Investments, wealth and risk tolerance

Lecture V

Topics

- Utility-based measurement of performance
- Variational and stochastic components of optimal utility volume
- Optimal allocations and their stochastic evolution
- Efficient frontier

Utility-based measurement of performance

Stochastic environment

Important ingredients

- Time evolution concurrent with the one of the investment universe
- Consistency with up to date information
- Incorporation of available opportunities and constraints
- Meaningful optimal utility volume

Dynamic utility

$U(x,t)$ is an $\mathcal{F}_t$-adapted process

- As a function of x, U is increasing and concave

- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi},t) \mid \mathcal{F}_s) \leq U(X_s^{\pi},s) \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*},t) \mid \mathcal{F}_s) = U(X_s^{\pi^*},s) \qquad 0 \leq s \leq t$$

Traditional framework

A deterministic utility datum $u(x,T)$ is assigned at the end of a fixed investment horizon

$$U(x,T) = u(x,T)$$

Backwards in time generation of optimal utility volume

$$V(X_t,t) = \sup_{\pi} E_{\mathbb{P}}(u(X_t^{\pi},T)/\mathcal{F}_t)$$

$$V(X_t^{\pi^*},t) = E_{\mathbb{P}}(V(X_s^{\pi^*},s)/\mathcal{F}_t) \quad \text{(DPP)}$$

$$\Downarrow$$

$$U(x,t) \equiv V(x,t) \qquad 0 \le t < T$$

The dynamic utility coincides with the traditional value function.

It remains constant for times beyond T.

Alternative framework

A deterministic utility datum $u(x, 0)$ is assigned at the beginning of the trading horizon, $t = 0$

$$U(x, 0) = u(x, 0)$$

Forward in time generation of optimal utility volume

$$U(X_s^{\pi^*}, x) = E_{\mathbb{P}}(U(X_t^{\pi^*}, t)/\mathcal{F}_s) \qquad 0 \leq s \leq t$$

- Dynamic utility can be defined for all trading horizons
- Utility and allocations more intuitive
- Difficulties due to the "inverse in time" nature of the problem

Construction of a class of forward dynamic utilities

Creating the martingale that yields the optimal utility volume

Minimal model assumptions

Stochastic optimization problem “inverse” in time

Key idea

Stochastic input

Market

Variational input

Individual

Maximal utility — Optimal allocation

Variational utility input

- Key ingredients : wealth and risk tolerance

- Risk tolerance solves a fast diffusion equation posed inversely in time

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ r(x,0) = -\frac{u_0'(x)}{u_0''(x)} \end{cases}$$

- Utility surface generated by a transport equation

$$\begin{cases} u_t + \frac{1}{2} r(x,t) u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Stochastic market input

Investment universe

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P}) \quad ; \quad W = (W^1, \ldots, W^d)$ standard Brownian Motion

- Traded securities

$$1 \leq i \leq k \qquad \begin{cases} dS_t^i = S_t^i(\mu_t^i dt + \sigma_t^i \cdot dW_t) \ , & S_0^i > 0 \\ dB_t = r_t B_t dt \ , & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-measurable stochastic processes

- Postulate existence of a $\mathcal{F}_t$-measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t \, \mathbb{1} = \sigma_t^T \lambda_t$$

Investment universe

- Self-financing investment strategies $\pi_t^0, \pi_t^i, \quad i = 1, \ldots, k$

- Present value of this allocation

$$X_t = \sum_{i=0}^{k} \pi_t^i$$

$$dX_t = \sum_{i=0}^{k} \pi_t^i (\mu_t^i - r_t)\, dt + \sum_{i=0}^{k} \pi_t^i \sigma_t^i \cdot dW_t$$

$$= \sigma_t \pi_t \cdot (\lambda_t\, dt + dW_t)$$

$$\pi_t = (\pi_t^1, \ldots, \pi_t^k), \qquad \mu_t - r_t\, \mathbb{1} = \sigma_t^T \lambda_t$$

Market input processes

(σ_t, λ_t) and (Y_t, Z_t, A_t)

These $\mathcal{F}_t$-mble processes do **not** depend on the investor's variational utility

They reflect and represent, respectively

(λ_t, σ_t) : dynamics of traded securites

Y_t : benchmark
numeraire

Z_t : market view away from market equilibrium
feasibility and trading constraints

A_t : subordination

The processes (Y_t, Z_t, A_t)

- Benchmark and/or numeraire

A "replicable" process Y_t satisfying

$$\begin{cases} dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\ Y_0 = 1 \end{cases}$$

$$\delta_t \in \mathcal{F}_t \ , \qquad \sigma_t \sigma_t^+ \delta_t = \delta_t$$

σ_t^+ : Moore-Penrose matrix inverse

Market input processes

- Market views, feasibility and trading constraints

An exponential martingale Z_t satisfying

$$\begin{cases} dZ_t = Z_t \phi_t \cdot dW_t \\ Z_0 = 1 \, , \quad \phi_t \in \mathcal{F}_t \end{cases}$$

- Subordination

A non-decreasing process A_t solving

$$\begin{cases} dA_t = |\delta_t - \sigma_t \sigma_t^{+} (\lambda_t + \phi_t)|^2 \, dt \\ A_0 = 0 \end{cases}$$

Optimal utility volume
Optimal asset allocation

Optimal utility volume

Stochastic input : (Y_t, Z_t, A_t)

Variational input : $u(x,t)$

Benchmark Y_t

Time change A_t

$$u_t u_{xx} = \frac{1}{2} u_x^2$$

$$u(x,0) = u_0(x)$$

Market view Z_t

$$U(x,t) = u(\frac{x}{Y_t}, A_t) Z_t$$

Optimal utility volume

Stochastic market input

$\lambda_t,\ \sigma_t$

$\downarrow$

benchmark, views

subordination

$(Y_t,\ Z_t,\ A_t)$

Variational input

$x, r_0(x) = -\dfrac{u_0'(x)}{u_0''(x)}$

$\downarrow$

$r_t + \frac{1}{2}r^2 r_{xx} = 0$ (FDE)

$u_t + \frac{1}{2}r u_x = 0$ (TE)

$u(x,t)$

$$U(x,t) = u\left(\frac{x}{A_t}, Y_t\right)Z_t$$

Model independent construction!

What is the optimal allocation?

Optimal portfolio processes

$$\pi_t = (\pi_t^0, \pi_t^1, \ldots, \pi_t^k)$$

can be directly and explicitly characterized

along with the construction of the forward utility!

The structure of optimal portfolios

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t)$$

Stochastic input
Market

$(Y_t, \, Z_t, \, A_t)$
$\lambda_t, \, \sigma_t, \, \delta_t, \, \phi_t$

Variational input
Individual

wealth x
risk tolerance $r(x,t)$

$\dfrac{1}{Y_t}\pi_t^*$ is a *linear* combination

of (benchmarked) optimal wealth

and subordinated (benchmarked) risk tolerance

Optimal asset allocation

- Let X_t^* be the optimal wealth, Y_t the benchmark and A_t the subordination processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$

$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

- Define r_t^* the subordinated (benchmarked) risk tolerance

$$r_t^* = r\left(\frac{X_t^*}{Y_t}, A_t\right)$$

Optimal (benchmarked) portfolios

$$\boxed{\frac{1}{Y_t}\pi_t^* = \sigma_t^+ \left((\lambda_t + \phi_t) r_t^* + \delta_t \left(\frac{X_t^*}{Y_t} - r_t^*\right)\right)}$$

Wealth-risk tolerance stochastic evolution

A system of SDEs at optimum utility volume

$$\widehat{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \widehat{r}_t^* = r(\widehat{X}_t^*, A_t)$$

$$\begin{cases} d\widehat{X}_t^* = \widehat{r}_t^* (\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \\ d\widehat{r}_t^* = r_x(\widehat{X}_t^*, A_t) d\widehat{X}_t^* \end{cases}$$

- Separability of wealth dynamics in terms of risk tolerance and market input
- Sensitivity of risk tolerance in terms of its spatial gradient and changes in optimal wealth
- Utility functional has essentially vanished

Universal representation, no Markovian assumptions

An efficient frontier

Optimal wealth-risk tolerance $(\widehat{X}_t^*, \widehat{r}_t^*)$ system of SDEs in original market configuration

$$\begin{cases} d\widehat{X}_t^* = \widehat{r}_t^*(\sigma_t \sigma_t^+(\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \\ \\ d\widehat{r}_t^* = r_x(\widehat{X}_t^*, A_t)\, d\widehat{X}_t^* \end{cases}$$

$$\downarrow$$

change of measure	change of time
historical $\longrightarrow$ benchmarked	Levy's theorem

An efficient frontier

Optimal wealth-risk tolerance (x_t^1, x_t^2) system of SDEs

in canonical market configuration

$$x_t^1 = \left(\frac{X_t^*}{Y_t}\right)_{A_t^{(-1)}} \qquad x_t^2 = r\left(\frac{X_t^*}{Y_t}, A_t\right)_{A_t^{(-1)}}$$

$$\langle M_t \rangle = A_t \qquad w_t = M_{A^{(-1)}}$$

$$\downarrow$$

$$\begin{cases} dx_t^1 = x_t^2 \, dw_t \\ dx_t^2 = r_x(x_t^1, t) x_t^2 \, dw_t \\ x_0^1 = \frac{x}{y} \, , \quad x_0^2 = r_x(\frac{x}{y}, 0) \end{cases}$$

Analytic solution of the efficient frontier SDE system

$$\begin{cases} dx_t^1 = x_t^2 \, dw_t \\ dx_t^2 = r_x(x_t^1, t) x_t^2 \, dw_t \end{cases}$$

- Define the budget capacity function $h(x,t)$ via

$$x = \int_{\underline{x}}^{h(x,t)} \frac{du}{r(u,t)} = \int_{\underline{x}}^{h(x,t)} \gamma(u,t) du$$

$\underline{x}$: related to symmetry properties of risk tolerance, reflection point of its spatial derivative and risk aversion front

Analytic solutions

The budget capacity function h solves the (inverse) heat equation

$$\begin{cases} h_t + \frac{1}{2}h_{xx} - \frac{1}{2}r_x(\underline{x}, t)h_x = 0 \\ h(x,0) = h_0(x)\ , \quad x = \displaystyle\int_{\underline{x}}^{h_0(x)} \frac{du}{r(u,0)} \end{cases}$$

Solution of the efficient frontier SDE system

$$\begin{cases} x_t^1 = h(z_t, t) \\ x_t^2 = h_z(z_t, t) \end{cases}$$

$$z_t = h_0^{-1}(x) - \int_0^t \frac{1}{2} r_x(\underline{x}, s)ds + w_t$$

Using equivalent measure transformations and time change we recover the original pair of optimal (benchmarked) wealth and (benchmark) risk tolerance

Utility-based performance measurement

Market

Benchmark, views, constraints

Market input processes

Subordination

Investor

Wealth, risk tolerance

Fast diffusion eqn

Transport eqn

Forward evolution
Y_t, Z_t, A_t
x, $r(x,t)$, $u(x,t)$

Optimal utility volume and optimal portfolios

measure change | time change

Efficient frontier SDE system

Heat eqn | Fast diffusion eqn

Universal analytic solutions

Dynamic exponential utility

Objective: Find an $\mathcal{F}_t-$adapted process $U_t(x)$ such that

$$\begin{cases} U_0(x) = -\exp\left(-\frac{x}{y}\right) \\ E_{\mathbb{P}}\left(U_s\left(X_s^{\pi}\right)|\mathcal{F}_t\right) \leq U_t\left(X_t^{\pi}\right) \\ E_{\mathbb{P}}\left(U_s\left(X_s^{\pi^*}\right)|\mathcal{F}_t\right) = U_t\left(X_t^{\pi^*}\right), \qquad s \geq t \end{cases}$$

Solution

- Variational input

$$u(x, y, z) = -\exp\left(-\frac{x}{y} + z\right)$$

Dynamic exponential utility (continued)

- Stochastic market input consists of a pair of Ito processes, (Y, Z), solving, respectively,

$$\begin{cases} dY_t = Y_t \delta_t \cdot (\kappa_t dt + dW_t) \\ \\ Y_0 = y > 0 \end{cases}$$

and

$$\begin{cases} dZ_t = \eta_t dt + \xi_t \cdot dW_t \\ \\ Z_0 = 0. \end{cases}$$

- Moore-Penrose pseudo-inverse matrix σ^+ of σ $(d \times k)$ is the unique $k \times d$ matrix satisfying

$$\sigma\sigma^+\sigma = \sigma \qquad \sigma^+\sigma\sigma^+ = \sigma^+$$

$$\left(\sigma\sigma^+\right)^T = \sigma\sigma^+ \quad \left(\sigma^+\sigma\right)^T = \sigma^+\sigma.$$

- The processes $\delta, \kappa, \eta, \xi$ are taken to be bounded and $\mathcal{F}_t-$progressively measurable. It is, also, assumed that

$$\sigma\sigma^+\delta = \delta \quad \text{and} \quad \delta \cdot (\kappa - \lambda) = 0$$

- The drift η of the process Z satisfies

$$2\eta = \left|\delta - \sigma\sigma^+ (\lambda + \xi)\right|^2 - |\xi|^2$$

- Wlog, the dynamics of the benchmark process Y can be written as

$$dY_t = Y_t\delta_t \cdot (\lambda_t dt + dW_t)$$

Dynamic exponential performance

Solution

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} + Z_t\right)$$

Idea of the proof: applying Ito calculus and using the structural assumptions on the market input yields

$$dU(X) = U(X)\left(-Y^{-1}\beta \cdot dW + XY^{-1}\delta \cdot dW + \xi \cdot dW\right)$$

$$+\frac{1}{2}U(X)\left(-2Y^{-1}\beta \cdot \lambda + 2XY^{-1}\delta \cdot \kappa + 2\eta + Y^{-2}|\beta|^2 + 2\left(Y^{-1} - XY^{-2}\right)\delta \cdot \beta\right.$$

$$\left.-2Y^{-1}\xi \cdot \beta + 2XY^{-1}\delta \cdot \xi + \left(X^2Y^{-2} - 2XY^{-1}\right)|\delta|^2 + |\xi|^2\right) dt$$

$$= U(X)\left(-Y^{-1}\beta \cdot dW + XY^{-1}\delta \cdot dW + \xi \cdot dW\right)$$

$$+\frac{1}{2}U(X)\left(\left|Y^{-1}\beta - \left((\lambda + \xi) + \left(XY^{-1} - 1\right)\delta\right)\right|^2 + 2XY^{-1}\delta \cdot (\kappa - \lambda)\right.$$

$$\left.+2\eta + |\xi|^2 - |\delta - (\lambda + \xi)|^2\right) dt$$

Idea of the proof (continued)

and, in turn,

$$dU(X) = U(X)\left(-Y^{-1}B^{-1}\sigma\pi + XY^{-1}\delta + \xi\right)\cdot dW$$

$$+\frac{1}{2}U(X)\left(\left|Y^{-1}B^{-1}\sigma\pi - \sigma\sigma^{+}\left((\lambda+\xi) + \left(XY^{-1}-1\right)\delta\right)\right|^2 + 2XY^{-1}\delta\cdot(\kappa-\lambda)\right.$$

$$\left. + \left|\left(I - \sigma\sigma^{+}\right)(\lambda+\xi)\right|^2 + 2\eta + |\xi|^2 - |\delta - (\lambda+\xi)|^2\right) dt$$

..... and, finally,

$$dU(X) = U(X)\left(-Y^{-1}B^{-1}\sigma\pi + XY^{-1}\delta + \xi\right)\cdot dW$$

$$\frac{1}{2}U(X)\left|Y^{-1}B^{-1}\sigma\pi - \sigma\sigma^{+}\left(\lambda + \left(XY^{-1}-1\right)\delta + \xi\right)\right|^2 dt$$

At the optimum

- Feedback portfolio control process

$$\pi^* = YB\sigma^+ \left(\lambda + \left(X^* Y^{-1} - 1\right)\delta + \xi\right).$$

- Optimal wealth process

$$dX^* = B^{-1}\sigma\pi^* \cdot (\lambda dt + dW)$$

$$= \left(Y\left(\sigma\sigma^+ (\lambda + \xi) - \delta\right) + X^*\delta\right) \cdot (\lambda dt + dW).$$

- Optimal utility volume

$$dU(X^*) = U(X^*)\left(-Y^{-1}B^{-1}\sigma\pi^* + X^* Y^{-1}\delta + \xi\right) \cdot dW$$

$$= U(X^*)\left(\sigma\sigma^+ (\delta - \lambda) + \left(I - \sigma\sigma^+\right)\xi\right) \cdot dW.$$

Explicit solutions

- Optimal wealth process:

$$X_t^* = \mathcal{E}_t \left(x + \int_0^t \mathcal{E}_s^{-1} Y_s \left(\sigma_s \sigma_s^+ (\lambda_s + \xi_s) - \delta_s \right) \cdot ((\lambda_s - \delta_s)\, ds + dW_s) \right)$$

$$\mathcal{E}_t = \exp\left(\int_0^t \left(\delta_s \cdot \lambda_s - \tfrac{1}{2} |\delta_s|^2 \right) ds + \int_0^t \delta_s \cdot dW_s \right)$$

- Optimal portfolio process

$$\pi_t^* = B_t Y_t \sigma_t^+ (\lambda_t + \xi_t - \delta_t) + B_t X_t^* \sigma_t^+ \delta_t$$

$$= x \mathcal{E}_t B_t \sigma_t^+ \delta_t + B_t Y_t \sigma_t^+ (\lambda_t + \xi_t - \delta_t)$$

$$+ B_t \mathcal{E}_t \left(\int_0^t \mathcal{E}_s^{-1} Y_s \left(\sigma_s \sigma_s^+ (\lambda_s + \xi_s) - \delta_s \right) \cdot ((\lambda_s - \delta_s)\, ds + dW_s) \right) \sigma_t^+ \delta_t$$

- Optimal utility volume

$$U_t(X_t^*) = \exp\left(-\frac{x}{y} - \int_0^t \frac{1}{2} \left| \sigma_s \sigma_s^+ (\delta_s - \lambda_s) + \left(I - \sigma_s \sigma_s^+ \right) \xi_s \right|^2 ds \right.$$

$$\left. + \int_0^t \left(\sigma_s \sigma_s^+ (\delta_s - \lambda_s) + \left(I - \sigma_s \sigma_s^+ \right) \xi_s \right) \cdot dW_s \right)$$

Case I: No benchmark and 'no' views $\delta = \xi = 0$.

Then, $Y_t = y$, for $t \geq 0$.

- The forward performance process takes the form

$$U_t\left(x\right) = -\exp\left(-\frac{x}{y} + \int_0^t \frac{1}{2}\left|\sigma_s \sigma_s^+ \lambda_s\right|^2 ds\right)$$

Note that even in this simple case, the solution is equal to the classical exponential utility only at $t = 0$.

- The optimal discounted wealth and optimal asset allocation are given, respectively, by

$$X_t^* = x + \int_0^t y\left(\sigma_s \sigma_s^+ \lambda_s\right) \cdot \left(\lambda_s ds + dW_s\right)$$

and

$$\pi_t^* = y B_t \sigma_t^+ \lambda_t$$

Observe that π^* is **independent** of the initial wealth x.

Case 1: No benchmark and 'no' views $\delta = \xi = 0$ (continued)

- Optimal utility volume

$$U_t\left(X_t^*\right) = -\exp\left(-\frac{x}{y} - \int_0^t \frac{1}{2}\left|\sigma_s \sigma_s^+ \lambda_s\right|^2 ds - \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot dW_s\right)$$

Observe that π^* is *independent* of the initial wealth x.

- Total amount allocated in the risky assets

$$\mathbf{1} \cdot \frac{\pi_t^*}{B_t} = \mathbf{1} \cdot y \sigma_t^+ \lambda_t$$

- Amount invested in the riskless asset

$$\pi_t^{0,*} = X_t^* - \mathbf{1} \cdot y \sigma_t^+ \lambda_t$$

Such an allocation is rather **conservative** and is often viewed as an argument **against** the classical exponential utility.

Case 2: Static performance $\sigma\sigma^{+}(\delta-\lambda)+\left(I-\sigma\sigma^{+}\right)\xi=0.$

Then $\sigma^{+}(\delta-\lambda)=0$ and $\sigma\sigma^{+}\xi=\xi$, and $Z_t=\int_0^t \xi_s\cdot dW_s$

- Dynamic exponential utility

$$U_t(x)=-\exp\left(-\frac{x}{Y_t}+\int_0^t \xi_s\cdot dW_s\right)$$

- Optimal discounted wealth

$$X_t^*=\mathcal{E}_t\left(x+\int_0^t \mathcal{E}_s^{-1}Y_s\xi_s\cdot((\lambda_s-\delta_s)\,ds+dW_s)\right)$$

Case 2: Static performance $\sigma\sigma^{+}(\delta-\lambda)+\left(I-\sigma\sigma^{+}\right)\xi=0.$ (continued)

- Optimal allocation

$$\pi_t^* = x\mathcal{E}_t B_t \sigma_t^+ \delta_t + Y_t B_t \sigma_t^+ \xi_t + B_t \mathcal{E}_t \left(\int_0^t \mathcal{E}_s^{-1} Y_s \xi_s \cdot dW_s \right) \sigma_t^+ \delta_t$$

- Optimal utility volume

$$U_t\left(X_t^*\right) = U_0\left(x\right) = -\exp\left(-\frac{x}{y}\right)$$

Observe that the optimal level of forward performance remains constant across times.

Case 3: No benchmark and risk neutralization $\delta = 0$ and $\lambda + \xi = 0$.

Then, $\mathcal{E}_t = 1$, $Y_t = y > 0$ and $Z_t = -\int_0^t \frac{1}{2} |\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s$.

- Dynamic exponential utility

$$U_t(x) = -\exp\left(-\frac{x}{y} - \frac{1}{2}\int_0^t |\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s\right)$$

- Optimal discounted wealth

$$X_t^* = x$$

Case 3: No benchmark and risk neutralization $\delta = 0$ and $\lambda + \xi = 0$. (continued)

- Optimal allocations

$$\pi_t^* = 0 \qquad \text{and} \qquad \pi_t^{0,*} = X_t^* = x.$$

- Optimal utility volume

$$U_t\left(X_t^*\right) = U_t\left(x\right).$$

It is important to notice that, for all trading times, the optimal allocation consists of putting **zero** into the risky assets and, therefore, investing the entire wealth into the riskless asset. Such a solution seems to capture quite accurately the strategy of a **derivatives trader** for whom the underlying **objective is to hedge** as opposed to the asset manager whose objective is to invest.

Case 4: Following the benchmark $\delta = \lambda + \xi$ with $\lambda + \xi \neq 0$.

Then $\delta = \sigma\sigma^{+}(\lambda + \xi)$ and, in turn, $Z_t = -\int_0^t \frac{1}{2}|\xi_s|^2 ds + \int_0^t \xi_s \cdot dW_s$.

- Dynamic exponential utility

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2}|\xi_s|^2 ds + \int_0^t \xi_s \cdot dW_s\right) .$$

- Optimal wealth

$$X_t^* = x\mathcal{E}_t.$$

- Returns of wealth and of benchmark

$$\frac{dX_t^*}{X_t^*} = \frac{dY_t}{Y_t}$$

Case 4: Following the benchmark $\delta = \lambda + \xi$ with $\lambda + \xi \neq 0.$ (continued)

- Optimal allocation

$$\pi_t^* = B_t X_t^* \sigma_t^+ \delta_t$$

- Optimal performance level

$$U_t\left(X_t^*\right) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2}\left|\xi_s\right|^2 ds + \int_0^t \xi_s \cdot dW_s\right)$$

Observe that, contrary to what we have observed in traditional backward exponential utility problems, the optimal portfolio is a **linear functional of the wealth** and not independent of it.

Case 5: Generating arbitrary portfolio allocations

- Assume that $\mathbf{1} \cdot \sigma_t^{+} (\lambda_t + \xi_t) = 1$. Then

$$\mathbf{1} \cdot \frac{\pi_t^*}{B_t} = X_t^* \qquad \text{and} \qquad \pi_t^{0,*} = 0$$

Hence, the optimal allocation π^* puts *zero* amount in the riskless asset and invests **all** wealth in the risky assets, according to the weights specified by the vector $\sigma^{+} (\lambda + \xi)$.

Case 5: Generating arbitrary portfolio allocations (continued)

- Note, also, that for an arbitrary vector ν_t with $\mathbf{1} \cdot \sigma_t^+ \nu_t \neq 0$, the vector

$$\xi_t = \frac{1 - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{\mathbf{1} \cdot \sigma_t^+ \nu_t} \nu_t$$

  satisfies the above constraint since $\mathbf{1} \cdot \sigma_t^+ \left(\lambda_t + \frac{1 - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{\mathbf{1} \cdot \sigma_t^+ \nu_t} \nu_t \right) = 1$

Can we generate optimal portfolios that allocate **arbitrary**, but constant, fractions of wealth to the different accounts?

The answer is affirmative. Indeed, for $p \in \mathcal{R}$, set,

$$\mathbf{1} \cdot \sigma_t^+ \left(\lambda_t + \xi_t \right) = p$$

Then, the total investment in the risky assets and the allocation in the riskless bond are

$$\mathbf{1} \cdot \frac{\pi_t^*}{B_t} = p X_t^* \qquad \text{and} \qquad \frac{\pi_t^0}{B_t} = (1 - p) X_t^*$$