Comment on the paper "Macdonald polynomials and algebraic integrability" by O. A. Chalykh

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Chalykh computed in [1, §4.1] a Baker-Akhiezer (BA) function for root system A_1. It is given by his formulas (4.1), (4.4) and (4.6). Let me write his resulting BA function $\psi = \psi(x,z;m)$, depending on a parameter $m \in \mathbb{Z}_{\ge 0}$, as a ${}_2\phi_1$:

$$\psi(x,z;m) = (-1)^m \, q^{-\frac{3}{2}m^2-\frac{1}{2}m} \, q^{(x-m)(z-m)} \, (q^{2x+2};q^2)_m \; {}_2\phi_1\left(\begin{matrix} q^{-2m}, q^{-2m+2x} \\ q^{2x+2} \end{matrix} ; q^2, q^{2(m+z+1)}\right). \tag{1}$$

Observe the following properties of ψ.

- We have
$$\psi(x,z;m) = \psi(-x,-z;m). \tag{2}$$
Indeed, for generic x the ${}_2\phi_1$ in (1) can be written as
$$\sum_{j=0}^{m} \frac{(q^{-2m};q^2)_j \, (q^{-2m+2x};q^2)_j}{(q^{2x+2};q^2)_j \, (q^2;q^2)_j} \, (q^{2(m+z+1)})^j = \sum_{j=0}^{m} c_j = \sum_{j=0}^{m} c_{m-j},$$
and then the right-hand side can again be written as a ${}_2\phi_1$, see [3, (1.8)] (or more generally [2, Exercise 1.4(ii)]). Then the resulting identity can be rewritten as (2).

- We have
$$\psi(x,z;m) = \psi(x,-z;m) \qquad (x = 1,\ldots,m). \tag{3}$$
Indeed, now write the ${}_2\phi_1$ in (1) as
$$\sum_{j=0}^{m-x} c_j = \sum_{j=0}^{m-x} c_{m-x-j},$$
and write the right-hand side again as a ${}_2\phi_1$. Then the resulting identity can be rewritten as (3).

- We have
$$\psi(x,z;m) = \psi(z,x;m). \tag{4}$$
Indeed, (4) is a rewritten version of the transformation formula [2, (III.2)].

- We have
$$\psi(x,z;m) = \psi(x,-z;m) \qquad (z = 1,\ldots,m). \tag{5}$$
Indeed, combine (3), (2) and (5).

- Note that we can use (1) as a definition of $\psi(x,z;m)$ for general integer m by the convention

$$(q^{2x+2};q^2)_m = \frac{(q^{2x+2};q^2)_\infty}{(q^{2x+2m+2};q^2)_\infty}\,.$$

We have

$$\psi(x,z;-m) = -\frac{q^{(2m-1)(x+z)}}{(q^{2x-2m+2};q^2)_{2m-1}\,(q^{2z-2m+2};q^2)_{2m-1}}\,\psi(x,z;m-1) \quad (m\in\mathbb{Z}_{>0}). \quad (6)$$

Indeed, (6) is a rewritten version of [2, (III.3)]. Note that $\psi(x,z;-m)$ ($m\in\mathbb{Z}_{>0}$) has poles for $x\in\{1,2,\ldots,m-1\}$ and for $z\in\{1,2,\ldots,m-1\}$, by which there are no immediate analogues of (3) and (5) for $\psi(x,z;-m)$. However, (2) and (4) remain valid for negative integer m.

With ψ given by (1) we can now review [1, Proposition 4.2, formula (4.10) and Lemma 5.4]:

1. ψ is a normalized BA function. Indeed, (1) has the form [1, (4.1)]:

$$\psi(x,z;m) = q^{xz}\sum_{j=0}^{m}\psi_{-m+2j}(x;m)\,q^{(-m+2j)z},$$

and it satisfies the normalization condition [1, (4.10)]:

$$\psi_m(x;m) = \prod_{j=1}^{m}(q^{j-x}-q^{-j+x}).$$

The function ψ also satisfies condition [1, (4.2)], since this condition can be rewritten as (5).

2. ψ is symmetric in x and z, see (4).

3. ψ satisfies the difference equation

$$\frac{q^{x-m}-q^{-x+m}}{q^x-q^{-x}}\,\psi(x+1,z;m) + \frac{q^{x+m}-q^{-x-m}}{q^x-q^{-x}}\,\psi(x-1,z;m) = (q^z+q^{-z})\,\psi(x,z;m). \quad (7)$$

Indeed, (7) combined with (4) is a rewritten version of [2, Exercise 1.13], see also [3, p.15 below].

4. ψ satisfies [1, Lemma 5.4]:

$$\psi(wx,wz;m) = \psi(x,z;m) \qquad (w\in W).$$

Indeed, this turns down to (2).

Next I will discuss [1, Theorem 5.11] for root system A_1. First we identify Macdonald polynomials for that root system with continuous q-ultraspherical polynomials. Following (2.10), (2.11), (2.1) and (2.13) in [1] the Macdonald polynomial then has the form

$$P_n(x;q,t)=\sum_{j=0}^{n} f_{n-2j}\,q^{(n-2j)x},$$

where $f_{n-2j}=f_{2j-n}$ and $f_n=1$, and it satisfies

$$D\,P_n=(q^n t+q^{-n}t^{-1})P_n,$$

where D is the difference operator given by

$$(Df)(x):=\frac{tq^x-t^{-1}q^{-x}}{q^x-q^{-x}}\,f(x+1)+\frac{tq^{-x}-t^{-1}q^{x}}{q^x-q^{-x}}\,f(x-1).$$

By the reasoning in [4, (9.10) and following] we obtain

$$P_n(x;q,t)=\frac{(q^2;q^2)_n}{(t^2;q^2)_n}\,C_n(\tfrac12(q^x+q^{-x});t^2\mid q^2), \tag{8}$$

where C_n is a continuous q-ultraspherical polynomial, see [2, (7.4.2)]:

$$C_n(\cos\theta;t\mid q):=\sum_{j=0}^{n}\frac{(t;q)_j(t;q)_{n-j}}{(q;q)_j(q;q)_{n-j}}\,e^{i(n-2j)\theta}.$$

Now observe [2, (7.4.5)]:

$$C_n(\cos\theta;t\mid q)=\frac{(t;q)_\infty}{(t^2;q)_\infty}\,\frac{(t^2;q)_n}{(q;q)_n}\Big(\widetilde{D}_n(e^{i\theta};t\mid q)+\widetilde{D}_n(e^{-i\theta};t\mid q)\Big), \tag{9}$$

where

$$\widetilde{D}_n(e^{i\theta};t\mid q):=e^{in\theta}\,\frac{(te^{-2i\theta};q)_\infty}{(e^{-2i\theta};q)_\infty}\;{}_2\phi_1\!\left(\begin{matrix}t,\,te^{2i\theta}\\ qe^{2i\theta}\end{matrix};q,q^{n+1}\right). \tag{10}$$

(Note that, by [2, (III.1)], the ${}_2\phi_1$ in (10) coincides with the ${}_2\phi_1$ for $S_n(e^{i\theta};t^{1/2},(qt)^{1/2},-t^{1/2},-(qt)^{1/2})$ in [5, (3.4)].) Observe that

$$\widetilde{D}_{n+m}(q^x;q^{-2m}\mid q^2)=q^{mn}q^{-\frac12 m(m+1)}\psi(x,n;m). \tag{11}$$

After replacing in (9) $q,t,e^{i\theta},n$ by $q^2,q^{-2m},q^x,n+m$, respectively, we arrive at the first identity in [1, Theorem 5.11] for A_1:

$$\psi(x,n;m)+\psi(-x,n;m)=\prod_{j=1}^{m}(q^{j-n}-q^{-j+n})\,P_{n+m}(x;q,q^{-m}). \tag{12}$$

For the derivation of the second identitity in [1, Theorem 5.11] for A_1 we apply [2, (III.3)] to (10). Then we obtain in particular the identity

$$\widetilde{D}_{n-m-1}(q^x;q^{2m+2}\mid q^2)=\frac{1}{(q^{2n-2m};q^2)_{2m+1}}\,\frac{\widetilde{D}_{n+m}(q^x;q^{-2m}\mid q^2)}{\prod_{j=-m}^{m}(q^{j-x}-q^{-j+x})}\,.$$

Substitution in (9) yields

$$(q^2;q^2)_m\prod_{j=-m}^{m}(q^{j-x}-q^{-j+x})\,C_{n-m-1}(\tfrac12(q^x+q^{-x});q^{2m+2}\mid q^2)\\ =\widetilde{D}_{n+m}(q^x;q^{-2m}\mid q^2)-\widetilde{D}_{n+m}(q^{-x};q^{-2m}\mid q^2).$$

By substitution of (8) and (11) we arrive at [1, Theorem 5.11] for A_1:

$$\psi(x,n;m)-\psi(x,-n;m)=\prod_{j=1}^{m}(q^{j-n}-q^{-j+n})\prod_{j=-m}^{m}(q^{j-x}-q^{-j+x})\,P_{n-m-1}(x;q,q^{m+1}).\qquad(13)$$

References

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