Indag. Math. 51(1989), 97-11?

MATHEMATICS Proceedings A 92 (1), March 20, 1989

Representations of the twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials

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Communicated by Prof. T.A. Springer at the meeting of October 31, 1988

ABSTRACT

The matrix elements of the irreducible unitary representations of the twisted $SU(2)$ quantum group are computed explicitly. It is shown that they can be identified with two different classes of q-hypergeometric orthogonal polynomials: with the little q-Jacobi polynomials and with certain q-analogues of Krawtchouk polynomials. The orthogonality relations for these polynomials correspond to Schur type orthogonality relations in the first case and to the unitarity conditions for the representations in the second case. The paper also contains a new proof of Woronowicz' classification of the unitary irreducible representations of this quantum group. It avoids infinitesimal methods. Symmetries of the matrix elements of the irreducible unitary representations are proved without using the explicit expressions.

1. INTRODUCTION

Many special functions of hypergeometric type admit some group theoretic interpretation, see Vilenkin's book [26]. It is somewhat annoying that the big impetus of the last ten years in the study of q-hypergeometric series and corresponding orthogonal polynomials was not matched by a large number of group theoretic interpretations for these special functions. The only exceptions were some q-Hahn and q-Krawtchouk polynomials living on Chevalley groups, cf. Stanton [24] for a survey, and the q-ultraspherical polynomials [2] with $q=0$, for which Cartier [5] gave an interpretation as spherical functions on homogeneous trees and also on certain homogeneous spaces of SL_2 over a p-adic number field.

The quantum groups, recently introduced by Drinfeld [7] and Woronowicz [27], which are not groups in the proper sense, but which provide highly nontrivial deformations of compact semisimple Lie groups, offer a very promising area for interpretations of q-hypergeometric series. Indeed, in the present paper we prove the q-analogue of the well-known fact that the matrix elements of the irreducible unitary representations of $SU(2)$ are expressible in terms of Jacobi polynomials (cf. for instance [26, Ch. 3]) and of Krawtchouk polynomials (cf. [20, § 12.7] and [15, § 2]). Instead of $SU(2)$ we have now the quantum group $S_\mu U(2)$, which was studied in detail by Woronowicz [28], and the matrix elements will be expressed in terms of little q-Jacobi polynomials and of q-Krawtchouk polynomials, which were introduced by Hahn [10, p. 29] (class II and a special case of class III); note that the q-Krawtchouk polynomials are different from the ones discussed in Stanton [24]. The orthogonality relations for the little q-Jacobi polynomials (derived in Askey and Andrews [1]) turn out to be equivalent to the Schur type orthogonality relations for the matrix elements of the irreducible representations of $S_\mu U(2)$ and the orthogonality relations for the q-Krawtchouk polynomials (derivable from Askey and Wilson [3]) express the fact that the matrix representations of $S_\mu U(2)$ under consideration are unitary.

These new interpretations of q-orthogonal polynomials described above can really be productive for obtaining new results about these polynomials which would have been hard to find without such an interpretation. In a forthcoming paper we will obtain an addition formula for the little q-Legendre polynomials, just as the classical addition formula for Legendre polynomials can be derived from the expression of the matrix elements of the irreducible representations of $SU(2)$ in terms of Jacobi polynomials.

It can be expected that every known connection between special functions and compact semisimple Lie groups will have a q-analogue in the context of (compact) quantum groups. For this a necessary tool will be some explicit representation theory of quantum groups which are deformations of compact semisimple Lie groups. For $S_\mu U(n)$ this was done by Woronowicz [29], while for the dual of a quantum group (in the sense of Hopf algebras) associated with an arbitrary root system the representation theory is also available, cf. Jimbo [12], [13], Rosso [22] and Lusztig [17]. It seems that the compact quantum groups do not provide a setting for interpretations of q-Wilson polynomials and their specializations (cf. Askey and Wilson [4]) or their multi-variable analogues (cf. Macdonald [18]). It is my feeling (also motivated by [16]) that they will rather live on noncompact quantum groups (deformations of noncompact semisimple Lie groups).

When I obtained the explicit expression of the matrix elements of the irreducible unitary representations of $S_\mu U(2)$ in terms of little q-Jacobi polynomials, I was not yet aware that essentially the same result also occurs in the paper [25] by Vaksman and Soibelman and the announcement [19] by Masuda e.a., cf. Remarks 5.6 and 5.7. However, there are some features which make the present paper still useful. In contrast with [25] we emphasize non-infinitesimal methods. Symmetries of matrix elements and unitariness of representations are derived by the use of generating functions for the matrix elements, before these matrix elements are explicitly computed. The present paper also provides an alternative proof of Woronowicz' [28] classification of the irreducible unitary representations of $S_\mu U(2)$. The q-Krawtchouk result was probably not yet observed elsewhere. Finally, we point out that $S_\mu U(2)$ provides a nice setting for a q-analogue with noncommuting variables of the binomial formula (cf. Lemma 2.1 and the proofs of (4.12) and Proposition 5.2) and for an operational identity by Jackson (cf. Lemma 5.1 and (5.4)).

The contents of this paper are as follows. In § 2 we treat the necessary preliminaries about q-hypergeometric functions and related orthogonal polynomials. In § 3 we briefly recall the theory for $SU(2)$ as it will be generalized for $S_\mu U(2)$. Next, in section 4 we introduce a class of matrix representations of $S_\mu U(2)$, derive generating functions for their matrix elements, obtain symmetries for these matrix elements and prove that the representations are unitary and irreducible. In § 5 we obtain explicit expressions for the matrix elements in terms of little q-Jacobi polynomials and show that we have a complete system of inequivalent irreducible unitary representations of $S_\mu U(2)$. Finally, in § 6 we express the matrix elements in terms of q-Krawtchouk polynomials.

W.A. Al-Salam kindly helped me with some historical information concerning the q-Krawtchouk polynomials.

NOTATION. Throughout this paper $\mathbb{Z}_+$ will denote the set of nonnegative integers and $\mathbb{N}$ the set of positive integers. For real x, y we will denote by $x \wedge y$ the minimum and by $x \vee y$ the maximum of x and y.

2. SOME q-HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

Let $1 \neq q \in \mathbb{C}$. For $a \in \mathbb{C}$, $k \in \mathbb{Z}_+$ define the *q-shifted factorial* by

$$(2.1) \qquad (a;q)_k := \prod_{j=0}^{k-1} (1-aq^j).$$

If $|q|<1$ then this definition extends to the case $k=\infty$:

$$(2.2) \qquad (a;q)_\infty := \prod_{j=0}^{\infty} (1-aq^j).$$

The product of n q-shifted factorials $(a_j;q)_k$ $(j=1,\ldots,n)$ will also be written as $(a_1,\ldots,a_n;q)_k$. For integers n,k such that $n \geq k \geq 0$, the *q-combinatorial coefficient* is defined by

$$(2.3) \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n;q^{-1})_k}{(q;q)_k} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q .$$

Observe that

$$(2.4) \qquad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_q ,$$

while a simple computation yields the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q. \tag{2.5}$$

By complete induction with respect to the degree it can now be proved that:

LEMMA 2.1. Let x and y be indeterminates satisfying the relation

$$xy = qyx. \tag{2.6}$$

Then

$$\begin{cases} (x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} \\ \qquad\quad\; = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} x^k y^{n-k}. \end{cases} \tag{2.7}$$

For $r \in \mathbb{Z}_+$ the q-hypergeometric series of type ${}_{r+1}\phi_r$ is formally defined by

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_{r+1}; q)_k z^k}{(b_1; q)_k \cdots (b_r; q)_k (q; q)_k}. \tag{2.8}$$

Its radius of convergence equals 1 for generic values of the parameters. If $a_1 = q^{-n}$ $(n \in \mathbb{Z}_+)$ and $b_1, \ldots, b_r \neq 1, q^{-1}, \ldots, q^{-n}$ then the right hand side of (2.8) is a well-defined terminating series, with summation running over $k = 0, 1, \ldots, n$. A good reference for q-hypergeometric series is the forthcoming book by Gasper and Rahman [9].

Some elementary identities are the terminating q-binomial formula

$$ {}_1\phi_0(q^{-n}; -; q, z) = (q^{-n}z; q)_n, \tag{2.9}$$

a q-analogue of the Chu-Vandermonde formula

$$ {}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(b^{-1}c; q)_n}{(c; q)_n} b^n, \tag{2.10}$$

Heine's q-analogue of Euler's transformation formula

$$ {}_2\phi_1(a, b; c; q, z) = \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1(c/a, c/b; c; q, abz/c) \tag{2.11}$$

(all given in [9, Chapter 1]) and an identity for terminating ${}_2\phi_1$ series obtained by inverting the direction of summation:

$$\begin{cases} {}_2\phi_1(q^{-n}, b; c; q, z) \\ \quad = (-1)^n q^{-1/2n(n+1)} \dfrac{(b; q)_n}{(c; q)_n} z^n \, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-n+1}c^{-1} \\ q^{-n+1}b^{-1} \end{matrix}; q, \dfrac{q^{1+n}c}{bz}\right). \end{cases} \tag{2.12}$$

The *little q-Jacobi polynomials*

$$p_n(x; a, b|q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) \tag{2.13}$$

occurred as part of a classification of orthogonal polynomials satisfying q-difference equations by Hahn [10]. Their properties as orthogonal polynomials were worked out in more detail by Askey and Andrews [1]. They introduced the notation and definition as in (2.13); slight variations of this can be found in later papers. If $0<q<1$ and $a=q^\alpha$, $b=q^\beta$ with $\alpha, \beta>-1$ then the orthogonality relations can be written down in an elegant way as

$$\begin{cases} \dfrac{1}{B_q(\alpha+1,\beta+1)} \displaystyle\int_0^1 p_n(t; q^\alpha, q^\beta|q) p_m(t; q^\alpha, q^\beta|q) t^\alpha \frac{(qt; q)_\infty}{(q^{\beta+1}t; q)_\infty} d_q t \\ = \dfrac{q^{n(\alpha+1)}(1-q^{\alpha+\beta+1})(q^{\beta+1}; q)_n (q; q)_n}{(1-q^{2n+\alpha+\beta+1})(q^{\alpha+1}; q)_n (q^{\alpha+\beta+1}; q)_n} \delta_{n,m}, \end{cases} \tag{2.14}$$

where the q-integral is defined by

$$\int_0^1 f(t) d_q t := \sum_{k=0}^\infty f(q^k)(q^k - q^{k+1}) \tag{2.15}$$

and

$$B_q(\alpha+1,\beta+1) := \int_0^1 t^\alpha \frac{(qt; q)_\infty}{(q^{\beta+1}t; q)_\infty} d_q t = \frac{(1-q)(q; q)_\infty (q^{\alpha+\beta+2}; q)_\infty}{(q^{\alpha+1}; q)_\infty (q^{\beta+1}; q)_\infty} \tag{2.16}$$

denotes the q-beta function.

Later in this paper we will meet the little q-Jacobi polynomials in disguised form, as a ${}_3\phi_2$ series which can be brought back to (2.13) by an identity of Jackson

$$(q^{-n}bz/c; q)_n \, {}_3\phi_2\left(\begin{matrix} q^{-n}, c/b, 0 \\ c, cqb^{-1}z^{-1} \end{matrix}; q, q\right) = {}_2\phi_1(q^{-n}, b, c; q, z), \tag{2.17}$$

cf. [9, (3.2.4)]. Let us sketch an elementary proof of (2.17). The left hand side can be written as

$$(-1)^n q^{-1/2n(n+1)} (bc^{-1}z)^n \sum_{k=0}^n \frac{(q^{-n}; q)_k (c/b; q)_k q^k}{(c; q)_k (q; q)_k} \left(\frac{cq^{k+1}}{bz}; q\right)_{n-k}$$

Now apply (2.9) to

$$\left(\frac{cq^{k+1}}{bz}; q\right)_{n-k}$$

and interchange summations. Then we get

$$(-1)^n q^{-1/2(n+1)} (bc^{-1})^n \times \sum_{l=0}^n \sum_{k=0}^{n-l} \frac{(q^{-n}, c/b; q)_k}{(c, q; q)_k} q^k \frac{(q^{-(n-k)}; q)_l}{(q; q)_l} (cq^{n+1}/b)^l z^{n-l}.$$

Now substitute $l=n-m$ and recognize the summation over k as

$$\sum_{k=0}^{m} \frac{(q^{-m};q)_k(c/b;q)_k}{(c;q)_k(q;q)_k} q^k,$$

which can be evaluated by (2.10). The identity (2.17) now readily follows.

We will also need the *q-Krawtchouk polynomials*

$$(2.18)\qquad K_n(x)=K_n(x;b,N|q):={}_2\phi_1\left(\begin{matrix}q^{-n},x\\ q^{-N}\end{matrix};q,bq^{n+1}\right),$$

where $N\in\mathbb{Z}_+$ and $n\in\{0,1,\ldots,N\}$. These polynomials are the case $\varrho=-N$ of the q-Meixner polynomials introduced by Hahn as class III in [10, p. 29]. The q-Krawtchouk polynomials just defined should not be confused with D. Stanton's q-Krawtchouk polynomials introduced in [23, § 3] and Ph. Delsarte's affine q-Krawtchouk polynomials in [6, (16)], see also [24, § 4]. However all those variants of q-Krawtchouk polynomials can be obtained by specialization or limit transition from the *q-Hahn polynomials*

$$(2.19)\qquad Q_n(x)=Q_n(x;a,b,N|q):={}_3\phi_2\left(\begin{matrix}q^{-n},abq^{n+1},x\\ aq,q^{-N}\end{matrix};q,q\right),$$

where $N\in\mathbb{Z}_+$ and $n\in\{0,1,\ldots,N\}$. These polynomials were also introduced in Hahn [10] and their orthogonality relations

$$(2.20)\qquad \left\{\begin{aligned}&\sum_{x=0}^{N} Q_m(q^{-x})Q_n(q^{-x})\frac{(aq;q)_x(bq;q)_{N-x}}{(q,q)_x(q;q)_{N-x}}(aq)^{-x}\\ &=\frac{(abq^2;q)_N(aq)^{-N}}{(q;q)_N}\frac{(1-abq)(q,bq,abq^{N+2};q)_n}{(1-abq^{2n+1})(aq,abq,q^{-N};q)_n}\\ &\times(-aq)^n q^{1/2n(n-1)-Nn}\delta_{m,n}\end{aligned}\right.$$

were obtained by Askey and Andrews in unpublished work. Askey and Wilson [3] derived orthogonality relations for the more general class of q-Racah polynomials and they observed that these imply (2.20), see also [9, (7.2.22)]. It follows from (2.18) and (2.19) that

$$(2.21)\qquad K_n(x;b,N|q)=\lim_{a\to\infty} Q_n(x;a,b,N|q).$$

Hence we obtain from (2.20) that

$$(2.22)\qquad \left\{\begin{aligned}&\sum_{x=0}^{N} K_m(q^{-x})K_n(q^{-x})\frac{(bq;q)_{N-x}(-1)^{N-x}q^{1/2x(x-1)}}{(q;q)_{N-x}(q;q)_x}\\ &=\frac{(-1)^n b^N(q,bq;q)_n(q;q)_{N-n}}{(q,q;q)_N}q^{1/2N^2+Nn-1/2n^2+1/2N-1/2n}\delta_{m,n}.\end{aligned}\right.$$

Note that the weights are positive if $0<q<1$ and $b>q^{-N}$. The q-Krawtchouk polynomials are related to the little q-Jacobi polynomials by the formula

$$(2.23)\qquad \begin{cases} p_n(q^x;q^\alpha,q^\beta\,|\,q)=(-q^{x+1})^n q^{-1/2n(n+1)}\dfrac{(q^{n+\alpha+\beta+1};q)_n}{(q^{\alpha+1};q)_n} \\ \times K_n(q^{-n-\alpha};q^{-n-\beta-x-1},2n+\alpha+\beta\,|\,q), \end{cases}$$

where $\alpha,\beta\in\mathbb{Z}_+$. This follows from the identity (2.12).

3. REPRESENTATION THEORY OF $SU(2)$

The results summarized in this section are well-known, cf. for instance Vilenkin [26, Ch. 3], but we include this material since our proofs in the case of $S_\mu U(2)$ will be analogous to the ones sketched here

Fix $l\in\{0,\frac{1}{2},1,\dots\}$. Let t^l be the representation of $GL(2,\mathbb{C})$ on the vector space $\mathscr{H}_l$ of homogeneous polynomials of degree $2l$ in two complex variables defined by

$$(3.1)\qquad \left(t^l\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}f\right)(x,y):=f(\alpha x+\gamma y,\beta x+\delta y),\ f\in\mathscr{H}_l.$$

For $\mathscr{H}_l$ we take a basis $\{\psi_n^l\}_{n=-l,-l+1,\dots,l}$ defined by

$$(3.2)\qquad \psi_n^l(x,y):=\binom{2l}{l-n}^{1/2}x^{l-n}y^{l+n}.$$

Let t^l have matrix elements $t_{m,n}^l$ with respect to this basis:

$$(3.3)\qquad t^l(g)\psi_n^l=\sum_{m=-l}^{l}t_{m,n}^l(g)\psi_m^l,\ g\in GL(2,\mathbb{C}).$$

Then

$$(3.4)\qquad \begin{cases} \dbinom{2l}{l-n}^{1/2}(\alpha x+\gamma y)^{l-n}(\beta x+\delta y)^{l+n} \\ =\displaystyle\sum_{m=-l}^{l}t_{m,n}^l\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\binom{2l}{l-m}^{1/2}x^{l-m}y^{l+m}. \end{cases}$$

If we multiply both sides of the generating function (3.4) with

$$\binom{2l}{l-n}^{1/2}\xi^{l-n}\eta^{l+n}$$

and sum over n then we obtain the double generating function

$$(3.5)\qquad \begin{cases} (\alpha x\xi+\beta x\eta+\gamma y\xi+\delta y\eta)^{2l} \\ =\displaystyle\sum_{m,n=-l}^{l}t_{m,n}^l\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\binom{2l}{l-m}^{1/2}\binom{2l}{l-n}^{1/2}x^{l-m}y^{l+m}\xi^{l-n}\eta^{l+n}. \end{cases}$$

Formula (3.5) immediately implies the symmetry

$$(3.6)\qquad t^l_{m,n}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=t^l_{n,m}\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix},$$

while (3.4) yields

$$(3.7)\qquad t^l_{m,n}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=t^l_{-m,-n}\begin{pmatrix}\delta & \gamma\\ \beta & \alpha\end{pmatrix}.$$

From (3.6) and (3.7) we obtain a third symmetry

$$(3.8)\qquad t^l_{m,n}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=t^l_{-n,-m}\begin{pmatrix}\delta & \beta\\ \gamma & \alpha\end{pmatrix}.$$

Next we show that the representation t^l, restricted to $SU(2)$, is unitary if $\mathscr{H}_l$ is endowed with the hermitian inner product for which the basis $\{\psi^l_n\}$ is orthonormal. So we have to show that

$$\overline{t^l_{m,n}(g)}=t^l_{n,m}(g^{-1}),\ g\in SU(2),$$

i.e.,

$$(3.9)\qquad \overline{t^l_{m,n}\begin{pmatrix}\alpha & -\bar{\gamma}\\ \gamma & \bar{\alpha}\end{pmatrix}}=t^l_{n,m}\begin{pmatrix}\bar{\alpha} & \bar{\gamma}\\ -\gamma & \alpha\end{pmatrix},$$

where $\alpha,\gamma\in\mathbb{C}$, $|\alpha|^2+|\gamma|^2=1$. Since, by (3.4), $t^l_{m,n}$ is polynomial with real coefficients in $\alpha,\beta,\gamma,\delta$, formula (3.9) can be equivalently written as

$$(3.10)\qquad t^l_{m,n}\begin{pmatrix}\bar{\alpha} & -\gamma\\ \bar{\gamma} & \alpha\end{pmatrix}=t^l_{n,m}\begin{pmatrix}\bar{\alpha} & \bar{\gamma}\\ -\gamma & \alpha\end{pmatrix}.$$

Now (3.10) follows from (3.6).

The irreducibility of the representations t^l of $SU(2)$ is seen as follows. Since

$$t^l\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}\psi^l_n=e^{-2in\theta}\psi^l_n,$$

the representation t^l restricted to its diagonal subgroup $U(1)$ splits uniquely as a direct sum of inequivalent one-dimensional irreducible representations. Suppose that t^l is not an irreducible representation of $SU(2)$. Then there will be a proper linear subspace L of $\mathscr{H}_l$ which is irreducible under $SU(2)$ and which contains ψ^l_l and there will be some ψ^l_m orthogonal to ψ^l_l. But then $t^l_{m,l}(g)=0$ for all $g\in SU(2)$. However, this gives a contradiction since we see from (3.4) that

$$t^l_{m,l}\begin{pmatrix}\alpha & -\bar{\gamma}\\ \gamma & \bar{\alpha}\end{pmatrix}=\binom{2l}{l-m}^{1/2}(-\bar{\gamma})^{l-m}\bar{\alpha}^{l+m}.$$

From (3.4) we can derive an explicit expression for the matrix elements $t^l_{m,n}$:

$$
(3.11)\quad \begin{cases} t^l_{m,n}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \binom{2l}{l-m}^{-1/2}\binom{2l}{l-n}^{1/2} \\ \quad\times \sum_{i=0\vee(-m-n)}^{(l-n)\wedge(l-m)} \binom{l-n}{i}\binom{l+n}{l-m-i}\alpha^i\beta^{l-m-i}\gamma^{l-n-i}\delta^{m+n+i}. \end{cases}
$$

For each choice of the signs of $n-m$ and $n+m$ this can be written in terms of a terminating hypergeometric series and next in terms of Jacobi polynomials, cf. for instance [14, (2.10)], or in terms of Krawtchouk polynomials, cf. [20, § 12.7] and [15, § 2]. It will also follow from these explicit expressions that the matrix elements $t^l_{m,n}$, considered as functions on $SU(2)$, form a basis of the algebra of polynomials on $SU(2)$, so this yields the inequivalence and completeness of the representations t^l of $SU(2)$. Schur's orthogonality representations for the matrix elements $t^l_{m,n}$ on $SU(2)$ turn out to be equivalent to the orthogonality relations for Jacobi polynomials and the unitariness of the matrix representations t^l can be seen to be equivalent to the orthogonality relations for Krawtchouk polynomials.

4. A CLASS OF REPRESENTATIONS OF THE QUANTUM GROUP $S_\mu U(2)$

In the rest of this paper we fix $0\neq\mu\in[-1,1]$. Sometimes the cases $\mu=\pm 1$ have to be interpreted by taking a limit. Let A be the unital C^*-algebra generated by the two elements α and γ satisfying the relations

$$
(4.1)\quad \alpha^*\alpha+\gamma^*\gamma=I,\ \alpha\alpha^*+\mu^2\gamma\gamma^*=I,\ \gamma\gamma^*=\gamma^*\gamma,\ \mu\gamma\alpha=\alpha\gamma,\ \mu\gamma^*\alpha=\alpha\gamma^*.
$$

Let

$$
(4.2)\quad u:=\begin{pmatrix}\alpha & -\mu\gamma^*\\ \gamma & \alpha^*\end{pmatrix}.
$$

As shown by Woronowicz [27], [28], (A,u) is a compact matrix pseudogroup (briefly called quantum group) $S_\mu U(2)$, which can be identified with the compact group $SU(2)$ if $\mu=1$. In particular, the comultiplication is the unital C^*-algebra homomorphism $\Phi: A\to A\otimes A$ such that

$$
(4.3)\quad \Phi(\alpha)=\alpha\otimes\alpha-\mu\gamma^*\otimes\gamma,\ \Phi(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma
$$

(cf. [28, (1.13)]).

Let $\mathscr{A}$ be the unital $*$-subalgebra of A generated by the matrix elements of u. Fix $l\in\{0,\frac{1}{2},1,\dots\}$. Then, for $n\in\{-l,-l+1,\dots,l\}$ we have

$$
(4.4)\quad \begin{cases} \Phi\left(\begin{bmatrix}2l\\ l-n\end{bmatrix}_{\mu^{-2}}^{1/2}\alpha^{l-n}\gamma^{l+n}\right) \\ \quad = \begin{bmatrix}2l\\ l-n\end{bmatrix}_{\mu^{-2}}^{1/2}(\alpha\otimes\alpha-\mu\gamma^*\otimes\gamma)^{l-n}(\gamma\otimes\alpha+\alpha^*\otimes\gamma)^{l+n} \\ \quad = \sum_{m=-l}^{l} t^{l,\mu}_{n,m}\otimes\begin{bmatrix}2l\\ l-m\end{bmatrix}_{\mu^{-2}}^{1/2}\alpha^{l-m}\gamma^{l+m} \end{cases}
$$

for certain $t^{l,\mu}_{n,m} \in \mathscr{A}$. The first identity follows from (4.3). For the second identity observe that, in each term obtained from the decomposition of the left hand side, the second factor of the tensor product is a $2l$-fold product of factors α and γ, which can be rewritten as const. $\alpha^{l-m}\gamma^{l+m}$ for some m in view of the relation $\mu\gamma\alpha = \alpha\gamma$, cf. (4.1). The q-combinatorial coefficients (cf. (2.3)) are inserted for cosmetic reasons, as will dbecome clear later in this section. If we apply $\mathrm{id} \otimes \Phi$ and $\Phi \otimes \mathrm{id}$ to (4.4) and use the coassociativity

$$(4.5) \qquad (\mathrm{id} \otimes \Phi) \circ \Phi = (\Phi \otimes \mathrm{id}) \circ \Phi$$

(cf. [27, (1.7)]) then we obtain

$$(4.6) \qquad \Phi t^{l,\mu}_{n,k} = \sum_{m=-l}^{l} t^{l,\mu}_{n,m} \otimes t^{l,\mu}_{m,k},$$

so, by [27, (2.4)], the matrix $(t^{l,\mu}_{n,m})_{n,m=-l,\ldots,l}$ is a representation $t^{l,\mu}$ of $S_\mu U(2)$ on $\mathbb{C}^{2l+1}$.

In (4.1) substitute

$$(4.7) \qquad \delta := \alpha^*, \ \beta := -\mu\gamma^*.$$

Then the relations (4.1) can be rewritten as

$$(4.8) \qquad \begin{cases} \alpha\beta = \mu\beta\alpha, \ \alpha\gamma = \mu\gamma\alpha, \ \gamma\delta = \mu\delta\gamma, \ \beta\delta = \mu\delta\beta, \\ \alpha\delta - \mu^2\delta\alpha = (1-\mu^2)I, \ \beta\gamma = \gamma\beta, \ \alpha\delta - \mu\beta\gamma = I. \end{cases}$$

Let $\mathscr{A}(\alpha, \beta, \gamma, \delta)$ denote the unital algebra of polynomials in the non-commuting variables $\alpha, \beta, \gamma, \delta$ with relations (4.8). Consider also the unital algebras $\mathscr{A}(\xi, \eta)$ and $\mathscr{A}(x, y)$ generated by ξ, η respectively x, y with relations

$$(4.9) \qquad \xi\eta = \mu\eta\xi,$$

$$(4.10) \qquad xy = \mu yx.$$

Now the second identity in (4.4) can be equivalently written as

$$(4.11) \qquad \begin{cases} \begin{bmatrix} 2l \\ l-n \end{bmatrix}_{\mu^{-2}}^{1/2} (\alpha \otimes \xi + \beta \otimes \eta)^{l-n} (\gamma \otimes \xi + \delta \otimes \eta)^{l+n} \\ \quad = \displaystyle\sum_{m=-l}^{l} \begin{bmatrix} 2l \\ l-m \end{bmatrix}_{\mu^{-2}}^{1/2} t^{l,\mu}_{n,m} \otimes \xi^{l-m}\eta^{l+m}. \end{cases}$$

Formula (4.11) is an identity between two elements in $\mathscr{A}(\alpha, \beta, \gamma, \delta) \otimes \mathscr{A}(\xi, \eta)$. For each n it is a generating function with non-commuting variables for the matrix elements $t^{l,\mu}_{n,m}$ $(m = -l, -l+1, \ldots, l)$. It will be possible to compute these matrix elements explicitly from this generating function.

From (4.11) we can derive an identity which generates all $t^{l,\mu}_{n,m}$ $(m, n = -l, -l+1, \ldots, l)$ simultaneously. Tensor to the left both sides of (4.11) with

$$\begin{bmatrix} 2l \\ l-n \end{bmatrix}_{\mu^{-2}}^{1/2} x^{l-n} y^{l+n}$$

and sum both sides over n from $-l$ to l. Now observe that Lemma 2.1 can be applied with x, y being replaced by $x\otimes(\alpha\otimes\xi+\beta\otimes\eta)$ and $y\otimes(\gamma\otimes\xi+\delta\otimes\eta)$, respectively, and q by μ^2. Thus we obtain

$$(4.12)\quad \begin{cases} (x\otimes\alpha\otimes\xi+x\otimes\beta\otimes\eta+y\otimes\gamma\otimes\xi+y\otimes\delta\otimes\eta)^{2l} \\ \quad = \sum_{m,n=-l}^{l} \begin{bmatrix} 2l \\ l-n \end{bmatrix}_{\mu^{-2}}^{1/2} \begin{bmatrix} 2l \\ l-m \end{bmatrix}_{\mu^{-2}}^{1/2} x^{l-n}y^{l+n}\otimes t_{n,m}^{l,\mu}\otimes\xi^{l-m}\eta^{l+m}, \end{cases}$$

which is an identity in $\mathscr{A}(x,y)\otimes\mathscr{A}(\alpha,\beta,\gamma,\delta)\otimes\mathscr{A}(\xi,\eta)$.

It is clear from (4.8) that interchange of the generators β,γ generates an isomorphism of $\mathscr{A}(\alpha,\beta,\gamma,\delta)$ and interchange of α and δ generates an anti-isomorphism of this algebra. By abuse of notation we write $a(\alpha,\beta,\gamma,\delta)$ for some $a\in\mathscr{A}(\alpha,\beta,\gamma,\delta)$ with a specific algebraic expression in the noncommuting generators $\alpha,\beta,\gamma,\delta$ and $\tilde{a}(\alpha,\beta,\gamma,\delta)$ for the same expression with the order of the factors in each term inverted. Then we can denote the isomorphism which interchanges β and γ by

$$a(\alpha,\beta,\gamma,\delta)\mapsto a(\alpha,\gamma,\beta,\delta)$$

and the anti-isomorphism which interchanges α and δ by

$$a(\alpha,\beta,\gamma,\delta)\mapsto \tilde{a}(\delta,\beta,\gamma,\alpha).$$

PROPOSITION 4.1. The $t_{m,n}^{l,\mu}$ satisfy the following symmetry relations:

$$(4.13)\quad t_{n,m}^{l,\mu}(\alpha,\beta,\gamma,\delta)=t_{m,n}^{l,\mu}(\alpha,\gamma,\beta,\delta)$$

$$(4.14)\quad =(t_{-n,-m}^{l,\mu})^{\sim}(\delta,\gamma,\beta,\alpha)$$

$$(4.15)\quad =(t_{-m,-n}^{l,\mu})^{\sim}(\delta,\beta,\gamma,\alpha).$$

PROOF. Clearly, (4.15) follows from (4.13) and (4.14). For the proof of (4.13) reverse the ordering of the tensor products in (4.12) and interchange x with ξ, y with η. (By (4.9), (4.10) this preserves the relations.) Now interchange m with n and compare the new obtained identity with (4.12).

For the proof of (4.14) apply the anti-isomorphism

$$a(\alpha,\beta,\gamma,\delta,\xi,\eta)\mapsto a(\delta,\gamma,\beta,\alpha,\eta,\xi)$$

of $\mathscr{A}(\alpha,\beta,\gamma,\delta)\otimes\mathscr{A}(\xi,\eta)$ to both sides of (4.11) and replace n by $-n$, m by $-m$. Then the left hand side of (4.11) is preserved, so (4.14) follows. □

Next we will show that the matrix representations $t^{l,\mu}$ are unitary. By the definition of unitary representation (cf. [27, p. 650]) we have to prove the identities

$$(4.16)\quad \sum_{n=-l}^{l} t_{n,k}^{l,\mu\,*}t_{n,m}^{l,\mu}=\delta_{k,m}I$$

and

$$(4.17)\qquad \sum_{n=-l}^{l} t_{k,n}^{l,\mu} t_{m,n}^{l,\mu\,*} = \delta_{k,m} I.$$

If we compare with [27, (4.27), (4.28)] then we obtain

$$(4.18)\qquad t_{m,n}^{l,\mu\,*} = \kappa(t_{n,m}^{l,\mu}),\ n, m = -l, \ldots, l,$$

as a necessary and sufficient condition for unitariness of t^l. Here κ is the unital linear antimultiplicative mapping of $\mathscr{A}$ to $\mathscr{A}$ such that

$$(4.19)\qquad \kappa(\alpha) = \alpha^*,\ \kappa(-\mu\gamma^*) = \gamma^*,\ \kappa(\gamma) = -\mu\gamma,\ \kappa(\alpha^*) = \alpha,$$

cf. [28, (1.14)]. For the proof of (4.18) note that, by (4.11) and (4.8), $t_{n,m}^{l,\mu}$ can be written as a polynomial with real coefficients in the noncommuting variables $\alpha, \beta, \gamma, \delta$. So, in view of (4.7) we have

$$(4.20)\qquad (t_{m,n}^{l,\mu}(\alpha, \beta, \gamma, \delta))^* = (t_{m,n}^{l,\mu})^{\sim}(\delta, -\mu\gamma, -\mu^{-1}\beta, \alpha)$$

and

$$(4.21)\qquad \kappa(t_{n,m}^{l,\mu}(\alpha, \beta, \gamma, \delta)) = (t_{n,m}^{l,\mu})^{\sim}(\delta, -\mu^{-1}\beta, -\mu\gamma, \alpha).$$

Now (4.18) follows from (4.20), (4.21) and (4.13).

Next we give a proof that the representations $t^{l,\mu}$ are irreducible. For this we need the concept of a *quantum subgroup* of a quantum group $G := (A, u)$. (In the following we freely use the definitions as given in [27].) This is a quantum group $H := (B, v)$ together with a unital C^*-homomorphism $\pi : A \to B$ such that π is surjective and the comultiplications of G and H are related by

$$(4.22)\qquad \Phi_H \circ \pi = (\pi \otimes \pi) \circ \Phi_G.$$

Now it is easily seen that, if t^G is a matrix representation of G then t^H defined by

$$t_{i,j}^H := \pi t_{i,j}^G$$

is a matrix representation of H, which is unitary if t^G is unitary. Furthermore, each t^G-invariant subspace of the representation space of t^G is also t^H-invariant. If $\mathscr{A}$ and $\mathscr{B}$ denote the unital *-subalgebras generated by the $u_{i,j}$ and $v_{i,j}$, respectively, then it can be shown that π maps $\mathscr{A}$ onto $\mathscr{B}$ and that $\mathscr{B}$ is the unital *-subalgebra of B generated by the $\pi u_{i,j}$.

Now let $G := S_\mu U(2)$ and let H be the quantum group corresponding to the unit circle $\mathbb{T}$ in $\mathbb{C}$ considered as multiplicative group. Then $B = C(\mathbb{T})$. Let $\pi : A \to B$ be the unital C^*-homomorphism generated by

$$(\pi\alpha)(z) := z,\ (\pi\gamma)(z) := 0\ (z \in \mathbb{T}).$$

This makes H into a quantum subgroup of $S_\mu U(2)$ (as was earlier pointed out by Podles [21]). We find from (4.11) and (4.7) that

$$(\pi t_{n,m}^{l,\mu})(z) = \delta_{n,m} z^{-2n},\ z \in \mathbb{T}.$$

So, if $\{e_l, e_{l-1}, \ldots, e_{-l}\}$ is the standard orthonormal basis of $\mathbb{C}^{2l+1}$ with respect to which we consider the matrix representation $t^{l,\mu}$, then $\pi t^{l,\mu}$ uniquely splits as a direct sum of inequivalent representations of $\mathbb{T}$ on the subspaces $\mathbb{C}e_j$. Now we can prove the irreducibility of $t^{l,\mu}$. Consider the linear subspace L of $\mathbb{C}^{2l+1}$ which is irreducible under $t^{l,\mu}$ and which contains e_l. (This subspace L exists.) Then

$$\sum_{m=-l}^{l} t^{l,\mu}_{m,l} \otimes e_m \in A \otimes L.$$

Suppose L would not contain some of the e_m. Then L would be orthogonal to this e_m and, hence, $t^{l,\mu}_{m,l}=0$. But it can be derived from (4.4), Lemma 2.1 and (4.13) that

$$t^{l,\mu}_{m,l} = \begin{bmatrix} 2l \\ l-m \end{bmatrix}^{1/2}_{\mu^{-2}} (-\mu\gamma^*)^{l-m} \alpha^{*l+m},$$

which is clearly nonzero. This establishes the irreducibility of $t^{l,\mu}$.

5. THE MATRIX ELEMENTS $t^{l,\mu}_{n,m}$ EXPRESSED AS LITTLE q-JACOBI POLYNOMIALS

Let $\mathscr{A}(\alpha, \beta, \gamma, \delta \,|\, \mu)$ be the unital algebra generated by $\alpha, \beta, \gamma, \delta$ under relations (4.8). Note that the mapping

$$a(\alpha, \beta, \gamma, \delta) \mapsto a(\delta, \beta, \gamma, \alpha) : \mathscr{A}(\alpha, \beta, \gamma, \delta \,|\, \mu) \to \mathscr{A}(\alpha, \beta, \gamma, \delta \,|\, \mu^{-1})$$

is an algebra isomorphism.

LEMMA 5.1. Under relations (4.8) we have for $k \in \mathbb{Z}_+$:

$$\alpha^k \delta^k = (-\mu\beta\gamma; \mu^2)_k, \tag{5.1}$$

$$\delta^k \alpha^k = (-\mu^{-1}\beta\gamma; \mu^{-2})_k. \tag{5.2}$$

PROOF. Formula (5.2) follows from (5.1) in view of the above algebra isomorphism. Formula (5.1) is proved by complete induction with respect to k:

$$\alpha^{k+1}\delta^{k+1} = \alpha^k(\alpha\delta)\delta^k = \alpha^k(I+\mu\beta\gamma)\delta^k = \alpha^k\delta^k(I+\mu^{2k+1}\beta\gamma). \quad \square$$

There is a model for $\mathscr{A}(\alpha, \beta, \gamma, \delta \,|\, \mu)$ as an algebra of operators acting on the space of entire holomorphic functions on $\mathbb{C}$ (or some other suitable function space). Fix $0 \neq t \in \mathbb{C}$. Put

$$\begin{cases} (\alpha f)(z) := \dfrac{f(z) - f(\mu^2 z)}{z}, \\ (\beta f)(z) := -\mu t^{-1} f(\mu z), \\ (\gamma f)(z) := t f(\mu z), \\ (\delta f)(z) := z f(z). \end{cases} \tag{5.3}$$

Then the operators $\alpha, \beta, \gamma, \delta$ satisfy (4.8). If we put $q := \mu^2$ and

$$(\eta f)(z) := f(qz), \; (D_q f)(z) := \frac{f(z) - f(qz)}{z},$$

then (5.2) takes the form

$$z^k D_q^k = (\eta; q^{-1})_k. \tag{5.4}$$

This identity was first obtained by Jackson [11].

Let $t_{n,m}^{l,\mu}$ be determined by (4.11) together with relations (4.8) and (4.9). It is not difficult to derive an explicit expression for $t_{n,m}^{l,\mu}$:

PROPOSITION 5.2. We have

$$\left\{\begin{aligned} t_{n,m}^{l,\mu} = &\begin{bmatrix} 2l \\ l-n \end{bmatrix}_{\mu^{-2}}^{1/2} \begin{bmatrix} 2l \\ l-m \end{bmatrix}_{\mu^{-2}}^{-1/2} \sum_{i=0\vee(m-n)}^{(l-n)\wedge(l+m)} \mu^{(l-n-i)(n-m+2i)} \mu^{-i(n-m+i)} \\ &\times \begin{bmatrix} l-n \\ i \end{bmatrix}_{\mu^{-2}} \begin{bmatrix} l+n \\ l+m-i \end{bmatrix}_{\mu^{-2}} \beta^i \gamma^{n-m+i} \alpha^{l-n-i} \delta^{l+m-i}. \end{aligned}\right. \tag{5.5}$$

PROOF. By (4.8) and (4.9) we have

$$(\alpha\otimes\xi)(\beta\otimes\eta) = \mu^2(\beta\otimes\eta)(\alpha\otimes\xi),$$

$$(\gamma\otimes\xi)(\delta\otimes\eta) = \mu^2(\delta\otimes\eta)(\gamma\otimes\xi).$$

Hence, by Lemma 2.1,

$$(\alpha\otimes\xi + \beta\otimes\eta)^{l-n} = \sum_{i=0}^{l-n} \begin{bmatrix} l-n \\ i \end{bmatrix}_{\mu^{-2}} \alpha^{l-n-i}\beta^i \otimes \xi^{l-n-i}\eta^i,$$

$$(\gamma\otimes\xi + \delta\otimes\eta)^{l+n} = \sum_{j=0}^{l+n} \begin{bmatrix} l+n \\ j \end{bmatrix}_{\mu^{-2}} \gamma^{l+n-j}\delta^j \otimes \xi^{l+n-j}\eta^j.$$

Thus, again by (4.8) and (4.9):

$$\begin{aligned} &(\alpha\otimes\xi + \beta\otimes\eta)^{l-n}(\gamma\otimes\xi + \delta\otimes\eta)^{l+n} \\ &= \sum_{i=0}^{l-n} \sum_{j=0}^{l+n} \mu^{(l-n-i)(i+l+n-j)} \mu^{-i(l+n-j)} \begin{bmatrix} l-n \\ i \end{bmatrix}_{\mu^{-2}} \begin{bmatrix} l+n \\ j \end{bmatrix}_{\mu^{-2}} \\ &\quad \times \beta^i \gamma^{l+n-j} \alpha^{l-n-i} \delta^j \otimes \xi^{2l-i-j} \eta^{i+j} \\ &= \sum_{m=-l}^{l} \sum_{i=0\vee(m-n)}^{(l-n)\wedge(l+m)} \mu^{(l-n-i)(n-m+2i)} \mu^{-i(n-m+i)} \\ &\quad \times \begin{bmatrix} l-n \\ i \end{bmatrix}_{\mu^{-2}} \begin{bmatrix} l+n \\ l+m-i \end{bmatrix}_{\mu^{-2}} \beta^i \gamma^{n-m+i} \alpha^{l-n-i} \delta^{l+m-i} \otimes \xi^{l-m} \eta^{l+m}, \end{aligned}$$

where we substituted $j = l+m-i$. Now compare with (4.11). □

We can next rewrite (5.5) in terms of q-hypergeometric series, but depending on the signs of $n-m$ and $n+m$. This will yield the expressions in terms of little q-Jacobi polynomials (2.13). For $n \geq m$ define

$$(5.6)\qquad c^{l,\mu}_{n,m} := \begin{bmatrix} l-m \\ n-m \end{bmatrix}^{1/2}_{\mu^2} \begin{bmatrix} l+n \\ n-m \end{bmatrix}^{1/2}_{\mu^2} \mu^{-(n-m)(l-n)}.$$

THEOREM 5.3. We have

$$(5.7)\qquad t^{l,\mu}_{n,m} = c^{l,\mu}_{n,m}\delta^{n+m} p_{l-n}(-\mu^{-1}\beta\gamma; \mu^{2(n-m)}, \mu^{2(n+m)} | \mu^2)\gamma^{n-m}$$

if $n \geq m \geq -n$;

$$(5.8)\qquad t^{l,\mu}_{n,m} = c^{l,\mu}_{m,n}\delta^{m+n} p_{l-m}(-\mu^{-1}\beta\gamma; \mu^{2(m-n)}, \mu^{2(m+n)} | \mu^2)\beta^{m-n}$$

if $m \geq n \geq -m$;

$$(5.9)\qquad t^{l,\mu}_{n,m} = c^{l,\mu}_{-n,-m}\beta^{m-n} p_{l+n}(-\mu^{-1}\beta\gamma; \mu^{2(m-n)}, \mu^{-2(m+n)} | \mu^2)\alpha^{-m-n}$$

if $-n \geq m \geq n$;

$$(5.10)\qquad t^{l,\mu}_{n,m} = c^{l,\mu}_{-m,-n}\gamma^{n-m} p_{l+m}(-\mu^{-1}\beta\gamma; \mu^{2(n-m)}, \mu^{-2(n+m)} | \mu^2)\alpha^{-m-n}$$

if $-m \geq n \geq m$.

PROOF. Because of Proposition 4.1 it is sufficient to prove (5.7). Thus assume that $n \geq m \geq -n$. Then the summation in (5.5) runs from $i=0$ to $l-n$ and

$$\begin{aligned}\beta^i\gamma^{n-m+i}\alpha^{l-n-i}\delta^{l+m-i} &= \gamma^{n-m}(\beta\gamma)^i(\alpha^{l-n-i}\delta^{l-n-i})\delta^{n+m} \\ &= \gamma^{n-m}(\beta\gamma)^i(-\mu\beta\gamma;\mu^2)_{l-n-i}\delta^{n+m}.\end{aligned}$$

In the last identity we used Lemma 5.1. Thus formula (5.5) becomes

$$(5.11)\qquad \left\{\begin{aligned} t^{l,\mu}_{n,m} = &\begin{bmatrix} 2l \\ l-n \end{bmatrix}^{1/2}_{\mu^{-2}} \begin{bmatrix} 2l \\ l-m \end{bmatrix}^{-1/2}_{\mu^{-2}} \gamma^{n-m} \Bigg\{ \sum_{i=0}^{l-n} \mu^{(l-n-i)(n-m+2i)}\mu^{-i(n-m+i)} \\ &\times \begin{bmatrix} l-n \\ i \end{bmatrix}_{\mu^{-2}} \begin{bmatrix} l+n \\ l+m-i \end{bmatrix}_{\mu^{-2}} (-\mu\beta\gamma;\mu^2)_{l-n-i}(\beta\gamma)^i \Bigg\}\delta^{n+m}. \end{aligned}\right.$$

Now use that

$$\begin{bmatrix} 2l \\ n \end{bmatrix}^{1/2}_{\mu^{-2}} \begin{bmatrix} 2l \\ m \end{bmatrix}^{-1/2}_{\mu^{-2}} = \mu^{(n+m)(n-m)} \left(\frac{(\mu^{2l-2m};\mu^{-2})_{n-m}}{(\mu^{2l+2n};\mu^{-2})_{n-m}} \right)^{1/2},$$

$$\begin{bmatrix} l-n \\ i \end{bmatrix}_{\mu^{-2}} = (-1)^i\mu^{i(i+1)} \frac{(\mu^{-2(l-n)};\mu^2)_i}{(\mu^2;\mu^2)_i}$$

$$\begin{aligned}\begin{bmatrix} l+n \\ l+m-i \end{bmatrix}_{\mu^{-2}} &= (-1)^i\mu^{(n-m+i)(n-m+i+1)}\mu^{-(n-m)(n+m+2l+1)} \\ &\times \frac{(\mu^{2(l+n)};\mu^{-2})_{n-m}}{(\mu^2;\mu^2)_{n-m}} \frac{(\mu^{-2(l+m)};\mu^2)_i}{(\mu^{2(n-m+1)};\mu^2)_i},\end{aligned}$$

$$(-\mu\beta\gamma;\mu^2)_{l-n-i}(\beta\gamma)^i = \mu^{i(-2l+2n+i)} \frac{(-\mu\beta\gamma;\mu^2)_{l-n}}{(-\mu^{-2l+2n+1}(\beta\gamma)^{-1};\mu^2)_i} \text{ (formally)}.$$

Thus (5.11) becomes

$$\begin{aligned} t_{n,m}^{l,\mu} &= c_{n,m}^{l,\mu}\mu^{-(n-m)(n+m)}\gamma^{n-m}\Bigg\{(-\mu\beta\gamma;\mu^2)_{l-n} \\ &\quad \times \sum_{i=0}^{l-n} \frac{(\mu^{-2(l-n)};\mu^2)_i(\mu^{-2(l+m)};\mu^2)_i\mu^{2i}}{(\mu^{2(n-m+1)};\mu^2)_i(-\mu^{-2l+2n+1}(\beta\gamma)^{-1};\mu^2)_i(\mu^2;\mu^2)_i}\Bigg\}\delta^{n+m} \quad \text{(formally)} \\ &= c_{n,m}^{l,\mu}\mu^{-(n-m)(n+m)}\gamma^{n-m}\Bigg\{\sum_{i=0}^{l-n} \frac{(\mu^{-2(l-n)};\mu^2)_i(\mu^{2(l+n+1)};\mu^2)_i}{(\mu^{2(n-m+1)};\mu^2)_i(\mu^2;\mu^2)_i} \\ &\quad \times(-\mu^{-2n-2m+1}\beta\gamma)^i\Bigg\}\delta^{n+m} \end{aligned}$$

by (2.17). The theorem now follows by moving δ^{n+m} to the extreme left and by substituting (2.13). □

THEOREM 5.4. *The representations $t^{l,\mu}$ ($l\in\frac{1}{2}\mathbb{Z}_+$) form a complete system of inequivalent irreducible unitary representations of the quantum group $S_\mu U(2)$.*

PROOF. It was already proved in § 4 that the representations $t^{l,\mu}$ are unitary and irreducible. The inequivalence and completeness will follow from the fact that the matrix elements $t_{n,m}^{l,\mu}$ ($l\in\frac{1}{2}\mathbb{Z}_+$; $m,n=-l,-l+1,\ldots,l$), after substitution of (4.7), form a basis of $\mathscr{A}$. This last statement is clear from the explicit expressions in Theorem 5.3 in view of the fact that all elements of the form $\alpha^k\gamma^n(\gamma^*)^m$ ($k,m,n\in\mathbb{Z}_+$) and $(\alpha^*)^k\gamma^n(\gamma^*)^m$ ($k\in\mathbb{N}$, $m,n\in\mathbb{Z}_+$) form a basis of $\mathscr{A}$ (cf. [28, Theorem 1.2]). □

REMARK 5.5. In [28, Theorem 5.8] Woronowicz showed that, for each $d\in\mathbb{N}$ there is a unique (up to equivalence) irreducible unitary representation of $S_\mu U(2)$ on a d-dimensional space. So our representation $t^{l,\mu}$ must be equivalent to Woronowicz' representation of dimension $2l+1$.

REMARK 5.6. Our formula (5.10) can be translated into the formula given by Vaksman and Soibelman [25] in their Corollary to Proposition 6.6, apart from a constant factor, which they do not specify. Here our α, β, γ, δ, μ, $t_{-n,-m}^{l,\mu}$ correspond to their t_{11}, t_{12}, t_{21}, t_{22}, e^{-h}, π_{nm}^l. Be aware that they use the notation for q-hypergeometric series introduced in [8]. For the verification that the two formulas are the same one has to rewrite a ${}_2\phi_1$ of base q in terms of a ${}_2\phi_1$ of base q^{-1}. Their method of proof is infinitesimal. In this way they obtain a second order q-difference equation with a little q-Jacobi polynomial as a solution.

REMARK 5.7. All statements of the Theorems 5.3 and 5.4 except for the unitariness of the representations were announced by Masuda e.a. [19, Theorems 1 and 2]. However, it is not yet clear from their announcement which method of proof they are using. In order to make the correspondence with our notation observe that they write Δ instead of Φ for the comultiplication, and q, y, v, u, x instead of our $\mu, \alpha, \beta, \gamma, \delta$, respectively. They also write ξ for $-\mu^{-1}\beta\gamma$. Their matrix elements $w_{n,m}^{(l)}$ can be expressed in terms of our matrix elements $t_{n,m}^{l,\mu}$ by the identity

$$(w_{n,m}^{(l)})^* = (-1)^{n-m}\mu^{(n-m)(n+m+1)}\begin{bmatrix} 2l \\ l-n \end{bmatrix}_{\mu^{-2}}^{-1/2}\begin{bmatrix} 2l \\ l-m \end{bmatrix}_{\mu^{-2}}^{1/2} t_{n,m}^{l,\mu}.$$

We conclude this section with the verification of the Schur type orthogonality relations

$$(5.12)\qquad h(t_{j,i}^{k,\mu *}t_{n,m}^{l,\mu}) = \delta_{k,l}\delta_{i,m}\frac{f_{-1}(t_{n,j}^{l,\mu})}{\sum_{r=-l}^{l} f_{-1}(t_{r,r}^{l,\mu})},$$

cf. [27, Theorem 5.7], where the Haar functional h on A and the multiplicative linear functional f_{-1} on $\mathscr{A}$ are determined by

$$h(\alpha^k\gamma^{*m}\gamma^n) = 0 = h(\alpha^{*k}\gamma^{*m}\gamma^n) \text{ if } k\in\mathbb{N} \text{ or } k=0,\ m\neq n,$$

$$h((\gamma^*\gamma)^m) = \frac{1-\mu^2}{1-\mu^{2m+2}},$$

$$f_{-1}(\alpha) = |\mu|,\ f_{-1}(\alpha^*) = |\mu|^{-1},\ f_{-1}(\gamma) = f_{-1}(\gamma^*) = 0,\ f_{-1}(I) = 1,$$

cf. [27, Appendix A1].

Now consider (5.12). By use of the explicit expressions in Theorem 5.3 and by (4.7), (4.10) and Lemma 5.1 we can write $t_{j,i}^{k,\mu *}t_{n,m}^{l,\mu}$ as the product of a power of α or α^*, a power of γ or γ^*, an elementary polynomial in $\gamma\gamma^*$ and two little q-Jacobi polynomials depending on $\gamma\gamma^*$. If $(j,i)\neq(n,m)$ then not both the powers of α or α^* and of γ or γ^* will be zero, so then the left hand side of (5.12) will vanish. But then the right hand side of (5.12) will also vanish, since either $i\neq m$ or $i=m$ but $j\neq n$. If $j\neq n$ then, by Theorem 5.3, $t_{n,j}^{l,\mu}$ will contain a nonzero power of γ or γ^*, so $f_{-1}(t_{n,j}^{l,\mu})$ will vanish.

Thus the only case of (5.12) which has yet to be verified is

$$(5.13)\qquad h(t_{n,m}^{k,\mu *}t_{n,m}^{l,\mu}) = \delta_{k,l}\frac{f_{-1}(t_{n,n}^{l,\mu})}{\sum_{r=-l}^{l} f_{-1}(t_{r,r}^{l,\mu})}.$$

We will restrict ourselves to the case $n\geq m\geq -n$; the other cases can be handled in a similar way. In the following write

$$p_l(\gamma\gamma^*) := p_l(\gamma\gamma^*;\mu^{2(n-m)},\mu^{2(n+m)}|\mu^2).$$

Observe that, for any polynomial p in one variable,

$$h(p(\gamma\gamma^*)) = (1-\mu^2)\sum_{j=0}^{\infty}\mu^{2j}p(\mu^{2j}).$$

Then, by (5.7),

$$\begin{aligned} &h(t_{n,m}^{k,\mu *}t_{n,m}^{l,\mu}) \\ &= c_{n,m}^{k,\mu}c_{n,m}^{l,\mu}h(\gamma^{*n-m}p_{k-n}(\gamma\gamma^*)\alpha^{n+m}\alpha^{*n+m}p_{l-n}(\gamma\gamma^*)\gamma^{n-m}) \end{aligned}$$

$$= c^{k,\mu}_{n,m} c^{l,\mu}_{n,m} h((\gamma\gamma^*)^{n-m}(\mu^2\gamma\gamma^*;\mu^2)_{n+m} p_{k-n}(\gamma\gamma^*) p_{l-n}(\gamma\gamma^*))$$

$$= c^{k,\mu}_{n,m} c^{l,\mu}_{n,m} (1-\mu^2) \sum_{j=0}^{\infty} \mu^{2j}(\mu^{2j})^{n-m}(\mu^{2j+2};\mu^2)_{n+m} p_{k-n}(\mu^{2j}) p_{l-n}(\mu^{2j})$$

$$= c^{k,\mu}_{n,m} c^{l,\mu}_{n,m} \int_0^1 t^{n-m}(\mu^2 t;\mu^2)_{n+m} p_{k-n}(t) p_{l-n}(t) d_{\mu^2} t,$$

where we used (2.15). We can evaluate this last expression by use of the orthogonality relations (2.14) for the little q-Jacobi polynomials. After substitution of (2.16) and (5.6) we obtain

$$h(t^{k,\mu}_{n,m} {}^* t^{l,\mu}_{n,m}) = \delta_{k,l} \frac{\mu^{2(l-n)}(1-\mu^2)}{1-\mu^{2(2l+1)}} = \delta_{k,l} \frac{\mu^{-2n}}{\sum_{r=-l}^{l} \mu^{-2r}} = \delta_{k,l} \frac{f_{-1}(t^{l,\mu}_{n,n})}{\sum_{r=-l}^{l} f_{-1}(t^{l,\mu}_{r,r})}.$$

Thus we have identified (5.13) with the orthogonality relations for the little q-Jacobi polynomials.

6. THE MATRIX ELEMENTS $t^{l,\mu}_{n,m}$ EXPRESSED AS q-KRAWTCHOUK POLYNOMIALS

Consider (5.7), express β and δ in terms of γ^* and α^* by (4.7), apply (2.13) and next (2.12) and move the factor α^{*n+m} to the right by using (4.1). Then we obtain for $n \geq m \geq -n$:

$$(6.1) \quad \begin{cases} t^{l,\mu}_{n,m} = a^{l,\mu}_{n,m}(\mu^{-1/2} i^{-1}\gamma)^{l-m}(\mu^{1/2} i\gamma^*)^{l-n} \\ \times {}_2\phi_1\left(\begin{matrix} \mu^{-2l+2n}, \mu^{-2l+2m} \\ \mu^{-4l} \end{matrix}; \mu^2, \frac{1}{\gamma\gamma^*}\right)\tau^{\mu}_{n+m}, \end{cases}$$

where

$$(6.2) \quad a^{l,\mu}_{n,m} := \frac{i^{2l-n-m}\mu^{-(l-n-1/2)(l-m-1/2)+1/4}\mu^{(n+m)(n+m-2l)}(\mu^2;\mu^2)_{2l}}{((\mu^2;\mu^2)_{l-n}(\mu^2;\mu^2)_{l+n}(\mu^2;\mu^2)_{l-m}(\mu^2;\mu^2)_{l+m})^{1/2}}$$

and

$$(6.3) \quad \tau^{\mu}_{n+m} = \alpha^{*n+m}.$$

The negative powers of $\gamma\gamma^*$ in the ${}_2\phi_1$ series are harmless, since they are dominated by the positive powers of γ and γ^* in front. Because of the symmetry (4.13), formula (6.1) together with (6.2) and (6.3) is valid both for $n \geq m \geq -n$ and $m \geq n \geq -m$.

If we apply the symmetry (4.14) to the case $n \geq m \geq -n$ of (6.1) then we obtain an expression for $t^{l,\mu}_{n,m}$ in the case $-n \geq m \geq n$:

$$\text{const.}\ \alpha^{-n-m}\gamma^{l+n}(\gamma^*)^{l+m}{}_2\phi_1\left(\begin{matrix} \mu^{-2l-2n}, \mu^{-2l-2m} \\ \mu^{-4l} \end{matrix}; \mu^2, \frac{1}{\gamma\gamma^*}\right).$$

When we move α^{-n-m} in this expression to the right and apply the identity (2.11) then we obtain again (6.1) but with

$$(6.4)\qquad \tau^{\mu}_{m+n}=\frac{1}{(\mu^2\gamma\gamma^*;\mu^2)_{-m-n}}\,\alpha^{-m-n}.$$

Although the expression (6.1) with (6.4) looks very formal, it is, with $-n\geq \geq m\geq n$, well defined as a polynomial in γ, γ^* and α. Again by the symmetry (4.13) we see that (6.1) together with (6.4) is also valid if $-m\geq n\geq m$. Thus we have proved:

THEOREM 6.1. The matrix elements $t^{l,\mu}_{n,m}$ can be expressed by (6.1) with (6.2), where τ^{μ}_{m+n} is given by (6.3) if $m+n\geq 0$ and by (6.4) if $m+n\leq 0$.

If we compare (6.1) with (2.18) then we see that $t^{l,\mu}_{n,m}$ contains the q-Krawtchouk polynomial

$$K_{l-n}\left(\mu^{-2l+2m};\frac{\mu^{-2l+2n-2}}{\gamma\gamma^*},2l\,|\,\mu^2\right)$$

as a factor. The only draw-back of this expression is that the parameter b of (2.18) occurs here only in a formal way. Nevertheless we will identify the unitariness (cf. (4.16), (4.17)) of the matrix representation $t^{l,\mu}$ with the orthogonality relations and dual orthogonality relations of the q-Krawtchouk polynomials. Because of the self-duality of the q-Krawtchouk polynomials it is sufficient to identify (4.17) with (2.22).

We will use the faithful representation of the C^*-algebra A on a Hilbert space $\mathscr{H}$ with orthonormal basis $\{e_{\nu,\kappa}\,|\,\nu\in\mathbb{Z}_+,\kappa\in\mathbb{Z}\}$ such that

$$(6.5)\qquad \begin{cases} \alpha e_{\nu,\kappa}=(1-\mu^{2\nu})^{1/2}e_{\nu-1,\kappa},\ \alpha^* e_{\nu,\kappa}=(1-\mu^{2\nu+2})^{1/2}e_{\nu+1,\kappa},\\ \gamma e_{\nu,\kappa}=\mu^{\nu}e_{\nu,\kappa+1},\ \gamma^* e_{\nu,\kappa}=\mu^{\nu}e_{\nu,\kappa-1}, \end{cases}$$

cf. [28, Theorem 1.2]. Thus (4.17) is equivalent to the Hilbert space identities

$$(6.6)\qquad \Big(\sum_{m=-l}^{l} t^{l,\mu}_{n,m}t^{l,\mu}_{k,m}{}^*\Big)e_{\nu,\kappa}=\delta_{n,k}e_{\nu,\kappa}$$

for all $n,k=-l,\dots,l$ and for all basis vectors $e_{\nu,\kappa}$.

It follows from (6.5), (6.3) and (6.4) that

$$\tau^{\mu}_{m+n}e_{\nu,\kappa}=\left(\frac{(\mu^{2\nu+2};\mu^2)_\infty}{(\mu^{2\nu+2m+2n+2};\mu^2)_\infty}\right)^{1/2}e_{\nu+m+n,\kappa},$$

$$\tau^{\mu}_{m+n}{}^*e_{\nu,\kappa}=\left(\frac{(\mu^{2\nu-2m-2n+2};\mu^2)_\infty}{(\mu^{2\nu+2};\mu^2)_\infty}\right)^{1/2}e_{\nu-m-n,\kappa},$$

independent of the sign of $m+n$. Of course, for $m+n<0$, these formal results have to be interpreted with τ^{μ}_{m+n} as a factor in (6.1) so that the dependence on $\alpha,\alpha^*,\gamma,\gamma^*$ is polynomial. Now we find for the left hand side of (6.6):

$$
(6.7)\quad \left\{
\begin{aligned}
&\sum_{m=-l} a_{n,m}^{l,\mu} a_{k,m}^{l,\mu} (i^{-1}\mu^{\nu+n-k-1/2})^{l-m} (i\mu^{\nu+n-k+1/2})^{l-n}\\
&\times {}_2\phi_1\left(\begin{matrix}\mu^{-2l+2n}, \mu^{-2l+2m}\\ \mu^{-4l}\end{matrix}; \mu^2, \mu^{-2\nu-2n+2k}\right)\\
&\times \left(\frac{(\mu^{2\nu-2m-2k+2};\mu^2)_\infty}{(\mu^{2\nu+2n-2k+2};\mu^2)_\infty}\right)^{1/2} \left(\frac{(\mu^{2\nu-2m-2k+2};\mu^2)_\infty}{(\mu^{2\nu+2};\mu^2)_\infty}\right)^{1/2}\\
&\times {}_2\phi_1\left(\begin{matrix}\mu^{-2l+2k}, \mu^{-2l+2m}\\ \mu^{-4l}\end{matrix}; \mu^2, \mu^{-2\nu}\right) (i^{-1}\mu^{\nu+1/2})^{l-k} (i\mu^{\nu-1/2})^{l-m} e_{\nu+n-k,\kappa+n-k}
\end{aligned}\right.
$$

Substitute in (6.7)

$$
(6.8)\qquad b := \mu^{-2\nu-2l+2k-2}.
$$

Then the first ${}_2\phi_1$ in (6.7) becomes

$$
{}_2\phi_1\left(\begin{matrix}\mu^{-2l+2n}, \mu^{-2l+2m}\\ \mu^{-4l}\end{matrix}; \mu^2, b\mu^{2l-2n+2}\right) = K_{l-n}(\mu^{-2l+2m}; b, 2l \mid \mu^2)
$$

and the second ${}_2\phi_1$ similarly, with n replaced by k. If we also substitute (6.2) in (6.7) then we can rewrite (6.7) as

$$
(6.9)\quad \left\{
\begin{aligned}
&\frac{(-1)^{n-k}(\mu^2,\mu^2;\mu^2)_{2l} b^{-2l}}{((\mu^2;\mu^2)_{l-n}(\mu^2;\mu^2)_{l+n}(\mu^2;\mu^2)_{l-k}(\mu^2;\mu^2)_{l+k})^{1/2}}\\
&\times \frac{\mu^{-7l^2+(n+k-1)l+1/2n(n-1)+1/2k(k-1)}}{((-1)^{2l-n-k}(b\mu^2;\mu^2)_{l-n}(b\mu^2;\mu^2)_{l-k})^{1/2}}\\
&\times \sum_{m=-l}^{l} (-1)^{m+l} \mu^{(l-m)(l-m+1)} \frac{(b\mu^2;\mu^2)_{l+m}}{(\mu^2;\mu^2)_{l+m}(\mu^2;\mu^2)_{l-m}}\\
&\times K_{l-n}(\mu^{-2l+2m}; b, 2l\mid\mu^2) K_{l-k}(\mu^{-2l+2m}; b, 2l\mid\mu^2) e_{\nu+n-k,\kappa+n-k}.
\end{aligned}\right.
$$

In view of (2.22), the expression (6.9) is equal to the right hand side of (6.6). On the other hand, the validity of (6.6) for all l, m, n, k, ν, κ implies (2.20) with $q=\mu^2$ for all N, n, m and all $b=\mu^{-2j}$ with $j \in \mathbb{N}$. Thus our promises are fulfilled.

NOTE ADDED IN PROOF. The results announced in [19] are now fully described in the preprint "Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials" by the same five authors. Lemma 2.1 was earlier given in J. Cigler, Monatsh. Math. 88, 87–105 (1979), Satz 1, and in Ph. Feinsilver, Rocky Mountain J. Math. 12, 171–183 (1982), p. 177.

REFERENCES

1. Andrews, G.E. and R. Askey – Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials. Higher combinatorics (ed. M. Aigner), Reidel, 3–26 (1977).

2. Askey, R. and M.E.H. Ismail – A generalization of ultraspherical polynomials. Studies in Pure Mathematics (ed. P. Erdös), 55–78, Birkhäuser, 1983.
3. Askey, R. and J. Wilson – A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols, SIAM J. Math. Anal. **10**, 1008–1016 (1979).
4. Askey, R. and J. Wilson – Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Mem. Amer. Math. Soc. **319** (1985).
5. Cartier, P. – Harmonic analysis on trees. Proc. Sympos. Pure Math. **26**, 419–424 (1973).
6. Delsarte, Ph. – Properties and applications of the recurrence $F(i+1,k+1,n+1) = q^{k+1}F(i,k+1,n) - q^kF(i,k,n)$. SIAM J. Appl. Math. **31**, 262–270 (1976).
7. Drinfeld, V.G. – Quantum groups. Proceedings of the International Congress of Mathematicians, Berkeley, 1986, American Mathematical Society, 798–820 (1987).
8. Exton, H. – q-Hypergeometric functions and applications. Ellis Horwood (1983).
9. Gasper, G. and M. Rahman – Basic hypergeometric series. Cambridge University Press (to appear).
10. Hahn, W. – Über Orthogonalpolynome, die q-Differenzengleichungen genügen. Math. Nachr. **2**, 4–34, 379 (1949).
11. Jackson, F.H. – On generalized functions of Legendre and Bessel. Trans. Roy. Soc. Edinburgh **41**, 1–28 (1904).
12. Jimbo, M. – q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985).
13. Jimbo, M. – A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra and the Yang-Baxter equation. Lett. Math. Phys. **11**, 247–252 (1986).
14. Koornwinder, T.H. – Clebsch-Gordan coefficients for $SU(2)$ and Hahn polynomials. Nieuw Arch. Wisk. (3) **29**, 140–155 (1981).
15. Koornwinder, T.H. – Krawtchouk polynomials, a unification of two different group theoretic interpretations. SIAM J. Math. Anal. **13**, 1011–1023 (1982).
16. Koornwinder, T.H. – Jacobi functions as limit cases of q-ultraspherical polynomials. CWI Rep. PM-R8807 (1988).
17. Lusztig, G. – Quantum deformations of certain simple modules over enveloping algebras. Adv. in Math. **70**, 237–249 (1988).
18. Macdonald, I.G. – Orthogonal polynomials associated with root systems. preprint (1988).
19. Masuda, T., K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno – Representations of quantum groups and a q-analogue of orthogonal polynomials. C.R. Acad. Sci. Paris, Sér. I Math. **307**, 559–564 (1988).
20. Nikiforov, A.F. and V.B. Uvarov – Special functions of mathematical physics. Birkhäuser (1988).
21. Podles, P. – Quantum spheres. Lett. Math. Phys. **14**, 193–202 (1987).
22. Rosso, M. – Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra. Comm. Math. Phys. **117**, 581–593 (1988).
23. Stanton, D. – Some q-Krawtchouk polynomials on Chevalley groups. Amer. J. Math. **102**, 625–662 (1980).
24. Stanton, D. – Orthogonal polynomials and Chevalley groups. Special functions: group theoretic aspects and applications (ed. R.A. Askey, T.H. Koornwinder, W. Schempp), Reidel, 87–128 (1984).
25. Vaksman, L.L. and Ya.S. Soibelman – Function algebra on the quantum group $SU(2)$. Funktsional Anal. i. Prilozhen **22** (3), 1–14 (1988). In Russian.
26. Vilenkin, N.Ya. – Special functions and the theory of group representations. Translations of Mathematical Monographs **22**, American Mathematical Society (1968).
27. Woronowicz, S.L. – Compact matrix pseudogroups. Comm. Math. Phys. **111**, 613–665 (1987).
28. Woronowicz, S.L. – Twisted $SU(2)$ group. An example of a non-commutative differential calculus. Publ. Res. Inst. Math. Sci. **23**, 117–181 (1987).
29. Woronowicz, S.L. – Tanaka-Krein duality for compact matrix pseudo-groups. Twisted $SU(N)$ groups. Invent. Math. **93**, 35–76 (1988).