Fundamentals in Optimal Investments

Lecture I

Portfolio choice

- Portfolio allocations and their ordering
- Performance indices
- Fundamentals in optimal portfolio choice
- Expected utility theory and its limitations
- Forward utilities and risk budgeting

Allocations and their ordering

Contents

- What is an allocation
- How do we order them?
- Stochastic dominance
- Indices of satisfaction

Readings: W. F. Sharpe: *Investors and Markets (2007)*

A. Meucci: *Risk and Asset and Allocation (2005)*

C.-F. Huang and R. H. Litzenberger: *Foundations of Mathematical Economics (1988)*

Portfolio allocations

Investment horizon $[0,T]$, $T \leq +\infty$

Market uncertainty: $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$

- Securities available for trading $\mathbf{S}_t = \left(S_t^1, ..., S_t^n\right)$

- Allocation $\boldsymbol{\alpha}_t = \left(\alpha_t^1, ..., \alpha_t^n\right)$

  $\boldsymbol{\alpha}_t \in \mathcal{F}_t$ and self-financing (trading constraints/transaction costs)

- Portfolio $X_t^{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_t \cdot \mathbf{S}_t$

Objectives

Investment time: $t \le s \le T, \ T \le \infty$

- Absolute wealth

$$X_s^{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_s \cdot \mathbf{S}_s$$

Personal wealth management : $T \sim 30$ years

- Performance relative to a benchmark $Y_s = \boldsymbol{\beta}_s \cdot \mathbf{S}_s$

$$X_s^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \boldsymbol{\alpha}_s \cdot \mathbf{S}_s - \frac{\boldsymbol{\alpha}_t \cdot \mathbf{S}_t}{Y_t} Y_s$$

Fund managers : $T \sim$ a year/ quarter

Objectives (continued)

- Net profit

$$X^{\boldsymbol{\alpha}}_{s,s+\delta} = \boldsymbol{\alpha}_{s+\delta} \cdot \mathbf{S}_{s+\delta} - \boldsymbol{\alpha}_s \cdot \mathbf{S}_s$$

Derivatives trader (P&L) : $T \sim$ one day

How do we order the allocations?

- Stochastic Dominance

- Performance/satisfaction indices

Stochastic Dominance

Static criteria

- Fix time at t_0 : $t \leq t_0 \leq T$

- Allocations at t_0: $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

- Objectives: $X^{\boldsymbol{\alpha}}$ and $X^{\boldsymbol{\beta}}$

  The objectives are expressed in certain units

How can we compare $X^{\boldsymbol{\alpha}}$ and $X^{\boldsymbol{\beta}}$?

Degrees of stochastic dominance

Strong Dominance (zeroth degree)

$$X^{\boldsymbol{\alpha}} \;\geq\; X^{\boldsymbol{\beta}} \quad \text{a.s.}$$

Example: Objectives $X^{\boldsymbol{\alpha}}$ and $X^{\boldsymbol{\beta}}$ have, respectively, chi-square distribution with two and one degree of freedom

$$X^{\boldsymbol{\alpha}} \sim \chi_2^2 \quad \text{and} \quad X^{\boldsymbol{\beta}} \sim \chi_1^2$$

Assume $X^{\boldsymbol{\alpha}} = X^{\boldsymbol{\beta}} + Y$ with $Y \sim \chi_1^2$

If Y is independent of $X^{\boldsymbol{\beta}}$ then $X^{\boldsymbol{\alpha}}$ strongly dominates $X^{\boldsymbol{\beta}}$

Strong dominance is **inapplicable** since it needs the joint distribution of the objectives. It will also lead to "free-lunches" . Need to **weaken the dominance criteria**

Weak Dominance

First degree stochastic dominance

$$X^{\boldsymbol{\alpha}} \;\geq\; X^{\boldsymbol{\beta}} \qquad \text{iff} \qquad \mathbb{P}\left(X^{\boldsymbol{\alpha}} \leq x\right) \leq \mathbb{P}\left(X^{\boldsymbol{\beta}} \leq x\right) \quad \forall x \in \mathcal{R}$$

$$\Updownarrow$$

$$F_{X^{\boldsymbol{\alpha}}}(x) \leq F_{X^{\boldsymbol{\beta}}}(x) \qquad \text{and} \qquad Q_{X^{\boldsymbol{\alpha}}}(p) \geq Q_{X^{\boldsymbol{\beta}}}(p) \quad \forall x \in \mathcal{R},\;\; p \in (0,1)$$

Example

$$X^{\boldsymbol{\alpha}} \sim N(1,1) \quad \text{and} \quad X^{\boldsymbol{\beta}} \sim N(0,1)$$

$$F_{X^{\boldsymbol{\alpha}}}(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x-1}{\sqrt{2}}\right)\right) \leq \frac{1}{2}\left(1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right) = F_{X^{\boldsymbol{\beta}}}(x)$$

If $X^{\boldsymbol{\alpha}}$ and $X^{\boldsymbol{\beta}}$ are independent, $X^{\boldsymbol{\alpha}} - X^{\boldsymbol{\beta}} \sim N(1,2)$

Then

$$F_{X^{\boldsymbol{\alpha}}-X^{\boldsymbol{\beta}}}(0) = \frac{1}{2}\left(1 + \text{erf}\left(-\frac{1}{\sqrt{2}}\right)\right) > 0$$

Therefore, $X^{\boldsymbol{\alpha}}$ weakly dominates $X^{\boldsymbol{\beta}}$ but **not** strongly

Still, first degree of stochastic dominance **rarely** occurs in practice

Alternative formulation via expected utility criteria

Let assets A and B have returns r_A and r_B. Then $A \succsim^{FSD} B$ if all investors having utility increasing and continuous prefer A to B i.e.

$$Eu(1 + r_A) \geq Eu(1 + r_B)$$

Equivalent results

$$A \succsim^{FSD} B$$
$$\updownarrow$$
$$F_A(z) \leq F_B(z)$$
$$\updownarrow$$
$$r_A =^d r_B + a, \quad a \geq 0 \text{ a.s.}$$

Weak Dominance

Second degree stochastic dominance

$$X^{\boldsymbol{\alpha}} \;\geq\; X^{\boldsymbol{\beta}} \quad \text{iff} \quad E\left(\left(X^{\boldsymbol{\alpha}} - x\right)^{-}\right) \geq E\left(\left(X^{\boldsymbol{\beta}} - x\right)^{-}\right) \quad \forall x \in \mathcal{R}$$

Equivalently: $\mathcal{I}^{2}\left(f_{X^{\boldsymbol{\alpha}}}\right)(x) \leq \mathcal{I}^{2}\left(f_{X^{\boldsymbol{\beta}}}\right)(x)$

Iterated integral: $\mathcal{I}^{n}(v)(x) = \int_{-\infty}^{x_n} v\left(x_1, x_2, ..., z_n, ..., x_N\right) dz_n$

Observe: $\mathcal{I}^{2}\left(f_{X^{\boldsymbol{\alpha}}}\right)(x) \equiv \mathcal{I}\left(F_{X^{\boldsymbol{\alpha}}}\right)(x)$

We then have

$$X^{\boldsymbol{\alpha}} \;\geq\; X^{\boldsymbol{\beta}} \quad \text{iff} \quad \int_{-\infty}^{x} F_{X^{\boldsymbol{\alpha}}}(y)\, dy \leq \int_{-\infty}^{x} F_{X^{\boldsymbol{\beta}}}(y)\, dy$$

Alternative formulation via expected utility criteria

Let assets A and B have returns r_A and r_B. Then $A \succsim^{SSD} B$ if all risk averse investors having differentiable a.e. utility prefer A to B i.e.

$$Eu(1 + r_A) \geq Eu(1 + r_B)$$

Equivalent results

$$A \succsim^{SSD} B$$
$$\updownarrow$$
$$E(r_A) = E(r_B) \quad \text{and} \quad S(z) = \int_0^z (F_A(z) - F_B(z))\, dz \leq 0, \quad \forall z \in [0,1]$$
$$\updownarrow$$
$$r_B =^d r_A + \epsilon, \quad E(\epsilon | r_A) = 0$$
$$\downarrow$$
$$E(r_A) = E(r_B) \quad \text{and} \quad Var(r_A) \leq Var(r_B)$$

Weak Dominance (highest degrees)

q^{th} **degree stochastic dominance**

$$X^{\boldsymbol{\alpha}} \;\geq\; X^{\boldsymbol{\beta}} \qquad \text{iff} \qquad \mathcal{I}^n\left(f_{X^{\boldsymbol{\alpha}}}\right)(x) \leq \mathcal{I}^n\left(f_{X^{\boldsymbol{\beta}}}\right)(x) \quad \forall x \in \mathcal{R}$$

$$\mathcal{I}^n(v)(x) = \int_{-\infty}^{x_n} v\left(x_1, x_2, ..., z_n, ..., x_N\right) dz_n$$

Equivalences

$$\boxed{0^{th}... \Longrightarrow 1^{rst} \Longrightarrow .... \Longrightarrow q^{th}}$$

Limitations of stochastic dominance criteria

- Limited intuition (unless utility-type argumentation is incorporated)
- Computation of the q^{th} cumulative distribution not easy
- Stochastic ordering might not be feasible since allocations might not be comparable
- Inapplicability when we move to dynamic settings

Need to construct more useful/applicable criteria

$$\Downarrow$$

Indices of satisfaction/performance/ophelima

Satisfaction indices

Indices ν are "real numbers" (not always unitless) that are used to order allocations

$$\boldsymbol{\alpha} \to \nu(\boldsymbol{\alpha})$$

Examples: $\nu(\boldsymbol{\alpha}) = E(X^{\boldsymbol{\alpha}})$ or $\nu(\boldsymbol{\alpha}) = \frac{E(X^{\boldsymbol{\alpha}})}{SD(X^{\boldsymbol{\alpha}})}$ (different units)

Performance indices and their properties

- *Estimability*: the satisfaction level associated with an allocation $\boldsymbol{\alpha}$ is **fully** specified by the marginal distribution of $X^{\boldsymbol{\alpha}}$. Then, two allocations with the **same distributions** are **indistinguishable** to the investor

Performance indices and their properties (continued)

- *Sensibility*

$$\text{if } X^{\boldsymbol{\alpha}} \;\geq\; X^{\boldsymbol{\beta}} \text{ a.e. then } \nu(\boldsymbol{\alpha}) \geq \nu(\boldsymbol{\beta})$$

The larger, in a strong-sense, the investor's objective, the more 'satisfied' the investor should be

- *Consistency with stochastic dominance*

$$\text{if } \forall p \in (0,1) \quad Q_{X^{\boldsymbol{\alpha}}}(p) \geq Q_{X^{\boldsymbol{\beta}}}(p) \Longrightarrow \nu(\boldsymbol{\alpha}) \geq \nu(\boldsymbol{\beta})$$

The larger, in a weak-sense, the investor's objective, the more 'satisfied' the investor should be

Example: $\nu(\boldsymbol{\alpha}) = E(X^{\boldsymbol{\alpha}})$

$$E(X^{\boldsymbol{\alpha}}) \equiv \int_{-\infty}^{+\infty} x f_{X^{\boldsymbol{\alpha}}}(x)\,dx \equiv \int_0^1 Q_{X^{\boldsymbol{\alpha}}}(y)\,dy$$

Note that consistency with weak dominance is a stronger requirement on the index than consistency with strong dominance. However, if indices are estimable, the two are equivalent.

Generalizations

$q^{th}-$dominance consistency... $\Longrightarrow$ ....$1^{rst}-$dominance consistency

$\Longrightarrow$ $0^{th}-$dominance consistency

Example: The expected value index is consistent with $2^{nd}-$dominance and, thus, it is consistent with $1^{rst}-$dominance and, in turn, with $0^{th}-$dominance.

Performance indices (continued)

- *Constancy:* if $X^{\boldsymbol{\alpha}} \equiv x^{\boldsymbol{\alpha}}$ then $\nu(\boldsymbol{\alpha}) = x^{\boldsymbol{\alpha}}$

- *Positive homogeneity:* if $X^{\lambda\boldsymbol{\alpha}} = \lambda X^{\boldsymbol{\alpha}}$ then $\nu(\lambda\boldsymbol{\alpha}) = \lambda\nu(\boldsymbol{\alpha})$

- *Translation invariance:* if $X^{\boldsymbol{\beta}} = 1$ then $\nu(\boldsymbol{\alpha} + \boldsymbol{\lambda}\boldsymbol{\beta}) = \nu(\boldsymbol{\alpha}) + \lambda$

- *Sub- and super-additivity:* $\nu(\boldsymbol{\alpha} + \boldsymbol{\beta}) \ \substack{\leq \\ \geq} \ \nu(\boldsymbol{\alpha}) + \nu(\boldsymbol{\beta})$

Performance indices (continued)

- *Co-monotonic additivity:* one objective is a deterministic transformation of the other

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ co-monotonic} \quad \Longrightarrow \quad \nu(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \nu(\boldsymbol{\alpha}) + \nu(\boldsymbol{\beta})$$

- *Concavity-convexity:*

$$\forall \lambda \in (0,1)\,,\; \nu(\lambda \boldsymbol{\alpha} + (1-\lambda)\,\boldsymbol{\beta}) \;\begin{matrix} \le \\ \ge \end{matrix}\; \lambda \nu(\boldsymbol{\alpha}) + (1-\lambda)\,\nu(\boldsymbol{\beta})$$

How is diversification quanitified via these properties?

Performance indices (continued)

- *Risk-aversity:* a risk-free allocation $\mathbf{b}$ is preferred to a risky allocation $\mathbf{b}+\mathbf{f}$, with $\mathbf{f}$ being a fair bet

$$\nu(\mathbf{b}) \geq \nu(\mathbf{b}+\mathbf{f})$$

- Risk-premium: $RP(\mathbf{a}) = E\left(X^{\boldsymbol{\alpha}}\right) - \nu(\mathbf{a})$

Risk aversion: $RP(\mathbf{a}) \geq 0$

Risk propensity: $RP(\mathbf{a}) \leq 0$

Risk neutrality: $RP(\mathbf{a}) = 0$

Expected Utility and its certainty equivalent

Utility functional

A utility u is a mapping from outcomes to real numbers. A utility-linked index of performance links the utility of an outcome with the chances of this outcome occurring. This leads to the Von Neumann-Mongerstern expected utility

$$\boldsymbol{\alpha} \to E\left(u\left(X^{\boldsymbol{\alpha}}\right)\right) \to \int_{\mathcal{R}} u\left(x\right) f_{X^{\boldsymbol{\alpha}}}\left(x\right) dx$$

Example: $u\left(x\right) = -e^{-\gamma x}$ $\quad \gamma^{-1}$ is *not* unitless, it is measured in wealth units

In the Von Neumann-Mongerstern expected utility theory propabilities are exogeneously given (objective formulation). Alternatively, they might be determined by the investors. This is the case in Savage expected utility theory (subjective).

Certainty equivalent

$$\boldsymbol{\alpha} \to CE(\boldsymbol{\alpha}) \to u^{-1}\left(E\left(u\left(X^{\boldsymbol{\alpha}}\right)\right)\right)$$

- same units as the objective
- estimable performance index
- sensibility - Non-satiation implies that utilities need to be increasing.

Then, CE is an increasing function of the objective and consistent with strong dominance (sensible).

Monotonicity of utility also yields that CE is consistent with weak (1^{rst} degree stochastic dominance)

- consistency with 2^{nd} degree stochastic dominance holds if utility is increasing *and* concave
- constancy

Case specific properties of CE

- positive homogeneity $\longleftrightarrow$ power utilities

$$CE(\lambda\boldsymbol{\alpha}) = \lambda CE(\boldsymbol{\alpha}) \quad \text{iff} \quad u(x) = x^{1-\gamma}, \quad \gamma < 1$$

- translation invariance $\longleftrightarrow$ exponential utilities

for $X^{\mathbf{b}} \equiv 1$, $CE(\boldsymbol{\alpha} + \boldsymbol{\lambda}\mathrm{b}) = CE(\boldsymbol{\alpha}) + \lambda \quad \text{iff} \quad u(x) = -e^{-\gamma x}$

- sub- and super- additivity $\longleftrightarrow$ linear utilities
- co-monotonicity $\longleftrightarrow$ linear utilities
- concavity/ convexity $\longleftrightarrow$ fail

Risk aversion/ propensity and risk premium

$$RP(\boldsymbol{\alpha}) = E(X^{\boldsymbol{\alpha}}) - CE(\boldsymbol{\alpha})$$

- CE is a risk averse performance index iff u is concave

$$RP(\boldsymbol{\alpha}) \begin{array}{c} \geq \\ \leq \end{array} 0 \quad \text{iff} \quad u \text{ is } \begin{array}{c} concave \\ convex \end{array}$$

Observations

For concave and convex utilities, the risk premia do **not** change sign whatever the allocation is. In this sense, they are **global** indices.

This is not the case in $S-$ shape utilities arising in Prospect Theory

Local indices of risk aversion

Arrow-Pratt absolute risk aversion index

$$A(x) = -\frac{u''(x)}{u'(x)}$$

Consider an almost certain fair bet Z

$$u(x) = Eu(x + RP(Z) + Z)$$

$$\sim u(x) + RP(Z)\,u'(x) + E(Z)\,u'(x) + \frac{1}{2}u''(x)\,Var(Z)$$

$$RP(Z) \sim \frac{1}{2}\left(-\frac{u''(x)}{u'(x)}\right) Var(Z)$$

An example of an $S-$ shape utility

Assume that the investor objective is net profits

$$X_{t,T}^{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_T \cdot \mathbf{S}_T - \boldsymbol{\alpha}_t \cdot \mathbf{S}_t$$

$$u(x) = \operatorname{erf}\left(\frac{x}{\sqrt{2\delta}}\right) \qquad \delta \sim (\text{wealth})^2$$

$$A(x) = \frac{x}{\delta}$$

When net gains are occur, $x > 0$, $A(x) > 0$ while when net losses occur, $x < 0$, $A(x) < 0$

A power law Prospect Theory index

Assume that a bet Z yields outcomes x, y with $y \leq 0 \leq x$ and probabilities $p, 1-p$

The Prospect Theory performance index associated with this bet and utility u is given by

$$\nu(Z) = \pi(p)\, u(x) + \pi(1-p)\, u(y)$$

- Weighted probabilities $\pi(p)$, $\pi(1-p)$ are given, for $\delta \in (0, 1)$, by

$$\pi(p) = \frac{p^{\delta}}{p^{\delta} + (1-p)^{\delta}} \quad \text{and} \quad \pi(1-p) = \frac{(1-p)^{\delta}}{(1-p)^{\delta} + p^{\delta}}$$

A power law Prospect Theory index (continued)

- Utility, for $\alpha < 1$, by

$$u(z) = \begin{cases} z^{\alpha} & \text{for } z = x \\ -\lambda |z|^{\alpha} & \text{for } z = y \end{cases}$$

$$A(z) = -\frac{u''(z)}{u'(z)} = \left\{ \begin{array}{ll} \frac{1-\alpha}{z} > 0 & \text{for } z = x > 0 \\ -\frac{1-\alpha}{|z|} < 0 & \text{for } z = y < 0 \end{array} \right\}$$

Building utility functions from their risk aversion

Pearson specification: $A(x) = \frac{x}{\gamma x^2 + \zeta x + \eta}$

Pearson (1895), LiCalzi-Serato (2004)

$\eta = 0$: Hyperbolic absolute risk aversion (**HARA**) utilities

- $\zeta > 0$ and $\gamma = 0$: $u(x) = -e^{-\frac{1}{\zeta}x}$
- $\zeta > 0$ and $\gamma = -1$: $u(x) = x - \frac{1}{2\zeta}x^2$

Building utility functions from their risk aversion (continued)

Pearson specification: $A(x) = \frac{x}{\gamma x^2 + \zeta x + \eta}$

$\eta = 0$: Hyperbolic absolute risk aversion (**HARA**) utilities

- $\zeta = 0$ and $\gamma \geq 1$: $u(x) = x^{1-\frac{1}{\gamma}}$
- $\zeta = 0$ and $\gamma \to 1$: $u(x) = \ln x$
- $\zeta = 0$ and $\gamma \to \infty$: $u(x) = x$

All above utilities are concave

Need to capture realistic situations for behavior w.r.t. losses/gains

$\eta \neq 0$ $\zeta = 0$ and $\gamma = 0$: $u(x) = \text{erf}\left(\frac{x}{\sqrt{2\eta}}\right)$

Observations

- Static quantities

- Difficult implementation

- Utilities are not calibrated to the investment opportunities and constraints

- Monotonicity and concavity are taken into account but there is no notion of impatience (time decay of preferences)

Quantile Index of satisfaction

The aim is to control probabilities of large losses

$$\mathbb{P}\left(\boldsymbol{\alpha}_t' \cdot \mathbf{S}_t - \boldsymbol{\alpha}_T' \cdot \mathbf{S}_T < L_{\mathsf{max}}\right) \leq c$$

$$\mathcal{Q}_{X_{\boldsymbol{\alpha}}}(1-c) \geq -L_{\mathsf{max}}$$

$$VaR(\boldsymbol{\alpha}) \equiv -\mathcal{Q}_{X_{\boldsymbol{\alpha}}}(1-c)$$

Properties of Quantile Index

- Money equivalence
- Consistency with stochastic dominance (1st degree)

$$\text{If } \mathcal{Q}_{X_{\boldsymbol{\alpha}}}(p) \geq \mathcal{Q}_{X_{\boldsymbol{\beta}}}(p) \text{ for all } p \in (0,1) \Longrightarrow \mathcal{Q}_c(\boldsymbol{\alpha}) \geq \mathcal{Q}_c(\boldsymbol{\beta})$$

However, it fails to satisfy the 2nd degree stochastic dominance and, therefore, all higher degrees

- Translation invariance

$$X_{\boldsymbol{\beta}} \equiv 1 \Longrightarrow \mathcal{Q}_c(\boldsymbol{\alpha} + \lambda\boldsymbol{\beta}) = \mathcal{Q}_c(\boldsymbol{\alpha}) + \lambda$$

- Super-additivity

$$\mathcal{Q}_c(\boldsymbol{\alpha} + \boldsymbol{\beta}) \not\geq \mathcal{Q}_c(\boldsymbol{\alpha}) + \mathcal{Q}_c(\boldsymbol{\beta})$$

Properties of Quantile Index (continued)

- Co-monotonic additivity

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ co-monotonic} \quad \Longrightarrow \quad \mathcal{Q}_c(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \mathcal{Q}_c(\boldsymbol{\alpha}) + \mathcal{Q}_c(\boldsymbol{\alpha})$$

The quantile index does **not** promote diversification which is problematic for applications. However, it is very widely used index among practitioners !

- Risk aversion/propensity

The quantile index is neither convex or concave

$$X_{\boldsymbol{\alpha}} \sim Ca\left(\mu, \sigma^2\right) \qquad \text{(Cauchy distribution)}$$

$$RP(\boldsymbol{\alpha}) = E(X_{\boldsymbol{\alpha}}) - \mathcal{Q}_{X_{\boldsymbol{\alpha}}}(1 - c) = -c \tan\left(\pi\left(\tfrac{1}{2} - c\right)\right)$$

The sign changes depending on the confidence level c

Coherent indices of satisfaction

Properties imposed for their definition

- Sensibility

$$X_{\boldsymbol{\alpha}} \geq X_{\boldsymbol{\beta}} \longrightarrow \rho\left(\boldsymbol{\alpha}\right) \geq \rho\left(\boldsymbol{\beta}\right)$$

- Positive homogeneity

$$CE^{P}\left(\lambda X_{\boldsymbol{\alpha}};\gamma\right) = \frac{1}{\gamma}\left(E\left(\frac{\lambda^{\gamma}x^{\gamma}}{\gamma}\right)\right)^{\frac{1}{\gamma}} = \frac{1}{\gamma}\left(E\left(\frac{x^{\gamma}}{\gamma}\right)\right)^{\frac{1}{\gamma}} = \lambda CE^{P}\left(X_{\boldsymbol{\alpha}};\gamma\right)$$

- Translation invariance

$$CE^{E}\left(X_{\boldsymbol{\alpha}} + \lambda\boldsymbol{\beta};\gamma\right) = \frac{1}{\gamma}\log Ee^{\gamma\left(X_{\boldsymbol{\alpha}}+\lambda\boldsymbol{\beta}\right)} = \frac{1}{\gamma}\log Ee^{\gamma\left(X_{\boldsymbol{\alpha}}\right)} + \lambda = CE^{E}\left(X_{\boldsymbol{\alpha}};\gamma\right) + \lambda$$

- Super-additivity

$$\rho\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\right) \geq \rho\left(\boldsymbol{\alpha}\right) + \rho\left(\boldsymbol{\beta}\right)$$

Examples

One-sided moment criterion

$$\rho(\boldsymbol{\alpha}) = E(X_{\boldsymbol{\alpha}}) - \gamma \left\| \min\left(0, X_{\boldsymbol{\alpha}} - E(X_{\boldsymbol{\alpha}})\right) \right\|_{M;p} \qquad \gamma \geq 0$$

$$\|g\|_{M;p} = \left(\int |g(\mathbf{m})| \, f_{\mathbf{M}}(\mathbf{m}) \, d\mathbf{m} \right)^{\frac{1}{p}}$$

Special cases

$$\gamma = 0 \Longrightarrow \rho(\boldsymbol{\alpha}) = E(X_{\boldsymbol{\alpha}})$$

$\gamma = 1$ and $p = 2 \Longrightarrow$ mean/semi-standard deviation

Most widely used index among practitioners

Coherent indices of satisfaction (continued)

- Money-equivalence
- Concavity

$$\rho\left(\lambda\boldsymbol{\alpha}+(1-\lambda)\boldsymbol{\beta}\right) \geq \lambda\rho\left(\boldsymbol{\alpha}\right)+(1-\lambda)\rho\left(\boldsymbol{\beta}\right)$$

Spectral indices of satisfaction

- Co-monotonic additivity
- Consistency with weak stochastic dominance
- Consistency with risk aversion

Conclusions on the three indices of satisfaction

$$CE, \; VaR, \; Coh$$

- Static criteria
- Not all desired properties hold simultaneously
- Exogeneously defined, they are not calibrated to the investment opportunities
- Often, difficult to be implemented