Some more detailed comments to the book "A guide to quantum groups" by V. Chari and A. Pressley, Cambridge University Press, 1994, ISBN 0 521 43305 3

by Tom H. Koornwinder

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1. Left A-module algebra and left A-module coalgebra (p.109)

Let A be a bialgebra over over a commutative ring k with identity element. So $\lambda_V\colon A \to \mathrm{End}_k(V)$ is an algebra homomorphism and we write $a.v := \lambda_V(a)\,v$ ($a \in A$, $v \in V$). If W is another left A-module then $V \otimes W$ becomes a left A-module with

$$\lambda_{V\otimes W}(a) := (\lambda_V \otimes \lambda_W)(\Delta(a)), \quad \text{so} \quad a.(v\otimes w) = \sum_{(a)} (a_{(1)}.v) \otimes (a_{(2)}.w).$$

Also, k becomes a left A-module (the trivial A-module) by

$$\lambda_k(a) := \varepsilon(a), \quad \text{so} \quad a.\alpha := \varepsilon(a)\,\alpha.$$

I will give the definitions of left A-module algebra and left A-module coalgebra.

1.1. Left A-module algebra. V is moreover an algebra such that the following two diagrams commute for each $a \in A$:

$$\begin{array}{ccc} V\otimes V & \overset{\lambda_{V\otimes V}(a)}{\longrightarrow} & V\otimes V \\ \big\downarrow \mu_V & & \big\downarrow \mu_V \\ V & \overset{\lambda_V(a)}{\longrightarrow} & V \end{array} \qquad\qquad \begin{array}{ccc} k & \overset{\varepsilon(a)}{\longrightarrow} & k \\ \big\downarrow \iota_V & & \big\downarrow \iota_V \\ V & \overset{\lambda_V(a)}{\longrightarrow} & V \end{array}$$

Equivalently, $\mu_V\colon V\otimes V \to V$ and $\iota_V\colon k \to V$ are are left A-module homomorphisms. Another equivalent way to write this is by:

$$a.(vw) = \sum_{(a)} (a_{(1)}.v)\,(a_{(2)}.w), \qquad a.1_V = \varepsilon_A(a)\,1_V.$$

1.2. Left A-module coalgebra. V is moreover a coalgebra such that the following two diagrams commute for each $a \in A$:

$$\begin{array}{ccc} V & \overset{\lambda_V(a)}{\longrightarrow} & V \\ \big\downarrow \Delta_V & & \big\downarrow \Delta_V \\ V\otimes V & \overset{\lambda_{V\otimes V}(a)}{\longrightarrow} & V\otimes V \end{array} \qquad\qquad \begin{array}{ccc} V & \overset{\lambda_V}{\longrightarrow} & V \\ \big\downarrow \varepsilon_V & & \big\downarrow \varepsilon_V \\ k & \overset{\varepsilon(a)}{\longrightarrow} & k \end{array}$$

Equivalently, $\Delta_V\colon V \to V\otimes V$ and $\varepsilon_V\colon V \to k$ are left A-module homomorphisms. Another equivalent way to write this is by:

$$\Delta_V(a.v) = a.\Delta_V(v), \qquad \varepsilon_V(a.v) = \varepsilon_A(a)\,\varepsilon_V(v).$$

2. Quantum trace and quantum character (p.126)

Assume now that A is a Hopf algebra over a field k. Let all left A-modules under consideration be finite dimensional. Let V be a left A-module and write $V^* := \mathrm{Hom}_k(V,k)$. This gives a pairing between V^* and V:

$$\langle \xi, v\rangle := \xi(v) \quad (v \in V,\ \xi \in V^*).$$

Then V^* becomes a left A-module by

$$\lambda_{V^*}(a) := (\lambda_V(S(a))^*, \quad \text{so} \quad \langle a.\xi, v\rangle = \langle \xi, S(a).v\rangle.$$

V and V^{**} can be naturally identified as k-modules. However, for the left A-module structures we have

$$\langle \xi, \lambda_{V^{**}}(a)\, v\rangle = \langle \lambda_{V^*}(S(a))\, \xi, v\rangle = \langle \xi, \lambda_V(S^2(a))\, v\rangle.$$

So

$$\lambda_{V^{**}}(a) = \lambda_V(S^2(a)).$$

From now on we assume that there is an invertible element $u \in A$ such that $S^2(a) = uau^{-1}$. By Proposition 4.2.3 the element $u := \mu(S \otimes \mathrm{id})(R_{21})$ satisfies this property if A is an almost cocommutative Hopf algebra as in Definition 4.2.1.

Now it follows that the following diagram is commutative for each $a \in A$:

$$\begin{array}{ccc} V & \xrightarrow{\lambda_V(a)} & V \\ \big\downarrow \lambda_V(u) & & \big\downarrow \lambda_V(u) \\ V^{**} & \xrightarrow{\lambda_{V^{**}}(a)} & V^{**} \end{array}$$

Identify $W \otimes V^*$ and $\mathrm{Hom}_k(V,W)$ as k-modules such that $w \otimes \xi \in W \otimes V^*$ corresponds with $\langle \xi, .\,\rangle\, w \in \mathrm{Hom}_k(V,W)$. The k-module structure of $W \otimes V^*$ is carried by this identification to $\mathrm{Hom}_k(V,W)$. We obtain

$$(\lambda_{W\otimes V^*}(a)\, f)(v) = (a.f)(v) = \sum_{(a)} a_{(1)}.\, f(S(a_{(2)}).v) \quad (f \in \mathrm{Hom}_k(V,W),\ a \in A,\ v \in V).$$

The adjoint representation of A on A, denoted by ad, is defined by

$$\mathrm{ad}(a)\, b := \sum_{(a)} a_{(1)}\, b\, S(a_{(2)}) \quad (a, b \in A).$$

Now the following diagram commutes for each $a \in A$:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{ad}(a)} & A \\ \big\downarrow \lambda_V & & \big\downarrow \lambda_V \\ \mathrm{End}_k(V) & \xrightarrow{\lambda_{V\otimes V^*}(a)} & \mathrm{End}_k(V) \end{array} \qquad (2.1)$$

So $\lambda_V\colon A \to \mathrm{End}_k(V)$ is an intertwining operator for the representations ad on A and $\lambda_{V\otimes V^*}$ on $\mathrm{End}_k(V)$. For the proof note that

$$(a.\lambda_V(b))(v) = \sum_{(a)} a_{(1)}.\,\lambda_V(b)(S(a_{(2)}).v) = \sum_{(a)} \lambda_V(a_{(1)}bS(a_{(2)}))\,v = \lambda_V(\mathrm{ad}(a)\,b)\,v.$$

The mapping $\mathrm{tr}\colon \xi\otimes v \mapsto \langle \xi, v\rangle \colon V^*\otimes V \to k$ is a homomorphism of left A-modules:

$$\mathrm{tr}\,(a.(\xi\otimes v)) = \varepsilon(a)\,\langle \xi, v\rangle = \varepsilon(a)\,\mathrm{tr}\,(\xi\otimes v).$$

However, the mapping $\mathrm{tr}\colon v\otimes\xi \mapsto \langle \xi, v\rangle\colon V\otimes V^* \to k$ is generally not a homomorphism of left A-modules. Under the identification of $V\otimes V^*$ and $\mathrm{End}_k(V)$ the mapping $\mathrm{tr}\colon V^*\otimes V \to k$ is carried to the usual trace mapping from $\mathrm{End}_k(V)$ to k, but this mapping will neither be a homomorphism of left A-modules in general.

As k-modules we can identify $V\otimes V^*$ and $V^{**}\otimes V^*$, but they are generally different as left A-modules. We have

$$\lambda_{V^{**}\otimes V^*}(a)\,(v\otimes\xi) = \sum_{(a)} (S^2(a_{(1)}).v)\otimes(a_{(2)}.\xi).$$

Hence

$$\begin{aligned}\mathrm{tr}\,\big(\lambda_{V^{**}\otimes V^*}(a)\,(v\otimes\xi)\big) &= \sum_{(a)}\langle a_{(2)}.\xi, S^2(a_{(1)}).v\rangle = \sum_{(a)}\langle \xi, \lambda_V\big(S(a_{(2)}\,S^2(a_{(1)})\big)\,v\rangle \\ &= \sum_{(a)}\langle \xi, \lambda_V\big(S(S(a_{(1)})\,a_{(2)})\big)\,v\rangle = \langle \xi, \lambda_V(\varepsilon(a)1\,v\rangle = \varepsilon(a)\langle \xi, v\rangle = \varepsilon(a)\,\mathrm{tr}\,(v\otimes\xi).\end{aligned}$$

We conclude that the mapping $\mathrm{tr}\colon V^{**}\otimes V^* \to k$ is a homomorphism of left A-modules.

Now identify $\mathrm{End}_k(V)$ and $V^{**}\otimes V^*$ as k-modules. Carrying the left A-module structure of $V^{**}\otimes V^*$ to $\mathrm{End}(V)$ yields

$$(\lambda_{V^{**}\otimes V^*}(a)\,f)(v) = \sum_{(a)} S^2(a_{(1)}).\,f(S(a_{(2)}).v) \quad (f\in \mathrm{End}_k(V),\ a\in A,\ v\in V).$$

Then $\mathrm{tr}\colon \mathrm{End}_k(V)\to k$ intertwines the representations $\lambda_{V^{**}\otimes V^*}$ on $\mathrm{End}_k(V)$ and ε on k. So the following diagram is commutative for each $a\in A$:

$$\begin{array}{ccc} \mathrm{End}_k(V) & \overset{\lambda_{V^{**}\otimes V^*}(a)}{\longrightarrow} & \mathrm{End}_k(V) \\ \big\downarrow \mathrm{tr} & & \big\downarrow \mathrm{tr} \\ k & \overset{\varepsilon(a)}{\longrightarrow} & k \end{array} \tag{2.2}$$

Let $u\in A$ be as before. Then

$$\begin{aligned}\lambda_V(u)\,(\lambda_{V\otimes V^*}(a)\,f)\,(v) = \sum_{(a)} u.a_{(1)}.\,f(S(a_{(2)}).\,v) &= \sum_{(a)} S^2(a_{(1)}).u.\,f(S(a_{(2)}),\,v) \\ &= \lambda_{V^{**}\otimes V^*}(a)\,(\lambda_V(u)\,f(v)).\end{aligned}$$

Hence the following diagram commutes for each $a \in A$:

$$\begin{array}{ccc} \mathrm{End}_k(V) & \xrightarrow{\lambda_{V\otimes V^*}(a)} & \mathrm{End}_k(V) \\ \Big\downarrow \lambda_V(u)\,. & & \Big\downarrow \lambda_V(u)\,. \\ \mathrm{End}_k(V) & \xrightarrow{\lambda_{V^{**}\otimes V^*}(a)} & \mathrm{End}_k(V) \end{array} \tag{2.3}$$

Here $\lambda_v(u)\,.$ means left multiplication by $\lambda_V(u)$ in $\mathrm{End}_k(V)$.

Combination of the diagrams (2.1), (2.3) and (2.2) yields the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{ad}(a)} & A \\ \Big\downarrow \lambda_V & & \Big\downarrow \lambda_V \\ \mathrm{End}_k(V) & \xrightarrow{\lambda_{V\otimes V^*}(a)} & \mathrm{End}_k(V) \\ \Big\downarrow \lambda_V(u)\,. & & \Big\downarrow \lambda_V(u)\,. \\ \mathrm{End}_k(V) & \xrightarrow{\lambda_{V^{**}\otimes V^*}(a)} & \mathrm{End}_k(V) \\ \Big\downarrow \mathrm{tr} & & \Big\downarrow \mathrm{tr} \\ k & \xrightarrow{\varepsilon(a)} & k \end{array}$$

Define the *quantum trace*, *quantum character* and *quantum dimension* by

$$\begin{aligned} \mathrm{qtr}_V(f) &:= \mathrm{tr}\,(\lambda_V(u)\,f) \quad (f \in \mathrm{End}_k(V)), \\ \mathrm{qch}_V(b) &:= \mathrm{qtr}_V(\lambda_V(b)) = \mathrm{tr}\,(\lambda_V(ub)) \quad (b \in A), \\ \mathrm{qdim}(V) &:= \mathrm{qch}_V(1) = \mathrm{qtr}_V(I_V) = \mathrm{tr}\,(\lambda_V(u)). \end{aligned}$$

Then the following two diagrams commute for each $a \in A$:

$$\begin{array}{ccc} \mathrm{End}_k(V) & \xrightarrow{\lambda_{V\otimes V^*}(a)} & \mathrm{End}_k(V) \\ \Big\downarrow \mathrm{qtr}_V & & \Big\downarrow \mathrm{qtr}_V \\ k & \xrightarrow{\varepsilon(a)} & k \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\mathrm{ad}(a)} & A \\ \Big\downarrow \mathrm{qch}_V & & \Big\downarrow \mathrm{qch}_V \\ k & \xrightarrow{\varepsilon(a)} & k \end{array}$$

In particular, we have

$$\mathrm{qch}_V(\mathrm{ad}(a)\,b) = \varepsilon(a)\,\mathrm{qch}_V(b) \quad (a, b \in A).$$

Now assume that the element u satisfis moreover:

$$\Delta(u) = u \otimes u.$$

Then:

$$\begin{aligned} \lambda_{V\otimes W}(u) &= \lambda_V(u) \otimes \lambda_W(u) \\ \mathrm{qtr}_{V\otimes W}(f \otimes g) &= \mathrm{qtr}_V(f)\,\mathrm{qtr}_W(g), \\ \mathrm{qch}_{V\otimes W}(b) &= \mathrm{qch}_V(b)\,\mathrm{qch}_W(b), \\ \mathrm{qdim}(V \otimes W) &= \mathrm{qdim}(V)\,\mathrm{qdim}(W). \end{aligned}$$

So the quantum trace, quantum character and quantum dimension then have properties quite similar to their classical analogues.

Let A be quasitriangular and take $u := \mu(S \otimes \mathrm{id})(R_{21})$. If A is triangular then $\Delta(u) = u \otimes u$. Otherwise, A can be enlarged with a certain central element v such that $v^2 = u\,S(u)$ by which $v^{-1}u$ will have the required properties (cf. §4.2C).

If A is the Hopf $*$-algebra for a Woronowicz compact matrix group (or more generally a Dijkhuizen-Koornwinder CQG-algebra) then there is an invertible element u in the dual A° of A such that, for each irreducible unitary corepresentation λ_V of A, the operator $\lambda_V(u)$ intertwines λ_V and $\lambda_{V^{**}}$ and satisfies $\mathrm{tr}\,(\lambda_V(u)) = \mathrm{tr}\,(\lambda_V(u^{-1})) > 0$. This element u will have the required properties in A°. See §2.4 in the following reference:

T. H. Koornwinder, *Compact quantum groups and q-special functions*, in *Representations of Lie groups and quantum groups*, V. Baldoni & M. A. Picardello (eds.), Pitman Research Notes in Mathematics Series 311, Longman Scientific & Technical, 1994, pp. 46–128.

3. The inversion map on a Poisson-Lie group is an anti-Poisson map (p.21)

(personal communication by A. Pressley to T. H. Koornwinder)

In the Warning on p.21 it is stated that the inversion map ι on a Poisson-Lie group G satisfies

$$\{f_1 \circ \iota, f_2 \circ \iota\} = -\{f_1, f_2\} \circ \iota$$

for all $f_1, f_2 \in C^\infty(G)$. Here follows a proof. We have

$$\begin{aligned}\{f_1 \circ \iota, f_2 \circ \iota\}(g) &= \langle w_g, d(f_1 \circ \iota)_g \otimes d(f_2 \circ \iota)_g \rangle \\ &= \langle (\iota'_g \otimes \iota'_g)(w_g), (df_1)_{g^{-1}} \otimes (df_2)_{g^{-1}} \rangle.\end{aligned}$$

Differentiating the identity

$$\iota = L_{g^{-1}} \circ \iota \circ R_{g^{-1}}$$

at g, and noting that $\iota'_e = -\mathrm{id}$, gives

$$\begin{aligned}\iota'_g &= -(L_{g^{-1}})'_e (R_{g^{-1}})'_g \\ &= -[(L_g)'_{g^{-1}}]^{-1} (R_{g^{-1}})'_g.\end{aligned}$$

where the last equation was obtained by differentiating the identity $L_{g^{-1}} \circ L_g = \mathrm{id}$ at g^{-1}. So

$$(\iota'_g \otimes \iota'_g)(w_g) = \Big([(L_g)'_{g^{-1}}]^{-1} (R_{g^{-1}})'_g \otimes [(L_g)'_{g^{-1}}]^{-1} (R_{g^{-1}})'_g \Big)(w_g),$$

which, by taking $g' = g^{-1}$ in formula (8) on p.22, we see is exactly $-w_{g^{-1}}$. Thus,

$$\{f_1 \circ \iota, f_2 \circ \iota\}(g) = -\langle (df_1)_{g^{-1}} \otimes (df_2)_{g^{-1}}, w_{g^{-1}} \rangle = -\{f_1, f_2\}(g^{-1}).$$