Dual addition formula for Gegenbauer polynomials

Tom Koornwinder

Korteweg-de Vries Institute, University of Amsterdam
T.H.Koornwinder@uva.nl
https://staff.fnwi.uva.nl/t.h.koornwinder/

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Orthogonal polynomials

μ positive measure on $\mathbb{R}$ such that
$\forall n \in \mathbb{Z}_{\geq 0} \quad \int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty$.

For each $n \in \mathbb{Z}_{\geq 0}$ let $p_n(x)$ be polynomial of degree n such that

$$\int_{\mathbb{R}} p_m(x)\, p_n(x)\, d\mu(x) = 0 \quad (m \neq n) \tag{1}$$

(*orthogonal polynomials* with respect to μ). Then

$$x\, p_n(x) = A_n\, p_{n+1}(x) + B_n\, p_n(x) + C_n\, p_{n-1}(x) \tag{2}$$

(put $p_{-1}(x) = 0$) with $A_n C_n > 0$ (*3-term recurrence relation*).

Conversely, the 3-term recurrence (2) with $A_n C_n > 0$ and $p_{-1}(x) = 0$, $p_0(x) = 1$ generates a sequence of polynomials $\{p_n(x)\}_{n=0,1,\ldots}$ which satisfy (1) for some μ (*Favard theorem*).

classical orthogonal polynomials

We can write (2) as

$$J\,p_n(x) = x\,p_n(x)$$

where J is an operator (*Jacobi matrix*) acting on the variable n.
Can we dualize this? So this would be

$$L\,p_n(x) = \lambda_n\,p_n(x)$$

where L is an operator acting on the variable x.
Bochner (1929) classified the cases where L is a second order differential operator (*classical orthogonal polynomials*):

- $d\mu(x) = (1-x)^\alpha(1+x)^\beta dx$ on $[-1,1]$, $\alpha,\beta > -1$,
  $p_n(x) = P_n^{(\alpha,\beta)}(x)$, Jacobi polynomials
  (Gegenbauer polynomials for $\alpha = \beta$).
- $d\mu(x) = e^{-x}x^\alpha dx$ on $[0,\infty)$, $\alpha > -1$,
  $p_n(x) = L_n^{(\alpha)}(x)$, Laguerre polynomials.
- $d\mu(x) = e^{-x^2}\,dx$ on $(-\infty,\infty)$,
  $p_n(x) = H_n(x)$, Hermite polynomials.

bispectrality

$p_n(x)$ being a solution of two dual eigenvalue equations

$$J\,p_n(x) = x\,p_n(x), \quad L\,p_n(x) = \lambda_n\,p_n(x),$$

with J acting on the n-variable and L on the x-variable, is an example of *bispectrality*.
See Duistermaat & Grünbaum (1986) for continuous systems, with J and L both differential operators.
When sticking to OP's one can go beyond classical OP's if L is a higher order differential operator or a second order difference operator (W. Hahn, further extended in the (q-)Askey scheme).

Higher up in the Askey scheme one finds a more symmetric bispectrality than with the classical OP's.
Illustration: Jacobi polynomials as limits of Racah polynomials.

hypergeometric notation

Pochhammer symbol (shifted factorial):

$$(a)_k := a(a+1)\dots(a+k-1) \quad (k \in \mathbb{Z}_{>0}), \qquad (a)_0 := 1.$$

Some special hypergeometric series ($n \in \mathbb{Z}_{\geq 0}$):

$${}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; z\right) := \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{(c)_k\, k!}\, z^k.$$

$${}_4F_3\left(\begin{matrix}-n, a_2, a_3, a_4\\ b_1, b_2, b_3\end{matrix}; z\right) := \sum_{k=0}^{n} \frac{(-n)_k (a_2)_k (a_3)_k (a_4)_k}{(b_1)_k (b_2)_k (b_3)_k\, k!}\, z^k.$$

Racah polynomials

$$R_n(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)$$
$$:= {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix};1\right),\ \alpha \text{ or } \gamma=-N-1.$$

$$R_n(x(x-N+\delta);\alpha,\beta,-N-1,\delta)=R_x(n(n+\alpha+\beta+1);-N-1,\delta,\alpha,\beta)$$
$$(n,x=0,1,\ldots,N).$$

$$\sum_{x=0}^{N}(R_mR_n)(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)\,w_{\alpha,\beta,\gamma,\delta}(x)=h_{n;\alpha,\beta,\gamma,\delta}\,\delta_{m,n}\,.$$

Rescale $x\to Nx$ and $\delta\to\delta+N$ and let $N\to\infty$:

$$R_n(Nx(Nx+\delta);\alpha,\beta,-N-1,\delta+N)$$
$$= {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-Nx,Nx+\delta\\ \alpha+1,\beta+\delta+N+1,-N\end{matrix};1\right)$$
$$\to {}_2F_1\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix};x^2\right)=\mathrm{const.}\ P_n^{(\alpha,\beta)}(1-2x^2).$$

linearization formula and product formula

For OP's $p_n(x)$ there is a *linearization formula*

$$p_m(x)\,p_n(x) = \sum_{k=|m-n|}^{m+n} c_{m,n}(k)\,p_k(x).$$

In some cases we have explicit $c_{m,n}(k)$, in some cases we can prove that $c_{m,n}(k) \geq 0$.

Morally, in view of bispectrality, each explicit formula for classical OP's should have an explicit dual formula.
For a linearization formula the dual formula should be a *product formula*

$$p_n(x)\,p_n(y) = \int_{\mathbb{R}} p_n(z)\,d\nu_{x,y}(z)$$

for a suitable measure $\nu_{x,y}$.
Explicit? Positive? Support depending on x, y?

intermezzo: group theoretic background

Let G be a compact topological group (in particular compact Lie group or finite group). Let K be a closed subgroup of G such that, for any irreducible unitary representation of G on a finite dimensional complex vector space V with hermitian inner product, the space of K-fixed vectors in V has dimension at most 1. (Then (G, K) is called a *Gelfand pair*.)

Take such an irrep π of G on V with a K-fixed unit vector v in V. Then $\phi(g) := \langle \pi(g)v, v \rangle$ is called a *spherical function* on G with respect to K. Then

$$\phi(g_1)\phi(g_2) = \int_k \phi(g_1 k g_2)\, dk \ (dk \text{ normalized Haar measure on } K).$$

Also $\phi = \phi_\pi$ is a positive definite function on G, and therefore

$$\phi_\pi \phi_\rho = \sum_\sigma c_{\pi,\rho}(\sigma)\phi_\sigma \quad \text{with } c_{\pi,\rho}(\sigma) \geq 0.$$

For instance, for $G = SO(3)$, $K = SO(2)$, the $\phi_\pi(g)$ can be expressed in terms of Legendre polynomials.

Explicit formulas for Legendre polynomials $P_n(x)$

$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \quad (m \neq n), \qquad P_n(1) = 1.$

Linearization formula:

$$P_m(x)P_n(x) = \sum_{j=0}^{\min(m,n)} \frac{(\frac12)_j(\frac12)_{m-j}(\frac12)_{n-j}(m+n-j)!}{j!\,(m-j)!\,(n-j)!\,(\frac32)_{m+n-j}} \times \big(2(m+n-2j)+1\big)\,P_{m+n-2j}(x).$$

Product formula:

$$P_n(\cos\theta_1)P_n(\cos\theta_2) = \frac{1}{\pi}\int_0^\pi P_n(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi)\,d\phi,$$

or rewritten:

$$P_n(x)P_n(y) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}\,\sqrt{1-y^2}}^{\sqrt{1-x^2}\,\sqrt{1-y^2}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}}\,dz.$$

The first product formula is the constant term of a Fourier-cosine expansion of the integrand in terms of $\cos(k\phi)$. This expansion is called the *addition formula*.

Addition formula for Legendre polynomials

Addition formula:

$$
\begin{aligned}
&P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\\
&\quad = P_n(\cos\theta_1)P_n(\cos\theta_2)+2\sum_{k=1}^{n}\frac{(n-k)!\,(n+k)!}{2^{2k}(n!)^2}\\
&\qquad \times (\sin\theta_1)^k P_{n-k}^{(k,k)}(\cos\theta_1)\,(\sin\theta_2)^k P_{n-k}^{(k,k)}(\cos\theta_2)\cos(k\phi).
\end{aligned}
$$

Product formula:

$$P_n(\cos\theta_1)P_n(\cos\theta_2)=\frac{1}{\pi}\int_0^\pi P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\,d\phi.$$

Askey's question

Find an addition type formula corresponding to the linearization formula for Legendre polynomials just as the addition formula corresponds to the product formula.

A possible key for an answer

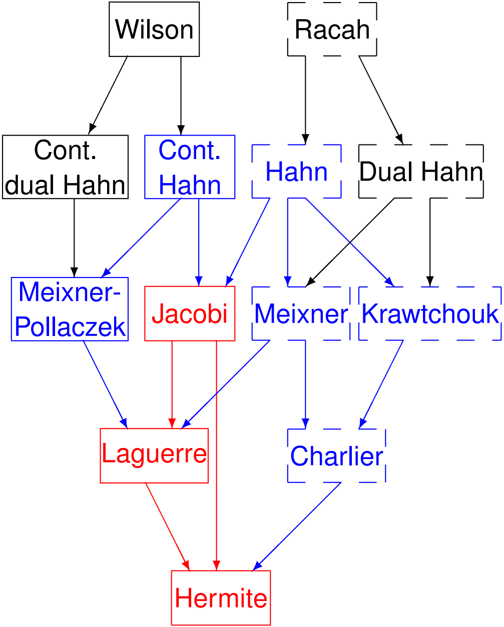
Chebyshev polynomials $T_n(\cos\phi) := \cos(n\phi)$. The $T_n(x)$ are OP's on [-1,1] with respect to the weight function $(1-x^2)^{-1/2}$.
The rewritten product formula

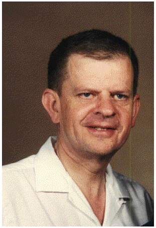
$$P_n(x)P_n(y) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}\,\sqrt{1-y^2}}^{\sqrt{1-x^2}\,\sqrt{1-y^2}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}}\,dz.$$

is the constant term of the Chebyshev expansion of $P_n(z+xy)$ in terms of $T_k\bigl(z(1-x^2)^{-1/2}(1-y^2)^{-1/2}\bigr)$, OP's with respect to the weight function $\bigl((1-x^2)(1-y^2)-z^2\bigr)^{-1/2}$ on the integration interval. This expansion is a rewriting of the addition formula. Can we recognize weights of discrete OP's in the coefficients of the linearization formula? So in the formula

$$P_l(x)P_m(x) = \sum_{j=0}^{\min(l,m)} \frac{(\frac12)_j(\frac12)_{l-j}(\frac12)_{m-j}(l+m-j)!}{j!\,(l-j)!\,(m-j)!\,(\frac32)_{l+m-j}} \times \bigl(2(l+m-2j)+1\bigr)\,P_{l+m-2j}(x).$$

Askey scheme





Dick Askey

discrete OP

quadratic lattice

Hahn class

very classical OP

A decisive hint

Jacobi functions:

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\begin{matrix}\frac12(\alpha+\beta+1+i\lambda),\frac12(\alpha+\beta+1-i\lambda)\\ \alpha+1\end{matrix};-\sinh^2 t\right).$$

Transform pair for suitable f or g $\quad(\alpha\ge\beta\ge-\frac12)$:

$$\begin{cases} g(\lambda)=\displaystyle\int_0^\infty f(t)\,\phi_\lambda^{(\alpha,\beta)}(t)\,(\sinh t)^{2\alpha+1}\,(\cosh t)^{2\beta+1}\,dt,\\ f(t)=\text{const.}\displaystyle\int_0^\infty g(\lambda)\,\phi_\lambda^{(\alpha,\beta)}(t)\,\frac{d\lambda}{|c(\lambda)|^2}\,. \end{cases}$$

Dual product formula for Jacobi functions ($\beta=-\frac12$) by Hallnäs & Ruijsenaars (2015) reveals weight function for Wilson polynomials with parameters $\pm i\lambda\pm i\mu+\frac12\alpha+\frac14$ (cases $\alpha=0$ and $\frac12$ due to Mizony, 1976): $\quad \phi_{2\lambda}^{(\alpha,-\frac12)}(t)\,\phi_{2\mu}^{(\alpha,-\frac12)}(t)$

$$=\text{const.}\int_0^\infty \phi_{2\nu}^{(\alpha,-\frac12)}(t)\left|\frac{\Gamma\left(i\nu\pm i\lambda\pm i\mu+\frac12\alpha+\frac14\right)}{\Gamma(2i\nu)}\right|^2 d\nu.$$

Linearization formula for Gegenbauer polynomials

Renormalized Jacobi polynomials $R_n^{(\alpha,\beta)}(x) := \dfrac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$.
Gegenbauer polynomials are Jacobi polynomials for $\alpha=\beta$.

Gegenbauer linearization formula (Rogers, 1895):

$$R_l^{(\alpha,\alpha)}(x)R_m^{(\alpha,\alpha)}(x)=\frac{l!\,m!}{(2\alpha+1)_l(2\alpha+1)_m}\sum_{j=0}^{\min(l,m)}\frac{l+m+\alpha+\frac12-2j}{\alpha+\frac12}$$
$$\times\frac{(\alpha+\frac12)_j(\alpha+\frac12)_{l-j}(\alpha+\frac12)_{m-j}(2\alpha+1)_{l+m-j}}{j!\,(l-j)!\,(m-j)!\,(\alpha+\frac32)_{l+m-j}}\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$
$$=\sum_{j=0}^{m}\frac{w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)}{h_{0;\,\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}}\,R_{l+m-2j}^{(\alpha,\alpha)}(x)\quad(l\ge m,\ \alpha>-\tfrac12),$$

where $w_{\alpha,\beta,\gamma,\delta}(x)$ are the weights for the Racah polynomials
$R_n\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)\quad(\gamma=-N-1,\,n=0,1,\ldots,N)$.

Racah polynomials

$$R_n\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)$$
$$:= {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix};1\right),\quad \gamma=-N-1,$$
$$\sum_{x=0}^{N}(R_mR_n)\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)\,w_{\alpha,\beta,\gamma,\delta}(x)=h_{n;\alpha,\beta,\gamma,\delta}\delta_{m,n}\,,$$

$$w_{\alpha,\beta,\gamma,\delta}(x)=\frac{\gamma+\delta+1+2x}{\gamma+\delta+1}\times\frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x(\delta+1)_x\,x!}\,,$$
$$\frac{h_{n;\,\alpha,\beta,\gamma,\delta}}{h_{0;\,\alpha,\beta,\gamma,\delta}}=\frac{\alpha+\beta+1}{\alpha+\beta+2n+1}\,\frac{(\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n\,n!}{(\alpha+1)_n(\alpha+\beta+1)_n(\beta+\delta+1)_n(\gamma+1)_n},$$
$$h_{0;\,\alpha,\beta,\gamma,\delta}=\sum_{x=0}^{N}w_{\alpha,\beta,\gamma,\delta}(x)=\frac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N}\,.$$

Racah coefficients of $R_{l+m-2j}^{(\alpha,\alpha)}(x)$

$$R_l^{(\alpha,\alpha)}(x)R_m^{(\alpha,\alpha)}(x)=\sum_{j=0}^{m}\frac{w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)}{h_{0;\,\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}}\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$

$(l\ge m,\ \alpha>-\frac12)$. More generally evaluate

$$S_n^\alpha(l,m):=\sum_{j=0}^{m}w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$
$$\times R_n\big(j(j-l-m-\alpha-\tfrac12);\,\alpha-\tfrac12,\alpha-\tfrac12,-m-1,-l-\alpha-\tfrac12\big).$$

By Racah Rodrigues formula, summation by parts, and a Gegenbauer difference formula we get

$$S_n^\alpha(l,m)=\frac{(2\alpha+1)_{l+n}(2\alpha+1)_{m+n}(\alpha+\frac12)_{l+m}}{2^{2n}(\alpha+\frac12)_l(\alpha+\frac12)_m(2\alpha+1)_{l+m}(\alpha+1)_n^2}\,(x^2-1)^n$$
$$\times R_{l-n}^{(\alpha+n,\alpha+n)}(x)\,R_{m-n}^{(\alpha+n,\alpha+n)}(x).$$

Then Fourier-Racah inversion gives:

Dual Gegenbauer addition formula

Theorem (Dual addition formula for Gegenbauer polynomials)

$$R^{(\alpha,\alpha)}_{l+m-2j}(x)=\sum_{n=0}^{m}\frac{\alpha+n}{\alpha+\frac12 n}\,\frac{(-l)_n(-m)_n(2\alpha+1)_n}{2^{2n}(\alpha+1)_n^2\,n!}$$
$$\times\,(x^2-1)^n\,R^{(\alpha+n,\alpha+n)}_{l-n}(x)\,R^{(\alpha+n,\alpha+n)}_{m-n}(x)$$
$$\times\,R_n\big(j(j-l-m-\alpha-\tfrac12);\,\alpha-\tfrac12,\alpha-\tfrac12,-m-1,-l-\alpha-\tfrac12\big)$$
$$(l\ge m,\;j=0,1,\ldots,m).$$

Compare with addition formula for Gegenbauer polynomials:

$$R^{(\alpha,\alpha)}_n(xy+z)=\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha+k}{\alpha+\frac12 k}\,\frac{(n+2\alpha+1)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2}\,(1-x^2)^{\frac12 k}$$
$$\times\,R^{(\alpha+k,\alpha+k)}_{n-k}(x)(1-y^2)^{\frac12 k}R^{(\alpha+k,\alpha+k)}_{n-k}(y)R_k^{(\alpha-\frac12,\alpha-\frac12)}\left(\frac{z}{\sqrt{1-x^2}\,\sqrt{1-y^2}}\right).$$

A common specialization

The two addition formulas have the common specialization

$$1=\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha+k}{\alpha+\frac{1}{2}k}\,\frac{(n+2\alpha+1)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2}\,(1-x^2)^k\big(R_{n-k}^{(\alpha+k,\alpha+k)}(x)\big)^2.$$

This implies that

$$|R_n^{(\alpha,\alpha)}(x)|\le 1\quad(-1\le x\le 1,\ \alpha>-\tfrac{1}{2}).$$

Commercial. My joint preprint with Aleksej Kostenko and Gerald Teschl: `arXiv:1602.08626`, *Jacobi polynomials, Bernstein-type inequalities and dispersion estimates for the discrete Laguerre operator*.

Addition formulas and inequalities as above play a certain role in this paper.

Further results

1. It turned out later that the dual addition formula for Gegenbauer polynomials is implicitly contained in formula (2.6) in Koelink, van Pruijssen & Román, Pub. RIMS Kyoto Univ. (2013).
2. The degenerate linearization formula is the well-known expansion of $R_n^{(\alpha,\alpha)}(\cos\theta)$ in terms of functions $\cos((n-2j)\theta)$. The corresponding degenerate dual addition formula involves Hahn polynomials.
3. The limit from Gegenbauer polynomials to Hermite polynomials can be applied to the dual addition formula for Gegenbauer polynomial in order to obtain a dual addition formula for Hermite polynomials.

Further perspective

1. Find dual addition formula for q-ultraspherical polynomials. Linearization formula also due to Rogers (1895). Probably q-Racah polynomials will pop up.
2. Find dual addition formula for Jacobi polynomials. Involves possibly a double summation, just as with the addition formula for Jacobi polynomials.
3. Find addition-type formula on a higher level which gives as limit cases for ultraspherical polynomials both the addition formula and the dual addition formula.
4. Find group theoretic interpretation of dual addition formula, for instance for $\alpha = \frac{1}{2}$ in connection with SU(2).

Some history: Jacobi

The orthogonal polynomials named after Carl Gustav Jacob **Jacobi** (1804–1851), and nowadays notated as $P_n^{(\alpha,\beta)}(x)$, first appeared in Jacobi's posthumous paper *J. Reine Angew. Math.* **56** (1859), 149–165.
This paper also observes the special case $\alpha = \beta$ (later called ultraspherical or Gegenbauer polynomials) and its simple generating function.

Moritz **Allé** (1837–1913) was an Austrian mathematician and astronomer who studied at the University of Vienna. He had successively positions at the observatory of Prague and at the *Technische Hochschulen* of Graz, Prague and Vienna. He published in *Sitz. Akad. Wiss. Wien Math. Natur. Kl. (Abt. II)* **51** (1865) on the ultraspherical polynomials, giving credit to Jacobi. In this paper he also derived the addition formula for these polynomials!

Some history: Leopold Gegenbauer

Leopold **Gegenbauer** (1849–1903) was an Austrian mathematician who was professor of Mathematics at the University of Vienna during 1893–1903. He often published in *Sitz. Akad. Wiss. Wien Math. Natur. Kl. (Abt. II)* (later *Abt. IIa*). His first paper on ultraspherical polynomials in Vol. **65** (1872) gives credit to Allé but not to Jacobi. He published proofs of the addition formula for these polynomials in Vol. **70** (1974) and Vol. **102** (1893), but he did not mention there Allé's earlier result.

Sources of pictures used:

- Askey: http://www.math.wisc.edu/~askey/
- Jacobi: https://en.wikipedia.org/wiki/Carl_Gustav_Jacob_Jacobi
- Allé: https://de.wikipedia.org/wiki/Moritz_All%C3%A9
- Gegenbauer: Acta Mathematica, Table Générale des Tomes 1–35 (1913).