Jacobi functions: the addition formula and the positivity of the dual convolution structure

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Abstract. We prove an addition formula for Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ $(\alpha \geqq \beta \geqq -\frac{1}{2})$ analogous to the known addition formula for Jacobi polynomials. We exploit the positivity of the coefficients in the addition formula by giving the following application. We prove that the product of two Jacobi functions of the same argument has a nonnegative Fourier-Jacobi transform. This implies that the convolution structure associated to the inverse Fourier-Jacobi transform is positive.

1. Introduction

For fixed α, β Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ form a continuous orthogonal system on $\mathbf{R}^+$ with respect to the measure $(\operatorname{sh} t)^{2\alpha+1} (\operatorname{ch} t)^{2\beta+1}\, dt$, generalizing the cosines $\varphi_\lambda^{(-1/2,-1/2)}(t) = \cos \lambda t$. In [3], [6], [10] the authors developed harmonic analysis for Jacobi function expansions, including a positivity result for the convolution product associated to these expansions.

The main result of the present paper is a similar positivity result for the dual case, i.e. the convolution product associated to the inverse Fourier-Jacobi transform $(\alpha \geqq \beta \geqq -\frac{1}{2})$. Equivalently, we prove that the Fourier-Jacobi transform $a(\lambda_1, \lambda_2, \cdot)$ of the product $t \mapsto \varphi_{\lambda_1}^{(\alpha,\beta)}(t)\varphi_{\lambda_2}^{(\alpha,\beta)}(t)$, $(\lambda_1, \lambda_2 \in \mathbf{R})$ is nonnegative. By using group theoretic considerations Flensted-Jensen [4] and Mizony [14] proved this result for special values of α, β. A similar positivity result of the dual convolution structure associated to Jacobi polynomial expansions was first proved by Gasper [8].

Koornwinder [13] applied the addition formula for Jacobi polynomials in order to obtain a new proof of the just mentioned result of Gasper's. Here we follow a similar approach and, therefore, we have first to derive the addition formula for Jacobi functions. This addition formula is an expansion of

$$\varphi_\lambda^{(\alpha,\beta)}(\operatorname{Arc\,ch} |\operatorname{ch} t_1 \cdot \operatorname{ch} t_2 + re^{i\psi} \operatorname{sh} t_1 \cdot \operatorname{sh} t_2|)$$

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in terms of certain orthogonal polynomials $R_{k,l}$ in the two variables $r\cos\varphi$ and r^2, with expansion coefficients

$$\gamma_{k,l}^{(\alpha,\beta)}(\lambda)\varphi_{\lambda,k,l}^{(\alpha,\beta)}(t_1)\varphi_{\lambda,k,l}^{(\alpha,\beta)}(t_2),$$

where the functions $\varphi_{\lambda,k,l}^{(\alpha,\beta)}$ are "associated Jacobi functions" and $\gamma_{k,l}^{(\alpha,\beta)}(\lambda)\geqq 0$ for real λ. The nonnegativity of the coefficients $\gamma_{k,l}$ is of crucial importance for our applications of the addition formula.

In a companion paper [7] we give another application of the addition formula for Jacobi functions. It presents a new approach to certain results of Kostant's dealing with a characterization of those values of λ for which a spherical function φ_λ on a given noncompact rank one symmetric space is positive definite.

In a forthcoming paper we will derive similar results as in the present paper for the functions

(1.1) $$\varphi_{\lambda,\nu}^{\alpha}(y,\theta) := (e^{i\theta}\operatorname{ch} y)^{\nu}\varphi_\lambda^{(\alpha,\nu)}(y),$$

which were studied by the first author [4], [5] in the cases $\alpha=0, 1, 2, \ldots$ by an interpretation as spherical functions.

2. The addition formula for Jacobi functions: statement of the result

The Jacobi function $\varphi_\lambda^{(\alpha,\beta)}$ is given (cf. [6]) by

(2.1) $$\varphi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\lambda), \tfrac{1}{2}(\alpha+\beta+1-i\lambda); \alpha+1; -(\operatorname{sh} t)^2),$$

$$t\in\mathbf{R},\ \lambda\in\mathbf{C},\ \alpha\in\mathbf{C}\setminus\{-\mathbf{N}\},\ \beta\in\mathbf{C}.$$

On writing

(2.2) $$R_\mu^{(\alpha,\beta)}(z) := {}_2F_1(-\mu, \mu+\alpha+\beta+1; \alpha+1; \tfrac{1}{2}(1-z))$$

we have

(2.3) $$\varphi_\lambda^{(\alpha,\beta)}(t) = R_{(i\lambda-\alpha-\beta-1)/2}^{(\alpha,\beta)}(\operatorname{ch} 2t).$$

If $\alpha,\beta>-1$, $n\in\mathbf{Z}_+$ then $R_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n in x satisfying $R_n^{(\alpha,\beta)}(1)=1$ and the orthogonality relations

(2.4)
$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} R_m^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx = (\pi_n^{(\alpha,\beta)})^{-1}\delta_{m,n},$$

where

(2.5) $$\pi_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)(\alpha+1)_n(\alpha+\beta+2)_n}{(n+\alpha+\beta+1)(\beta+1)_n\,n!}.$$

Here $(a)_k := a(a+1)\dots(a+k-1)$. Note that $P_n^{(\alpha,\beta)}(x) = \big((\alpha+1)_n/n!\big)R_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomial (cf. SZEGŐ [16]).

We shall need a family $R_{n,m}^{(\alpha,\beta)}$ of orthogonal polynomials on

$$\Omega := \{(x, y)\in \mathbf{R}^2 | x^2 \leqq y \leqq 1\}$$

with respect to the normalized measure

$$dm_{\alpha,\beta}(x, y) = \frac{\Gamma(\alpha+\beta+(5/2))}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(1/2)}(1-y)^\alpha (y-x^2)^\beta\, dx\, dy. \tag{2.6}$$

These polynomials can be defined in terms of Jacobi polynomials by

$$R_{n,m}^{(\alpha,\beta)}(x, y) := R_m^{(\alpha,\beta+n-m+(1/2))}(2y-1)\, y^{(n-m)/2} R_{n-m}^{(\beta,\beta)}(y^{-1/2}x), \tag{2.7}$$

$$n, m\in \mathbf{Z}, \quad n \geqq m \geqq 0 \quad \text{and} \quad \alpha, \beta > -1.$$

The orthogonality relations are given by

$$\iint_\Omega R_{n,m}^{(\alpha,\beta)}(x, y) R_{k,l}^{(\alpha,\beta)}(x, y)\, dm_{\alpha,\beta}(x, y) = (\pi_{n,m}^{(\alpha,\beta)})^{-1}\delta_{n,k}\delta_{m,l}, \tag{2.8}$$

where

(2.9)

$$\pi_{n,m}^{(\alpha,\beta)} = \frac{(2n-2m+2\beta+1)\big(n+m+\alpha+\beta+(3/2)\big)(\alpha+1)_m(2\beta+2)_{n-m}\big(\alpha+\beta+(5/2)\big)_n}{(n-m+2\beta+1)\big(n+\alpha+\beta+(3/2)\big)m!\,(n-m)!\,\big(\beta+(3/2)\big)_n},$$

cf. [12, § 3] and [13, § 2].

If $\alpha\downarrow -1$ or $\beta\downarrow -1$ then the measure $dm_{\alpha,\beta}(x, y)$ weakly tends to a measure with support on one of the edges of the orthogonality region. An easy calculation shows:

$$\pi_{n,m}^{(-1,\beta)} R_{n,m}^{(-1,\beta)}(x, 1) = \delta_{m,0}\pi_n^{(\beta,\beta)} R_n^{(\beta,\beta)}(x), \tag{2.10}$$

$$\pi_{n,m}^{(\alpha,-1)} R_{n,m}^{(\alpha,-1)}(x, x^2) = \begin{cases} \pi_{2n}^{(\alpha,\alpha)} R_{2n}^{(\alpha,\alpha)}(x) & \text{if } m = n, \\ \pi_{2n-1}^{(\alpha,\alpha)} R_{2n-1}^{(\alpha,\alpha)}(x) & \text{if } m = n-1, \\ 0 & \text{if } m \leqq n-2. \end{cases} \tag{2.11}$$

The associated Jacobi functions $\varphi_{\lambda,k,l}^{(\alpha,\beta)}$ are defined in terms of Jacobi functions by

$$\varphi_{\lambda,k,l}^{(\alpha,\beta)}(t) = (\operatorname{sh} t)^{k+l}(\operatorname{ch} t)^{k-l}\varphi_\lambda^{(\alpha+k+l,\beta+k-l)}(t), \tag{2.12}$$

$$k, l\in \mathbf{Z}, \quad k \geqq l \geqq 0.$$

Now we can state the addition theorem:

Theorem 2.1. *Let* $\alpha>\beta>-\frac{1}{2}$, *then*

(2.13) $$\varphi_\lambda^{(\alpha,\beta)}\big(\Lambda(-t_1, t_2, r, \psi)\big) =$$

$$= \sum_{k=0}^\infty \sum_{l=0}^k \gamma_{k,l}^{(\alpha,\beta)}(\lambda)\varphi_{\lambda,k,l}^{(\alpha,\beta)}(t_1)\varphi_{\lambda,k,l}^{(\alpha,\beta)}(t_2)\pi_{k,l}^{(\alpha-\beta-1,\beta-(1/2))}R_{k,l}^{(\alpha-\beta-1,\beta-(1/2))}(r\cos\psi, r^2),$$

where $t\in\mathbf{R}$, $\lambda\in\mathbf{C}$, $r\in[0,1]$, $\varphi\in[0,\pi]$,

(2.14) $$\Delta(t_1, t_2, r, \psi) := \operatorname{Arc\,ch}(|\operatorname{ch} t_1 \operatorname{ch} t_2 + re^{i\psi} \operatorname{sh} t_1 \operatorname{sh} t_2|),$$

(2.15) $$\gamma_{k,l}^{(\alpha,\beta)}(\lambda) =$$

$$= \frac{(1/2(\alpha+\beta+1+i\lambda))_k(1/2(\alpha+\beta+1-i\lambda))_k(1/2(\alpha-\beta+1+i\lambda))_l(1/2(\alpha-\beta+1-i\lambda))_l}{(\alpha+1)_{k+l}(\alpha+1)_{k+l}}.$$

Furthermore, the double series in (2.13) *converges absolutely, uniformly for* (t_1, t_2, r, ψ) *in compact subsets of* $\mathbf{R}^2\times[0,1]\times[0,\pi]$.

Remark 2.2. If $\alpha=\beta>-\frac{1}{2}$, $r=1$ or $\alpha>\beta=-\frac{1}{2}$, $\varphi=0,\pi$ then (2.13) still holds. In view of (2.10), (2.11) it then degenerates to a single series. The two cases are related by the quadratic transformation

(2.16) $$\varphi_\lambda^{(\alpha,-(1/2))}(2t) = \varphi_{2\lambda}^{(\alpha,\alpha)}(t),$$

cf. [10, (2.8)]. A further degeneracy in the addition formula occurs if $\alpha=\beta=-\frac{1}{2}$, $r=1$, $\varphi=0$ or π. Then (2.13) has only two terms at the right hand side.

Our addition formula was earlier proved by WHITTAKER & WATSON [18, § 15.71] in the case $\alpha=\beta=0$ and by HENRICI [9, (80)] in the case $\alpha=\beta$. A group theoretic derivation in the case $\alpha=\beta\in\{0,\frac{1}{2},\dots\}$ was given by VILENKIN [17, Chap. 10, § 3.5].

Corollary 2.3. *Let* $\lambda\in\mathbf{C}$, $\alpha>\beta>-\frac{1}{2}$. *Then* $\gamma_{k,l}^{(\alpha,\beta)}\geqq 0$ *for all* k, l *iff* $\lambda\in\mathbf{R}\cup i[-s_0, s_0]\cup\{\pm i(\alpha+\beta+1)\}$, *where* $s_0:=\min\{\alpha+\beta+1, \alpha-\beta+1\}$.

Proof. Use that

$$(\tfrac{1}{2}(\alpha+\beta+1+i\lambda))_k(\tfrac{1}{2}(\alpha+\beta+1-i\lambda))_k = \prod_{j=0}^{k-1}\tfrac{1}{4}((\alpha+\beta+j+1)^2+\lambda^2),$$

and similarly with β, k replaced by $-\beta$, l. □

3. Proof of the addition formula for Jacobi functions

The proof we will give below is analogous to the proof of the addition formula for Jacobi polynomials in [12]. The main difficulty compared to the Jacobi polynomial case is the convergence proof of the series in (2.13).

Fix α, β such that $\alpha>\beta>-\frac{1}{2}$ and let $d\tilde{m}_{\alpha,\beta}(r,\psi)$ be the measure on $[0,1]\times[0,\pi]$ such that

(3.1) $$\int_0^1\int_0^\pi f(r\cos\psi, r^2)\, d\tilde{m}_{\alpha,\beta}(r,\psi) = \iint_\Omega f(x,y)\, dm_{\alpha-\beta-1,\beta-(1/2)}(x,y)$$

for all continuous functions f on Ω (cf. (2.6)). For $\mu\in\mathbf{C}$ and $t_1, t_2\in\mathbf{R}$ we have the product formula (cf. [6, (4.1)]):

$$(3.2)\quad R_\mu^{(\alpha,\beta)}(\operatorname{ch} 2t_1)\, R_\mu^{(\alpha,\beta)}(\operatorname{ch} 2t_2) = \int_0^1\int_0^\pi R_\mu^{(\alpha,\beta)}(\operatorname{ch} 2\Lambda(t_1, t_2, r, \psi))\, d\tilde{m}_{\alpha,\beta}(r, \psi).$$

Lemma 3.1. *Let* $k\geqq l\geqq 0$, $k, l\in\mathbf{Z}$. *Then*

$$(3.3)\quad \int_0^1\int_0^\pi R_\mu^{(\alpha,\beta)}(\operatorname{ch} 2\Lambda(t_1, t_2, r, \psi))\, R_{k,l}^{(\alpha-\beta-1,\beta-(1/2))}(r\cos\psi, r^2)\, d\tilde{m}_{\alpha,\beta}(r, \psi)$$

$$= \frac{(-1)^{k+l}(-\mu)_k(-\mu-\beta)_l(\mu+\alpha+\beta+1)_k(\mu+\alpha+1)_l}{(\alpha+1)_{k+l}(\alpha+1)_{k+l}}$$

$$\times(\operatorname{sh} t_1 \operatorname{sh} t_2)^{k+l}(\operatorname{ch} t_1 \operatorname{ch} t_2)^{k-l}\, R_{\mu-k}^{(\alpha+k+l,\beta+k-l)}(\operatorname{ch} 2t_1)$$

$$\times R_{\mu-k}^{(\alpha+k+l,\beta+k-l)}(\operatorname{ch} 2t_2).$$

Proof. Apply Lemma 4.1 in [12] and observe that formula (4.7) in [12] can immediately be generalized to the case of noninteger n. Finally apply the product formula (3.2). □

The functions $(r, \psi)\to R_{k,l}^{(\alpha-\beta-1,\beta-(1/2))}(r\cos\psi, r^2)$ $(k, l\in\mathbf{Z},\ k\geqq l\geqq 0)$ form a complete orthogonal basis of the Hilbert space $L^2([0,1]\times[0,\pi], d\tilde{m}_{\alpha,\beta})$. The lemma gives the "Fourier" coefficients of $R_\mu^{(\alpha,\beta)}(\operatorname{ch} 2\Lambda(t_1, t_2, r, \psi))$ with respect to this basis. The corresponding expansion with t_1 replaced by $-t_1$, and μ by $\frac{1}{2}(i\lambda-\alpha-\beta-1)$ and with substitution of (2.3) and (2.12) shows that, for fixed $t_1, t_2, \lambda, \alpha, \beta$, (2.13) holds in L^2-sense with respect to the measure $d\tilde{m}_{\alpha,\beta}$.

The absolute and uniform convergence of (2.13) will follow from a general result for expansions of C^∞-functions $f(x, y)$ in terms of the polynomials $R_{n,m}^{(\alpha,\beta)}(x, y)$ (see Theorem 3.6 below). First we need estimates for $R_{n,m}^{(\alpha,\beta)}(x, y)$ as $n\to\infty$.

Lemma 3.2. *If* $\alpha\geqq\beta+\frac{1}{2}\geqq 0$, $n, m\in\mathbf{Z}$, $n\geqq m\geqq 0$ *and* $x^2\leqq y\leqq 1$ *then* $|R_{n,m}^{(\alpha,\beta)}(x, y)|\leqq 1$.

Proof. For fixed n, m, α, β with $\alpha\geqq|\beta+\frac{1}{2}|$ we have

$$(3.4)\qquad R_m^{(\alpha,\beta+n-m+(1/2))}(2y-1) = \sum_{l=0}^m c_l R_l^{(0,n-m)}(2y-1)$$

with $c_l\geqq 0$, cf. Askey & Gasper [1, Theorems 1 and 2]. Now using that

$$(3.5)\qquad |R_{n-m}^{(\beta,\beta)}(y^{-(1/2)}x)| \leqq R_{n-m}^{(\beta,\beta)}(1) = 1 \quad\text{if}\quad \beta\geqq -\tfrac{1}{2},\quad x^2\leqq y$$

(cf. Szegő [16, Theorem 7.32.1]), and

$$|R_l^{(0,n-m)}(2y-1)\, y^{(n-m)/2}| \leqq |R_l^{(0,n-m)}(2\cdot 1-1)\, 1^{(n-m)/2}| = 1$$

(cf. SZEGŐ [16, Theorem 7.2]), we obtain for $\alpha \geqq \beta+\frac{1}{2} \geqq 0$ and $x^2 \leqq y \leqq 1$ that

$$|R_{n,m}^{(\alpha,\beta)}(x,y)| = \left|\sum_{l=0}^{m} c_l R_l^{(0,n-m)}(2y-1)\, y^{(n-m)/2} R_{n-m}^{(\beta,\beta)}(y^{-(1/2)}x)\right|$$
$$\leqq \sum_{l=0}^{m} c_l |R_l^{(0,n-m)}(2y-1)\, y^{(n-m)/2}| \leqq \sum_{l=0}^{m} c_l = 1,$$

where the last equality is obtained by putting $y=1$ in (3.4). □

The inequality proved above was already announced in [11, (5.2)], however with slightly incorrect conditions on α and β.

Lemma 3.3. *For each $\alpha, \beta > -1$ there exists $\varkappa \geqq 0$ such that*

(3.6) $$R_{n,m}^{(\alpha,\beta)}(x,y) = \mathcal{O}(n^{\varkappa}) \quad \text{as} \quad n \to \infty,$$

uniformly for $m \in \{0, 1, \ldots, n\}$ and $(x,y) \in \Omega$.

Proof. For $\alpha \geqq \beta+\frac{1}{2} \geqq 0$ the result follows from the previous lemma. In the case $\alpha \geqq |\beta+\frac{1}{2}|$, $\beta < -\frac{1}{2}$ we reproduce the proof of Lemma 3.2 with (3.5) replaced by

$$R_{n-m}^{(\beta,\beta)}(y^{-(1/2)}x) = \mathcal{O}(n-m) \quad \text{as} \quad n-m \to \infty,$$

uniformly on Ω (cf. SZEGŐ [16, Theorem 7.32.1]). In order to handle the case $\alpha < |\beta+\frac{1}{2}|$ we use the recurrence relation

$$(\alpha+1)(n+m+\alpha+\beta+\tfrac{3}{2}) R_{n,m}^{(\alpha,\beta)}(x,y)$$
$$= (m+\alpha+1)(n+\alpha+\beta+\tfrac{3}{2}) R_{n,m}^{(\alpha+1,\beta)}(x,y) - m(n+\beta+\tfrac{1}{2}) R_{n-1,m-1}^{(\alpha+1,\beta)}(x,y),$$

which follows from ERDÉLYI [2, 10.8(35)]. Iteration of this identity reduces the problem to the case $\alpha \geqq |\beta+\frac{1}{2}|$ and the desired estimate follows. □

Next we introduce the partial differential operator

(3.7) $$D^{(\alpha,\beta)} := (1-x^2)\frac{\partial^2}{\partial x^2} + 4x(1-y)\frac{\partial^2}{\partial x\,\partial y} + 4y(1-y)\frac{\partial^2}{\partial y^2} - (2\alpha+2\beta+4)x\frac{\partial}{\partial x} + (2-(4\alpha+4\beta+10)y)\frac{\partial}{\partial y}.$$

Lemma 3.4. *For $f, g \in C^2(\Omega)$ we have*

(3.8) $$\iint_{\Omega} (D^{(\alpha,\beta)} f) g \, dm_{\alpha,\beta} = \iint_{\Omega} f (D^{(\alpha,\beta)} g) \, dm_{\alpha,\beta}.$$

Proof. Use integration by parts. If α, β are not too small then the vanishing of the stock terms is clear and (3.8) follows. The case of general $\alpha, \beta > -1$ then follows by analytic continuation of (3.8) with respect to α and β. □

Lemma 3.5.

(3.9) $$D^{(\alpha,\beta)} R_{n,m}^{(\alpha,\beta)}(x,y) = -(n+m)(n+m+2\alpha+2\beta+3) R_{n,m}^{(\alpha,\beta)}(x,y).$$

Proof. It is clear from (2.7) and (2.8) that $R_{n,m}^{(\alpha,\beta)}(x,y)$ is the polynomial with "highest" term const$\cdot x^{n-m}y^m$ which is obtained by orthogonalization of the sequence $1, x, y, x^2, xy, y^2, x^3, x^2y, \ldots$ with respect to the measure $dm_{\alpha,\beta}(x,y)$. Formula (3.7) implies that

(3.10) $$D^{(\alpha,\beta)}(x^{k-l}y^l) = -(k+l)(k+l+2\alpha+2\beta+3)x^{k-l}y^l + \text{"lower" terms.}$$

Application of (3.8) and (3.10) yields:

$$\iint_\Omega D^{(\alpha,\beta)}R_{n,m}^{(\alpha,\beta)}(x,y)x^{k-l}y^l\,dm_{\alpha,\beta}(x,y)$$
$$= \iint_\Omega R_{n,m}^{(\alpha,\beta)}(x,y)D^{(\alpha,\beta)}(x^{k-l}y^l)\,dm_{\alpha,\beta}(x,y) = 0$$

if $k<n$ or $k=n, l<m$. Formula (3.10) also implies that

$$D^{(\alpha,\beta)}R_{n,m}^{(\alpha,\beta)}(x,y) = -(n+m)(n+m+2\alpha+2\beta+3)R_{n,m}^{(\alpha,\beta)}(x,y) + \text{"lower" terms.}$$

Now (3.9) follows by orthogonality. □

For $f\in L^1(\Omega, dm_{\alpha,\beta})$ let

(3.11) $$f^\wedge(n,m) := \iint_\Omega f(x,y)R_{n,m}^{(\alpha,\beta)}(x,y)\,dm_{\alpha,\beta}(x,y).$$

As a consequence of Lemmas 3.3, 3.4 and 3.5 and the estimate $\pi_{n,m}^{(\alpha,\beta)} = \mathcal{O}(n^{2|\alpha|+2|\beta|+2})$ as $n\to\infty$, uniformly in m, we conclude:

Theorem 3.6. *Let $f\in C^\infty(\Omega)$. Then for each $\varkappa>0$ we have*

(3.12) $$f^\wedge(n,m) = \mathcal{O}(n^{-\varkappa}) \quad as \quad n\to\infty,$$

uniformly in m. Furthermore, the series

(3.13) $$\sum_{n=0}^\infty \sum_{m=0}^n f^\wedge(n,m)\pi_{n,m}^{(\alpha,\beta)}R_{n,m}^{(\alpha,\beta)}(x,y)$$

converges absolutely, uniformly on Ω, and its sum equals $f(x,y)$. If f depends on an additional parameter $s\in S$ such that, for each $k\in\mathbf{Z}_+$, $(D^{(\alpha,\beta)})^k f$ is uniformly bounded on $\Omega\times S$ then the estimate (3.12) *and the absolute convergence of* (3.13) *are also uniform on S.*

Application of this theorem to the series (2.13) completes the proof of Theorem 2.1. The cases $\alpha=\beta>-\frac{1}{2}$, $r=1$ and $\alpha>\beta=-\frac{1}{2}$, $\varphi=0, \pi$ can be proved in an analogous but more simple way.

4. Positivity of the convolution structure associated with the inverse Fourier-Jacobi transform

The Fourier-Jacobi transform $\mathscr{I}_{\alpha,\beta}$ of order (α,β), $\alpha>-1$, is defined by

$$(\mathscr{I}_{\alpha,\beta}f)(\lambda)=f^{\wedge}(\lambda):=\int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)\,d\mu_{\alpha,\beta}(t), \tag{4.1}$$

where

$$d\mu_{\alpha,\beta}(t):=(2\pi)^{-(1/2)}2^{2(\alpha+\beta+1)}(\operatorname{sh}t)^{2\alpha+1}(\operatorname{ch}t)^{2\beta+1}\,dt \tag{4.2}$$

and f belongs to the class C_0^∞ of even C^∞-functions of compact support on $\mathbf{R}$. We now assume that $\alpha\geqq\beta\geqq-\frac{1}{2}$. Then the inverse Fourier transform $\mathscr{I}_{\alpha,\beta}^{-1}$ is given by

$$f(t)=\int_0^\infty f^{\wedge}(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)\,d\nu_{\alpha,\beta}(\lambda), \tag{4.3}$$

where

$$d\nu_{\alpha,\beta}(\lambda):=(2\pi)^{-(1/2)}|c_{\alpha,\beta}(\lambda)|^{-2}\,d\lambda,\quad\text{and} \tag{4.4}$$

$$c_{\alpha,\beta}(\lambda):=\frac{2^{\alpha+\beta+1-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma((\alpha+\beta+1+i\lambda)/2)\Gamma((\alpha-\beta+1+i\lambda)/2)}. \tag{4.5}$$

Then $\mathscr{I}_{\alpha,\beta}$ extends to an isomorphism of

$$L^2([0,\infty),d\mu_{\alpha,\beta})\quad\text{onto}\quad L^2([0,\infty),d\nu_{\alpha,\beta}).$$

See [3] or [10] for a proof of these facts.

In [6] we calculated the kernel $K_{\alpha,\beta}(t_1,t_2,t_3)$ such that

$$\int_0^1\int_0^\pi f(\Lambda(t_1,t_2,r,\psi))\,d\tilde{m}_{\alpha,\beta}(r,\psi)=\int_0^\infty f(t_3)K_{\alpha,\beta}(t_1,t_2,t_3)\,d\mu_{\alpha,\beta}(t_3) \tag{4.6}$$

for all $f\in L^1_{\text{loc}}([0,\infty),d\mu_{\alpha,\beta})$, cf. (2.14), (3.1) and (4.2) for the definitions of Λ, $d\tilde{m}_{\alpha,\beta}$, $d\mu_{\alpha,\beta}$, respectively. Using this we defined the convolution of two functions $f,g\in L^1([0,\infty),d\mu_{\alpha,\beta})$ by

$$(f*g)(t_1):=\int_0^\infty\int_0^\infty f(t_2)g(t_3)K_{\alpha,\beta}(t_1,t_2,t_3)\,d\mu_{\alpha,\beta}(t_2)\,d\mu_{\alpha,\beta}(t_3). \tag{4.7}$$

Notice that we can also write this in the form

$$(f*g)(t_1)=\int_0^\infty f(t_2)\int_0^1\int_0^\pi g(\Lambda(t_1,t_2,r,\psi))\,d\tilde{m}_{\alpha,\beta}(r,\psi)\,d\mu_{\alpha,\beta}(t_2). \tag{4.8}$$

From now on let α and β be fixed such that $\alpha>\beta>-\frac{1}{2}$. For convenience, in subsequent formulas all indices α,β will be dropped.

Remember (cf. [3, Lemma 14]) that there exists $K>0$ such that

$$|\varphi_\lambda(t)|\leqq K(1+t)e^{(|\operatorname{Im}\lambda|-(\alpha+\beta+1))t}\quad\text{for all}\quad\lambda\in\mathbf{C},\ t\in[0,\infty). \tag{4.9}$$

Also, if $1 \leqq p < 2$ and $F \in L^p([0, \infty), d\mu)$ then $F^\wedge$ exists and is continuous on $[0, \infty)$ (cf. [6, Lemma 3.1]) and

$$(4.10) \qquad \|F * g\|_2 \leqq \text{const.}\, \|g\|_2 \quad \text{for all} \quad g \in L^2([0, \infty), d\mu),$$

cf. [6, Theorem 5.5].

Let $(f|g)$ denote the inner product of $f, g \in L^2([0, \infty), d\mu)$.

Lemma 4.1. *Let* $1 \leqq p < 2$ *and* $F \in L^p([0, \infty), d\mu)$. *Then* $F^\wedge(\lambda) \geqq 0$ *for all* $\lambda \in [0, \infty)$ *iff*

$$(4.11) \qquad (F * g|g) \geqq 0 \quad \textit{for all} \quad g \in C_0^\infty.$$

Proof. Since $\mathcal{J}_{\alpha,\beta}$ is a L^2-isomorphism and $(F*g)^\wedge = F^\wedge \cdot g^\wedge$ we have

$$(4.12) \qquad (F * g|g) = \int_0^\infty F^\wedge(\lambda) |g^\wedge(\lambda)|^2 \, d\nu(\lambda)$$

for all $g \in L^2([0, \infty), d\mu)$. If $F^\wedge \geqq 0$ then (4.11) follows. On the other hand assume (4.11) for all $g \in C_0^\infty$. By continuity, (4.11) holds for all $g \in L^2([0, \infty), d\mu)$. Hence, in view of (4.12), $F^\wedge \geqq 0$. □

In the proof of Lemma 4.3 we need an approximate identity with the following properties:

Lemma 4.2. *There is a family* $\{w_\varepsilon | \varepsilon > 0\}$ *of functions on* **R** *such that*

(i) $w_\varepsilon \in C_0^\infty$, $\operatorname{supp}(w_\varepsilon) \subset [-\varepsilon, \varepsilon]$, $w_\varepsilon \geqq 0$;

(ii) $\int_0^\infty w_\varepsilon(t)\, d\mu(t) = 1$;

(iii) $w_\varepsilon^\wedge \geqq 0$;

(iv) $\lim_{\varepsilon \downarrow 0} w_\varepsilon^\wedge(\lambda) = 1$, *uniformly for* λ *in compact subsets of* **R**.

Proof. Choose $v \in C_0^\infty$ such that $\operatorname{supp}(v) \subset [-1, 1]$, $v \geqq 0$ and $\int_0^\infty v(t)\, d\mu(t) = 1$. Just as in [3, Lemma 16] define

$$v_\varepsilon(t) := \varepsilon^{-1} \left(\frac{\operatorname{sh} \varepsilon^{-1} t}{\operatorname{sh} t} \right)^{2\alpha+1} \left(\frac{\operatorname{ch} \varepsilon^{-1} t}{\operatorname{ch} t} \right)^{2\beta+1} v(\varepsilon^{-1} t).$$

Then $v_\varepsilon \in C_0^\infty$, $\operatorname{supp}(v_\varepsilon) \subset [-\varepsilon, \varepsilon]$, $v_\varepsilon \geqq 0$, $\int_0^\infty v_\varepsilon(t)\, d\mu(t) = 1$ and $v_\varepsilon^\wedge$ is real-valued. Also $v_\varepsilon^\wedge(\lambda) \to 1$ as $\varepsilon \downarrow 0$, uniformly for λ in compact subsets of **R** (cf. [3, Lemma 16(i)]). Now let $w_\varepsilon := v_{\varepsilon/2} * v_{\varepsilon/2}$. Then (i) follows from (4.8) (observe that $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$) and (ii), (iii), (iv) are immediately obtained from $w_\varepsilon^\wedge(\lambda) = (v_{\varepsilon/2}^\wedge(\lambda))^2$. (Put $\lambda = i(\alpha+\beta+1)$ for the proof of (ii).) □

Lemma 4.3. *Let* $1 \leqq p < 2$. *Let* $F \in L^p([0, \infty), d\mu)$ *such that* F *is essentially bounded in some neighbourhood of* 0 *and* $F^\wedge \geqq 0$. *Then* $F^\wedge \in L^1([0, \infty), d\nu)$ *and*

$$(4.13) \qquad F(t) = \int_0^\infty F^\wedge(\lambda) \varphi_\lambda(t) \, d\nu(\lambda)$$

almost everywhere on $[0, \infty)$.

Proof. For some $\varepsilon_0>0$ we have $A:=\operatorname{ess\,sup}_{0\le t\le\varepsilon_0}|F(t)|<\infty$. Let w_ε be as in Lemma 4.2. Then $|(F|w_\varepsilon)|\le A\int_0^\varepsilon w_\varepsilon(t)\,d\mu(t)=A$ if $\varepsilon\le\varepsilon_0$. It follows from the proof of [6, Theorem 3.2] that $(F|w_\varepsilon)=\int_0^\infty F^\wedge(\lambda)w_\varepsilon^\wedge(\lambda)\,d\nu(\lambda)$. Hence, since $F^\wedge\ge 0$ and $w_\varepsilon^\wedge\ge 0$, we have for each $M>0$ and $\varepsilon\le\varepsilon_0$:

$$\int_0^M F^\wedge(\lambda)w_\varepsilon^\wedge(\lambda)\,d\nu(\lambda)\le\int_0^\infty F^\wedge(\lambda)w_\varepsilon^\wedge(\lambda)\,d\nu(\lambda)=(F|w_\varepsilon)\le A.$$

It follows from condition (iv) in Lemma 4.2 that $\int_0^M F^\wedge(\lambda)\,d\nu(\lambda)\le A$ for each $M>0$. Hence $F^\wedge\in L^1([0,\infty),d\nu)$. Finally, formula (4.13) is implied by the fact that

$$\int_0^\infty F(t)g(t)\,d\mu(t)=\int_0^\infty F^\wedge(\lambda)g^\wedge(\lambda)\,d\nu(\lambda)=\int_0^\infty\Big(\int_0^\infty F^\wedge(\lambda)\varphi_\lambda(t)\,d\nu(\lambda)\Big)g(t)\,d\mu(t)$$

for all $g\in C_0^\infty$. □

See Stein & Weiss [15, Cor. 1.26] for an analogous result for Fourier transforms.

Let $\lambda_1,\lambda_2\in\mathbf{R}$. By (4.9) the function $t\mapsto\varphi_{\lambda_1}(t)\varphi_{\lambda_2}(t)$ is in L^p for all $p>1$. Let

$$a(\lambda_1,\lambda_2,\lambda_3):=(\varphi_{\lambda_1}\varphi_{\lambda_2})^\wedge(\lambda_3)=\int_0^\infty\varphi_{\lambda_1}(t)\varphi_{\lambda_2}(t)\varphi_{\lambda_3}(t)\,d\mu(t). \tag{4.14}$$

Theorem 4.4. *The function a is nonnegative on $\mathbf{R}^3$.*

Proof. In view of Lemma 4.1 we have to prove that for all $g\in C_0^\infty$

$$(\varphi_{\lambda_1}\varphi_{\lambda_2}*g|g)\ge 0.$$

The left hand side of the above inequality equals

$$\int_0^\infty\int_0^\infty\int_0^\infty g(t_1)\overline{g(t_2)}\varphi_{\lambda_1}(t_3)\varphi_{\lambda_2}(t_3)K(t_1,t_2,t_3)\,d\mu(t_1)\,d\mu(t_2)\,d\mu(t_3). \tag{4.15}$$

We first compute, using (4.6),

$$\int_0^\infty\varphi_{\lambda_1}(t_3)\varphi_{\lambda_2}(t_3)K(t_1,t_2,t_3)\,d\mu(t_3)$$
$$=\int_0^1\int_0^\pi\varphi_{\lambda_1}(\Delta(t_1,t_2,r,\psi))\varphi_{\lambda_2}(\Delta(t_1,t_2,r,\psi))\,d\tilde m(r,\psi).$$

We can now use the addition formula (2.13) for φ_{λ_1} and φ_{λ_2}, and the orthogonality relations (2.8) for $R_{k,l}^{(\alpha-\beta-1,\beta-(1/2))}$ to find that this is equal to

$$\sum_{k=0}^\infty\sum_{l=0}^k\gamma_{k,l}(\lambda_1)\gamma_{k,l}(\lambda_2)\varphi_{\lambda_1,k,l}(-t_1)\varphi_{\lambda_1,k,l}(t_2)\varphi_{\lambda_2,k,l}(-t_1)\varphi_{\lambda_2,k,l}(t_2),$$

with absolute convergence, uniformly on compact subsets of $(t_1,t_2)\in\mathbf{R}^2$. Now inserting in (4.15) and using that $\varphi_{\lambda,k,l}(-t)=(-1)^{k+l}\varphi_{\lambda,k,l}(t)$ is real-valued we find that

$$(\varphi_{\lambda_1}\varphi_{\lambda_2}*g|g)=\sum_{k=0}^\infty\sum_{l=0}^k\gamma_{k,l}(\lambda_1)\gamma_{k,l}(\lambda_2)\left|\int_0^\infty g(s)\varphi_{\lambda_1,k,l}(s)\varphi_{\lambda_2,k,l}(s)\,d\mu(s)\right|^2\ge 0,$$

since $\gamma_{k,l}(\lambda)\ge 0$ if $\lambda\in\mathbf{R}$. □

Corollary 4.5.. *For real* λ_1, λ_2 *the function* $\lambda_3 \mapsto a(\lambda_1, \lambda_2, \lambda_3)$ *is in* $L^1([0, \infty), dv)$ *and*

$$\varphi_{\lambda_1}(t)\varphi_{\lambda_2}(t) = \int_0^\infty a(\lambda_1, \lambda_2, \lambda_3)\varphi_{\lambda_3}(t)\, dv(\lambda_3). \tag{4.16}$$

Proof. Use Lemma 4.3. □

Putting $t=0$ in (4.16) we get

$$\int_0^\infty a(\lambda_1, \lambda_2, \lambda_3)\, dv(\lambda_3) = 1. \tag{4.17}$$

For $\chi, \psi \in L^1([0, \infty), dv)$ define the dual convolution product

$$(\chi \circ \psi)(\lambda_1) := \int_0^\infty \int_0^\infty \chi(\lambda_2)\psi(\lambda_3) a(\lambda_1, \lambda_2, \lambda_3)\, dv(\lambda_2)\, dv(\lambda_3). \tag{4.18}$$

Application of (4.17) and (4.16) yields

Corollary 4.6. *If* $\chi, \psi \in L^1([0, \infty), dv)$ *then* $\chi \circ \psi \in L^1([0, \infty), dv)$,

$$\|\chi \circ \psi\|_1 \leqq \|\chi\|_1 \|\psi\|_1 \tag{4.19}$$

and

$$\mathcal{F}_{\alpha,\beta}^{-1}(\chi \circ \psi) = \mathcal{F}_{\alpha,\beta}^{-1}(\chi)\mathcal{F}_{\alpha,\beta}^{-1}(\psi). \tag{4.20}$$

Furthermore, if $\chi, \psi \geqq 0$ *then* $\chi \circ \psi \geqq 0$.

By standard arguments (cf. for instance [6, Theor. 5.4]), we get

Corollary 4.7. *If* $1 \leqq p, q, r \leqq \infty$ *and* $p^{-1}+q^{-1}-1=r^{-1}$, *then for* $\chi \in L^p(dv)$, $\psi \in L^q(dv)$, *the function* $\chi \circ \psi$ *is well-defined and satisfies*

$$\|\chi \circ \psi\|_r \leqq \|\chi\|_p \|\psi\|_q.$$

Remark 4.8. The previous results also hold if $\alpha=\beta>-\frac{1}{2}$ or $\alpha>\beta=-\frac{1}{2}$. They can be derived in the same way.

Remark 4.9. There is a striking contrast between the convolution product (4.7) and the dual convolution product (4.18). It follows from (4.9) and (4.14) that the kernel $a(\lambda_1, \lambda_2, \lambda_3)$ is analytic on

$$\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{C}^3 \,|\, |\mathrm{Im}\, \lambda_1| + |\mathrm{Im}\, \lambda_2| + |\mathrm{Im}\, \lambda_3| < \alpha+\beta+1\}.$$

In particular, for fixed $\lambda_1, \lambda_2 \in \mathbf{R}$, the function $\lambda_3 \mapsto a(\lambda_1, \lambda_2, \lambda_3)$ is analytic for $|\mathrm{Im}\, \lambda_3| < \alpha+\beta+1$. Hence, the restriction of this function to $\mathbf{R}$ has no compact support, in contrast with the function $t_3 \mapsto K(t_1, t_2, t_3)$. Also, if $\chi, \psi \in L^1([0, \infty), dv)$ then $\chi \circ \psi$ is analytic on the strip $\{\lambda \in \mathbf{C} \,|\, |\mathrm{Im}\, \lambda| < \alpha+\beta+1\}$.

Remark 4.10. The kernel $a(r, s, t)$ was explicitly calculated by Mizony [14] for $\beta=-\frac{1}{2}$ and $\alpha=0$ or $\frac{1}{2}$. It would be of interest to generalize his results to the case of general α $(\beta=-\frac{1}{2})$, or even to general (α, β).

Remark 4.11. By using a group theoretic interpretation of the functions (1.1) the following extension of our results was proved in [4, § 12]:

Let $\alpha\in\mathbf{Z}^+$, $\beta\geqq 0$. *Let* $D_{\alpha,\beta}$ *denote the finite set* $\{\lambda=i\eta\in i\mathbf{R}\,|\,\exists m\in\mathbf{Z}^+$ *such that* $|\eta|=\beta-\alpha-1-2m>0\}$. *Then*

$$\varphi_{\lambda_1}^{(\alpha,\beta)}(t)\varphi_{\lambda_2}^{(\alpha,\beta)}(t)=\int_0^\infty a(\lambda_1,\lambda_2,\lambda_3)\varphi_{\lambda_3}^{(\alpha,\beta)}(t)\,dv(\lambda_3) + \sum_{\lambda_3\in D_{\alpha,\beta}} a(\lambda_1,\lambda_2,\lambda_3)\varphi_{\lambda_3}^{(\alpha,\beta)}(t)\,\|\varphi_{\lambda_3}^{(\alpha,\beta)}\|_2^{-2} \tag{4.21}$$

with

$$a(\lambda_1,\lambda_2,\lambda_3)\geqq 0 \quad \textit{for} \quad \lambda_1,\lambda_2,\lambda_3\in\mathbf{R}\cup D_{\alpha,\beta}.$$

In a forthcoming paper we will prove this result by analytic methods, also for non-integer α.

Remark 4.12. Let $\alpha\geqq\beta\geqq-\frac{1}{2}$, $\gamma\geqq\delta\geqq-\frac{1}{2}$ and $\gamma+\delta<\alpha+\beta$. Then $\varphi_\lambda^{(\alpha,\beta)}\in L^p([0,\infty), d\mu_{\gamma,\delta})$ for some p, $1\leqq p<2$ (cf. (4.9)) and

$$b_{\alpha,\beta;\gamma,\delta}(\lambda_0,\lambda):=\int_0^\infty \varphi_{\lambda_0}^{(\alpha,\beta)}(t)\varphi_\lambda^{(\gamma,\delta)}(t)\,d\mu_{\gamma,\delta}(t) \tag{4.22}$$

is well-defined for $\lambda,\lambda_0\in\mathbf{R}$. As another application of the addition formula it is possible to prove the nonnegativity of $b_{\alpha,\beta;\gamma,\delta}(\lambda_0,\lambda)$ under certain conditions for α, β, γ, δ (cf. the analogous results in Theorem 4.2 and Corollary 6.1 of [13]). In particular, we obtain that

$$\varphi_{\lambda_0}^{(\alpha,\delta)}(t)=\int_0^\infty b_{\alpha,\delta;\gamma,\delta}(\lambda_0,\lambda)\varphi_\lambda^{(\gamma,\delta)}(t)\,dv(\lambda) \tag{4.23}$$

with nonnegative b for $\alpha>\gamma\geqq\delta\geqq-\frac{1}{2}$, and

$$\varphi_{\lambda_0}^{(\alpha,\beta)}(t)=2^{3/2}\pi^{-(1/2)}\int_0^\infty b_{\alpha,\beta;-(1/2),-(1/2)}(\lambda_0,\lambda)\cos\lambda t\,dt \tag{4.24}$$

with nonnegative b for $\alpha\geqq\beta\geqq-\frac{1}{2}$, $(\alpha,\beta)\neq(-\frac{1}{2},-\frac{1}{2})$. However, much stronger results on the nonnegativity of b (as conjectured in [4, § 12]) can be obtained by using the addition formula for the functions (1.1). Therefore, we postpone a more detailed discussion of this problem to our subsequent paper.

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