dx\,dy = 0$$

$$((m,n) \neq (k,l);\ D \text{ unit disk}).$$

Work with complex spherical harmonics on $\mathbb{C}^d$: refinement of ordinary spherical harmonics on $\mathbb{R}^{2d}$.

disk polynomials (cntd.)

The disk polynomials, introduced by the Dutch Nobel prize winner Zernike, find important applications at the Dutch world leading chip machine maker ASML.

Addition formula for disk polynomials

R. L. Šapiro (1968), K (1972):

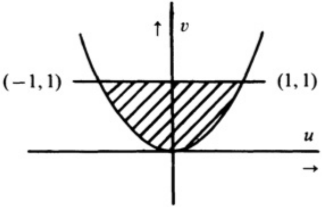
$$R^{\alpha}_{m,n}\big(z_1 z_2 + (1-|z_1|^2)^{\frac{1}{2}}(1-|z_2|^2)^{\frac{1}{2}} w\big)$$
$$= \sum_{k=0}^{m}\sum_{l=0}^{n} c^{\alpha}_{m,n,k,l}\,(1-|z_1|^2)^{\frac{1}{2}(k+l)} R^{\alpha+k+l}_{m-k,n-l}(z_1)$$
$$\times (1-|z_2|^2)^{\frac{1}{2}(k+l)} R^{\alpha+k+l}_{m-k,n-l}(z_2)\, R^{\alpha-1}_{k,l}(w).$$

This yields an addition formula for Jacobi polynomials $R_n^{(\alpha,0)}$ and next, by differentiation and by analytic continuation in the parameters, an addition formula for Jacobi polynomials $R_n^{(\alpha,\beta)}$ $(\alpha > \beta > -\frac{1}{2})$. It involves an expansion in terms of orthogonal polynomials in two variables on a parabolic biangle.

Orthogonal polynomials on the parabolic biangle

$$R_{n,k}^{(\alpha,\beta)}(x,y) := R_k^{(\alpha,\beta+n-k+\frac{1}{2})}(2y-1)\,y^{\frac{1}{2}(n-k)}\,R_{n-k}^{(\beta,\beta)}(y^{-\frac{1}{2}}x).$$

$$\int_{y=0}^{1}\int_{x=-y^{\frac{1}{2}}}^{y^{\frac{1}{2}}} R_{n,k}^{(\alpha,\beta)}(x,y)R_{m,l}^{(\alpha,\beta)}(x,y) \times (1-y)^{\alpha}\,(y-x^2)^{\beta}\,dx\,dy = 0 \quad ((n.k)\neq(m,l)).$$



Parametrize this region by

$(x,y) = (r\cos\phi, r^2)$

$(0 \le r \le 1,\ 0 \le \phi \le \pi)$.

Put $$d\nu_{\alpha,\beta}(r,\phi) := \frac{r^{2\beta+2}(1-r^2)^{\alpha}\,(\sin\phi)^{2\beta+1}\,dr\,d\phi}{\int_{r=0}^{1}\int_{\phi=0}^{\pi} r^{2\beta+2}(1-r^2)^{\alpha}\,(\sin\phi)^{2\beta+1}\,dr\,d\phi}.$$

Addition formula for Jacobi polynomials

$$\Lambda(x,y,r,\phi) := \tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)\,r^2 + (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}\,r\cos\phi - 1.$$

$$R_n^{(\alpha,\beta)}(\Lambda(x,y,r,\phi)) = \sum_{k=0}^{n}\sum_{l=0}^{k} c_{n,k,l}^{(\alpha,\beta)}\,(1-x)^{\frac{1}{2}(k+l)}(1+x)^{\frac{1}{2}(k-l)} \times R_{n-k}^{(\alpha+k+l,\beta+k-l)}(x)\,(1-y)^{\frac{1}{2}(k+l)}(1+y)^{\frac{1}{2}(k-l)}R_{n-k}^{(\alpha+k+l,\beta+k-l)}(y) \times R_{k,l}^{(\alpha-\beta-\frac{1}{2},\beta-\frac{1}{2})}(r\cos\phi,r^2).$$

Constant term in the expansion is the Jacobi product formula

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \int_{r=0}^{1}\int_{\phi=0}^{\pi} R_n^{(\alpha,\beta)}(\Lambda(x,y,r,\phi))\,d\nu_{\alpha-\beta-\frac{1}{2},\beta-\frac{1}{2}}(r,\phi).$$

Conversely, the product formula implies the addition formula by integration by parts and Rodrigues type formulas.

Laplace type integral representation

(Askey, 1974; K, 1974)

The Gegenbauer Laplace type integral representation is a degenerate case of the Gegenbauer product formula:

$$R_n^{(\alpha,\alpha)}(x) = \int_{-1}^{1} (x + i(1-x^2)^{\frac{1}{2}}\, t)^n \, d\mu_{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}(t).$$

Combine this with a fractional integral or degenerate the Jacobi product formula for obtaining

Jacobi Laplace type integral representation: $R_n^{(\alpha,\beta)}(x) =$

$$\int_{r=0}^{1}\int_{\phi=0}^{\pi} \left(\tfrac{1}{2}(1+x) - \tfrac{1}{2}(1-x)r^2 + i(1-x^2)^{\frac{1}{2}}\, r\cos\phi\right)^n d\nu_{\alpha-\beta-\frac{1}{2},\beta-\frac{1}{2}}(r,\phi).$$

One can go back and forth between this integral representation and the Jacobi product formula by Bateman's bilinear sum and its inverse.

Bateman's bilinear sum and its inverse

$$(x+y)^n\,R_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)=\sum_{k=0}^{n}a_{n,k}R_k^{(\alpha,\beta)}(x)R_k^{(\alpha,\beta)}(y),$$

$$\text{where}\quad (x+1)^n=\sum_{k=0}^{n}a_{n,k}R_k^{(\alpha,\beta)}(x);$$

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y)=\sum_{k=0}^{n}b_{n,k}(x+y)^k\,R_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right),$$

$$\text{where}\quad R_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}b_{n,k}(x+1)^k.$$

These connect

$$(x+y)^n\,R_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)=\int_{r=0}^{1}\int_{\phi=0}^{\pi}(\Lambda(x,y,r,\phi)+1)^n\,d\nu_{\alpha-\beta-\frac12,\beta-\frac12}(r,\phi)$$

$$\text{and } R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y)=\int_{r=0}^{1}\int_{\phi=0}^{\pi}R_n^{(\alpha,\beta)}(\Lambda(x,y,r,\phi))\,d\nu_{\alpha-\beta-\frac12,\beta-\frac12}(r,\phi).$$

Further developments

- **Hypergroups** (Dunkl, Jewett, Spector) axiomatize the structure associated with a positive convolution for an orthogonal system.
- Dunkl: Addition formulas for Krawtchouk, Hahn and q-Hahn polynomials from interpretation on finite groups.
- D. Stanton: Similarly for q-Krawtchouk polynomials.
- K: Addition formula for little q-Legendre polynomials from quantum group interpretation.
- Floris: Addition formula for q-disk polynomials in non-commuting variables from quantum group interpretation.
- Koelink: addition formulas in many q-cases, both from quantum groups and analytic.
- Rahman: analytic proofs of addition formulas in some q-cases.
- K & A. Schwartz: positive convolution for orthogonal polynomials on triangle and simplex.

The big open problem: Show the positivity of convolution for Heckman-Opdam Jacobi polynomials.

Partial results by Rösler and by Remling & Rösler.

Transmutation

$$q_n(x) = \int p_n(y)\, A(x,y)\, d\mu(y)$$

$$A(x,y) = \sum_{n=0}^{\infty} \frac{q_n(x)\, p_n(y)}{h_n}$$

Fractional integrals

Riemann-Liouville:

$$(R_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_0^x f(y)\,(x-y)^{\mu-1}\,dy \quad (\mathrm{Re}\,\mu > 0).$$

Weyl:

$$(W_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_x^\infty f(y)\,(y-x)^{\mu-1}\,dy \quad (\mathrm{Re}\,\mu > 0).$$

Askey & Fitch (1969) emphasized Bateman's integral:

$$\frac{x^{c+\mu-1}}{\Gamma(c+\mu)}\,{}_2F_1\left(\begin{matrix} a,b \\ c+\mu \end{matrix};x\right) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{y^{c-1}}{\Gamma(c)}\,{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};y\right)\,(x-y)^{\mu-1}\,dy,$$

$(\mathrm{Re}\,\mu, \mathrm{Re}\,c > 0)$. Hence, for $\mathrm{Re}\,\mu > 0$:

$$\frac{(1-x)^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)}\,R_n^{(\alpha+\mu,\beta-\mu)}(x) = \frac{1}{\Gamma(\mu)} \int_x^1 \frac{(1-y)^\alpha}{\Gamma(\alpha+1)}\,R_n^{(\alpha,\beta)}(y)\,(y-x)^{\mu-1}\,dy.$$

Transmutation property

Bateman's integral in kernel form:

$$R_n^{(\alpha+\mu,\beta-\mu)}(x) = \int R_n^{(\alpha,\beta)}(y)\,A(x,y)\,d\mu_{\alpha,\beta}(y),$$

where $\quad A(x,y) = \dfrac{2^{\alpha+\beta+1}\,\Gamma(\alpha+\mu+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)\,\Gamma(\mu)}\;\dfrac{(y-x)_+^{\mu-1}}{(1-x)^{\alpha+\mu}(1+y)^{\beta}}\,.$

Transmutation Theorem. Let $\{p_n\}$ and $\{q_n\}$ be complete orthogonal systems with respect to measures $d\mu$ and $d\nu$, respectively. Let D and E be operators having the p_n repectively the q_n as eigenfunctions with the same eigenvalue λ_n. Suppose that $q_n(x) = \int p_n(y)\,A(x,y)\,d\mu(y)$. Then the operator $\mathcal{A}$ given by $(\mathcal{A}f)(y) := \int f(x)\,A(x,y)\,d\nu(x)$ satisfies the transmutation property $\mathcal{A}\circ E = D\circ\mathcal{A}$.

Hence in case of Bateman's integral : $D = D_{\alpha,\beta}$, $E = D_{\alpha+\mu,\beta-\mu}$,
where $\quad D_{\alpha,\beta}R_n^{(\alpha,\beta)} = -n(n+\alpha+\beta+1)\,R_n^{(\alpha,\beta)}$.

Feldheim-Vilenkin integral
(not of the desired transmutation form):

$$\frac{(x-1)^{\alpha+\mu}}{\Gamma(\alpha+\mu+1)}\, x^{\frac{1}{2}n}\, R_n^{(\alpha+\mu,\alpha+\mu)}(x^{-\frac{1}{2}})$$
$$= \frac{1}{\Gamma(\mu)}\int_1^x \frac{(y-1)^{\alpha}}{\Gamma(\alpha+1)}\, y^{\frac{1}{2}n}\, R_n^{(\alpha,\alpha)}(y^{-\frac{1}{2}})\,(x-y)^{\mu-1}\,dy \quad (\mu>0).$$

Remark. Both the Bateman and Feldheim-Vilenkin integral can be obtained from spherical harmonics. For Bateman also use that

$$(x_1^2+\cdots+x_{q+p}^2)^n\, R_n^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}\left(\frac{(x_1^2+\cdots+x_q^2)-(x_{q+1}^2+\cdots+x_{q+p}^2)}{x_1^2+\cdots+x_{q+p}^2}\right)$$

is an $O(q)\times O(p)$-invariant homogeneous harmonic polynomial of degree $2n$ on $\mathbb{R}^{q+p}$.

Transmutation in the non-compact case

Jacobi functions (surveyed by K, 1984). These form a continuous orthogonal system of Gauss hypergeometric functions. They are noncompact analogues of Jacobi polynomials. They have richer transmutation properties.

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\begin{matrix}\frac{1}{2}(\rho+i\lambda), \frac{1}{2}(\rho-i\lambda)\\ \alpha+1\end{matrix}; -\sinh^2 t\right), \quad \rho := \alpha+\beta+1;$$

$$\widehat{f}(\lambda) = \int_0^\infty f(t)\,\Delta_{\alpha,\beta}(t)\,dt, \qquad f(t) = \int_0^\infty \widehat{f}(\lambda)\,|c_{\alpha,\beta}(\lambda)|^{-2}\,d\lambda.$$

$$D_{\alpha,\beta}\,\phi_\lambda^{(\alpha,\beta)} = -\lambda^2\,\phi_\lambda^{(\alpha,\beta)}; \qquad \phi_\lambda^{(-\frac{1}{2},-\frac{1}{2})}(t) = \cos(\lambda t).$$

Transmutation: $$\phi_\lambda^{(\alpha+\mu,\beta\pm\mu)}(t) = \int_0^t \phi_\lambda^{(\alpha,\beta)}(s)\,A(s,t)\,\Delta_{\alpha,\beta}(s)\,ds$$

with $A(s,t)$ positive and elementary if $\mu > 0$. Relationship with Abel transform on noncompact semisimple Lie groups. Generalization to Chébli-Trimèche hypergroups.

Linearization of products

$$p_m(x)\,p_n(x) = \sum_{k=|m-n|}^{m+n} c_{m,n,k}\,p_k(x)/h_k$$

$$c_{m,n,k} = \int p_m(x)\,p_n(x)\,p_k(x)\,d\mu(x)$$

Linearization of products (cntd.)

Jacobi polynomials:

$$R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(x) = \sum_{k=|m-n|}^{m+n} c_{m,n,k}^{(\alpha,\beta)}\,R_k^{(\alpha,\beta)}(x)/h_k^{(\alpha,\beta)}$$

Theorem (Gasper, 1970) (a) $\Longleftrightarrow$ (b) $\Longleftarrow$ (c)

(a) $c_{m,n,k}^{(\alpha,\beta)} \ge 0$ for all m, n, k.
(b) some quartic polynomial in α, β is nonnegative.
(c) $\alpha \ge \beta > -1$ and $\alpha + \beta > -1$.

General monic orthogonal polynomials p_n :

$$p_1(x)\,p_n(x) = p_{n+1}(x) + a_n p_n(x) + b_n p_{n-1}(x),$$
$$p_m(x)p_n(x) = \sum_{k=|m-n|}^{m+n} c_{m,n,k}\,p_k(x)/h_k.$$

Theorem (Askey, 1970)
$\forall n \quad a_n, b_n, a_{n+1} - a_n, b_{n+1} - b_n \ge 0 \Longrightarrow \forall m, n, k \quad c_{m,n,k} \ge 0.$

This covers: If $\alpha \ge \beta$ and $\alpha + \beta \ge 1$ then $c_{m,n,k}^{(\alpha,\beta)} \ge 0$.

Linearization of products (cntd.)

Remark 1. A function f on a group G is called *positive definite* if for all $x_1, \ldots, x_k \in G$ and all $c_1, \ldots, c_k \in \mathbb{C}$

$$\sum_{i,j=1}^{k} f(x_i x_j^{-1}) c_i \overline{c_j} \geq 0.$$

If (G, K) is a Gelfand pair with G, K compact and with spherical functions ϕ_λ then $\phi_\lambda \phi_\mu = \sum_\nu c_{\lambda,\mu,\nu} \phi_\nu$ with $c_{\lambda,\mu,\nu} \geq 0$.

Indeed, spherical functions are elementary positive definite functions, a product of positive definite functions is again positive definite, and a K-biinvariant positive definite function is a nonnegative linear combination of spherical functions.

Thus for special parameter values the theorems of Gasper and Askey also follow from group theory.

Linearization of products (cntd.)

Remark 2. K (1978):
An addition formula obtained for a spherical function on a Gelfand pair carries the essential information making it positive definite and leading to nonnegative linearization coefficients.

This last information is preserved in an addition formula for other parameter values which do not come from group theory. The addition formula needs to have certain properties. In particular, the expansion coefficients in the addition formula should be nonnegative. Then it implies the nonnegativity of the linearization coefficients.

This works in the Jacobi case for $\alpha \geq \beta \geq -\frac{1}{2}$.

An application to Laguere polynomials

This works also for disk polynomials.
If these are rewritten in terms of Jacobi polynomials and next the limit to the Laguerre case is taken then:

$$\int_0^\infty L_k^\alpha(x)\, L_m^\alpha(\lambda x)\, L_n^\alpha((1-\lambda)x)\, x^\alpha\, e^{-x}\, dx \geq 0 \quad (\alpha \geq 0,\ \lambda \in [0,1]).$$

By iteration:

$$\int_0^\infty L_{n_1}^\alpha(x)\, L_{n_2}^\alpha(x)\, L_{n_3}^\alpha(x)\, L_{n_4}^\alpha(x)\, x^\alpha\, e^{-2x}\, dx > 0 \quad (\alpha > 0).$$

This leads to the four boxes paper by Askey, Ismail & K (1978).

Connection formula

$$q_n(x) = \sum_{k=0}^{n} a_{n,k}\, p_k(x)/h_k$$

$$a_{n,k} = \int q_n(x)\, p_k(x)\, d\mu(x)$$

Connection formula (cntd.)

$$R_n^{(\gamma,\delta)}(x) = \sum_{k=0}^{n} a_{n,k}\, R_k^{(\alpha,\beta)}(x)/h_k^{(\alpha,\beta)} \Longrightarrow a_{n,k} = \text{stuff} \times {}_3F_2(1).$$

In particular, $a_{n,k}$ is elementary and nonnegative in the cases

$$R_n^{(\gamma,\gamma)}(x) = \sum_{k=0}^{n} a_{n,k}\, R_k^{(\alpha,\alpha)}(x)/h_k^{(\alpha,\alpha)} \quad (\gamma > \alpha > -1),$$

$$R_n^{(\gamma,\beta)}(x) = \sum_{k=0}^{n} a_{n,k}\, R_k^{(\alpha,\beta)}(x)/h_k^{(\alpha,\beta)} \quad (\gamma > \alpha > -1).$$

Askey & Gasper (1971) give sufficient conditions for nonnegativity of $a_{n,k} = a_{n,k}^{(\gamma,\delta),(\alpha,\beta)}$. For given (α, β) this includes an infinite region in the (γ, δ) plane bounded by three lines with $(\gamma, \delta) = (2\alpha + 1, 2\beta + 1)$ as one of the vertices.

Askey (1968): Certain of these positivity cases from isometric embeddings of projective spaces.

Connection formula (cntd.)

Nevai (1979): Connection coefficients for p_n in terms of Chebyshev polynomials T_k are limits of linearization coefficients for p_n.

Lasser (1994): Under certain assumptions positivity of linearization coefficients implies positivity of connection coefficients with Chebyshev.

Further work by Szwarc.

It seems that certain conditions on the coefficients in the three-term recurrence relation can identify a class of orthogonal polynomials giving rise to the dual case of the Chébli-Trimèche hypergroups.

De Branges' proof of the Bieberbach conjecture

Bieberbach (1916) conjectured that for a univalent function $F(z) = 1 + \sum_{n=2}^{\infty} a_n z^n$ on the unit disk there holds $|a_n| \leq n$. This was finally proved by Louis de Branges in Acta Math. (1985). During the preparation the last obstacle had been a proof of the inequality

$$
{}_3F_2\left(\begin{matrix} -n, n+\alpha+2, \frac{1}{2}(\alpha+1) \\ \alpha+1, \frac{1}{2}(\alpha+3) \end{matrix}; x\right) \geq 0
$$

$$
(0 \leq x \leq 1,\ n = 0, 1, 2, \ldots,\ \alpha = 2, 4, 6, \ldots).
$$

He had consulted Walter Gautschi, and Walter had called Dick Askey, who remembered that the inequality (for $\alpha \geq -2$) was in his paper with Gasper: *Positive Jacobi polynomial sums, II*, Amer. J. Math. (1976), and earlier in G. Gasper, *Positivity and special functions*, in: *Theory and application of special functions*, Academic Press, 1975.

Gasper's first proof of the inequality

Askey was interested when the following sum is nonnegative:

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \qquad (-1 \le x \le 1).$$

Gasper observed that for $\lambda = \beta = 0$

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(1-2x) = \frac{(\alpha+2)_n}{n!} \, {}_3F_2\left(\begin{matrix} -n, n+\alpha+2, \frac{1}{2}(\alpha+1) \\ \frac{1}{2}(\alpha+3), \alpha+1 \end{matrix}; x\right),$$

where the ${}_3F_2$ is close to Clausen's case

$${}_3F_2\left(\begin{matrix} -n, n+\alpha+1, \frac{1}{2}(\alpha+1) \\ \frac{1}{2}(\alpha+2), \alpha+1 \end{matrix}; x\right) = \left({}_2F_1\left(\begin{matrix} -\frac{1}{2}n, \frac{1}{2}(n+\alpha+1) \\ \frac{1}{2}(\alpha+2) \end{matrix}; x\right)\right)^2.$$

Gasper's first proof of the inequality (cntd.)

In fact, Gasper could expand the ${}_3F_2$ as a sum of Clausen ${}_3F_2$'s with nonnegative coefficients:

$$
{}_3F_2\left(\begin{matrix}-n, n+\alpha+2, \frac12(\alpha+1)\\ \frac12(\alpha+3), \alpha+1\end{matrix};x\right)
= \sum_{j=0}^{[\frac12 j]} c_{n,j}^{\alpha}\, {}_3F_2\left(\begin{matrix}-n+2j, n-2j+\alpha+1, \frac12(\alpha+1)\\ \frac12(\alpha+2), \alpha+1\end{matrix};x\right).
$$

by applying the operator $\quad f\mapsto \int_0^1 (t(1-t))^{\frac12(\alpha-1)} f(\cdot\, t)\,dt \quad$ to

$$
{}_2F_1\left(\begin{matrix}-n, n+\alpha+2,\\ \frac12(\alpha+3)\end{matrix};x\right) = \sum_{j=0}^{[\frac12 j]} c_{n,j}^{\alpha}\, {}_2F_1\left(\begin{matrix}-n+2j, n-2j+\alpha+1,\\ \frac12(\alpha+2)\end{matrix};x\right),
$$

i.e., to $\quad R_n^{(\frac12(\alpha+1),\frac12(\alpha+1))}(1-2x) = \sum_{j=0}^{[\frac12 n]} c_{n,j}^{\alpha}\, R_{n-2j}^{(\frac12\alpha,\frac12\alpha))}(1-2x)$

(connection formula for Gegenbauer polynomials).

Thank you, Dick,

and happy years to come with Liz, children and grandchildren.