Comment on the paper "A remarkable identity involving Bessel functions" by D. E. Dominici, P. M. W. Gill and T. Limpanuparb,
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Note by Tom H. Koornwinder, `T.H.Koornwinder@uva.nl`, *March 11, 2011*

A more conceptual proof of Corollary 1 is obtained by observing that for $f,g\in L^2([-\pi,\pi])$ we have

$$2\pi\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx=\int_{-\infty}^{\infty}\widehat{f}(y)\,\overline{\widehat{g}(y)}\,dy=\sum_{n=-\infty}^{\infty}\widehat{f}(n)\,\overline{\widehat{g}(n)},$$

where

$$\widehat{f}(y):=\int_{-\pi}^{\pi} f(x)\,e^{-ixy}\,dx.$$

Apply this to

$$f(x):=(1-x^2/a^2)^{\mu-k-\frac12}\,C_k^{\mu-k}(x/a)\quad(-a<x<a),$$
$$g(x):=(1-x^2/b^2)^{\overline{\nu}-\ell-\frac12}\,C_\ell^{\overline{\nu}-\ell}(x/b)\quad(-b<x<b),$$

and $f(x):=0$ outside $(-a,a)$, $g(x):=0$ outside $(-b,b)$. Assume that $a,b\in(0,\pi]$ and that the nonnegative integers k,ℓ satisfy $k<\operatorname{Re}\mu$ and $\ell<\operatorname{Re}\nu$. Then

$$\int_{-\infty}^{\infty} t^{k+\ell}\;{}_0F_1\left({-\atop \mu+1};-\tfrac14 a^2t^2\right)\;{}_0F_1\left({-\atop \nu+1};-\tfrac14 b^2t^2\right)dt$$
$$=\sum_{n=-\infty}^{\infty} n^{k+\ell}\;{}_0F_1\left({-\atop \mu+1};-\tfrac14 a^2n^2\right)\;{}_0F_1\left({-\atop \nu+1};-\tfrac14 b^2n^2\right).$$

By analytic continuation this remains valid and convergent for $k+l<\operatorname{Re}\mu+\operatorname{Re}\nu$. For $\mu+\nu=k+\ell+1$ we obtain the first equality in Corollary 2. Note that we need for this special case that $\mu+\nu$ is integer.