Weierstraß-Institut für Angewandte Analysis und Stochastik

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# Policy iterated lower bounds and linear MC upper bounds for Bermudan style derivatives

## Pricing complex structured products

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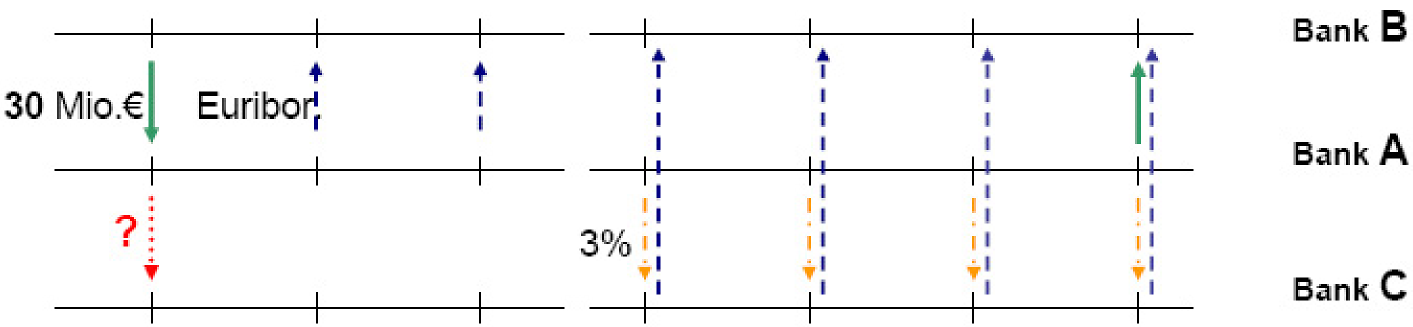
## Simple example: (Bermudan) callable interest rate swap

Euribor: Interest rate for a loan between banks

Contract I: A borrows from B 30 Mio. € over a period of 10 years and pays quarterly the 3M-Euribor.

Contract II: A buys from C a Bermudan swaption, i.e. the right to choose a payment date of contract I, from which on C pays quarterly the 3M-Euribor to B and receives a fixed payment of 3% from A.

**‘Exotic’ example: cancelable snowball swap**

**Snowball swap:** Instead of the floating spot rate the holder pays a starting coupon rate $\mathbf{I}$ over the first year and in the forthcoming years

$$(\mathbf{K} + \text{previous coupon} - \text{spot rate})^+ ,$$

where the first coupon $\mathbf{I}$ and the strike rate $\mathbf{K}$ are specified in the contract.

**Cancelable snowball swap:** The holder has the right to cancel this contract.

**What is the fair value of this cancelable product?**

▷ **Mathematical problem:**

Optimal stopping (calling) of a reward (cash-flow) process $Z$ depending on an underlying (e.g. interest rate) process $L$

▷ **Typical difficulties:**

– $L$ is usually high dimensional, for Libor interest rate models, $d = 10$ and up, so PDE methods do not work in general

– $Z$ may only be virtually known, e.g. $Z_i = E^{\mathcal{F}_i} \sum_{j \geq i} C(L_j)$ for some pay-off function $C$, rather than simply $Z_i = C(L_i)$

– $Z$ may be path-dependent

## The standard Bermudan pricing problem

Consider an underlying process $L$ in $\mathbb{R}^D$, e.g. a system of asset prices or Libor rates and a set of (future) dates $\mathbb{T} := \{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k\}$

Bermudan derivative: An option to exercise a cashflow $C(\mathcal{T}_\tau, L(\mathcal{T}_\tau))$ at a future time $\mathcal{T}_\tau \in \mathbb{T}$, to be decided by the option holder

Valuation: If $N$, with $N(0) = 1$, is some discounting numeraire and $P$ the with $N$ associated pricing measure, then with $Z_\tau := C(\mathcal{T}_\tau, L(\mathcal{T}_\tau))/N(\mathcal{T}_\tau)$, the $t = 0$ price of the option is given by the optimal stopping problem

$$V_0 = \sup_{\tau \in \{0,\ldots,k\}} E^{\mathcal{F}_0} Z^{(\tau)},$$

where the supremum runs over all stopping indexes $\tau$ with respect to $\{\mathcal{F}_{\mathcal{T}_i}, 0 \leq i \leq k\}$, where $(\mathcal{F}_t)_{t \geq 0}$ is the usual filtration generated by $L$.

At a future time point $t$, when the option is not exercised before $t$, the Bermudan option value is given by

$$V_t = N(t) \sup_{\tau \in \{\kappa(t),\dots,k\}} E^{\mathcal{F}_t} Z^{(\tau)}$$

with $\kappa(t) := \min\{m : \mathcal{T}_m \geq t\}$.

The process

$$Y_t^* := \frac{V_t}{N(t)},$$

called the *Snell-envelope* process, is a supermartingale, i.e.

$$E^{\mathcal{F}_s} Y_t^* \leq Y_s^*$$

## Canonical Solution by Backward Dynamic Programming

Set $Y^{*(i)} := Y^{*}(\mathcal{T}_i), \quad \mathcal{F}^{(i)} := \mathcal{F}_{\mathcal{T}_i}$. At the last exercise date $\mathcal{T}_k$ we have,

$$Y^{*(k)} = Z^{(k)}$$

and for $0 \leq j < k$,

$$Y^{*(j)} = \max\left(Z^{(j)}, E^{\mathcal{F}_j} Y^{*(j+1)}\right).$$

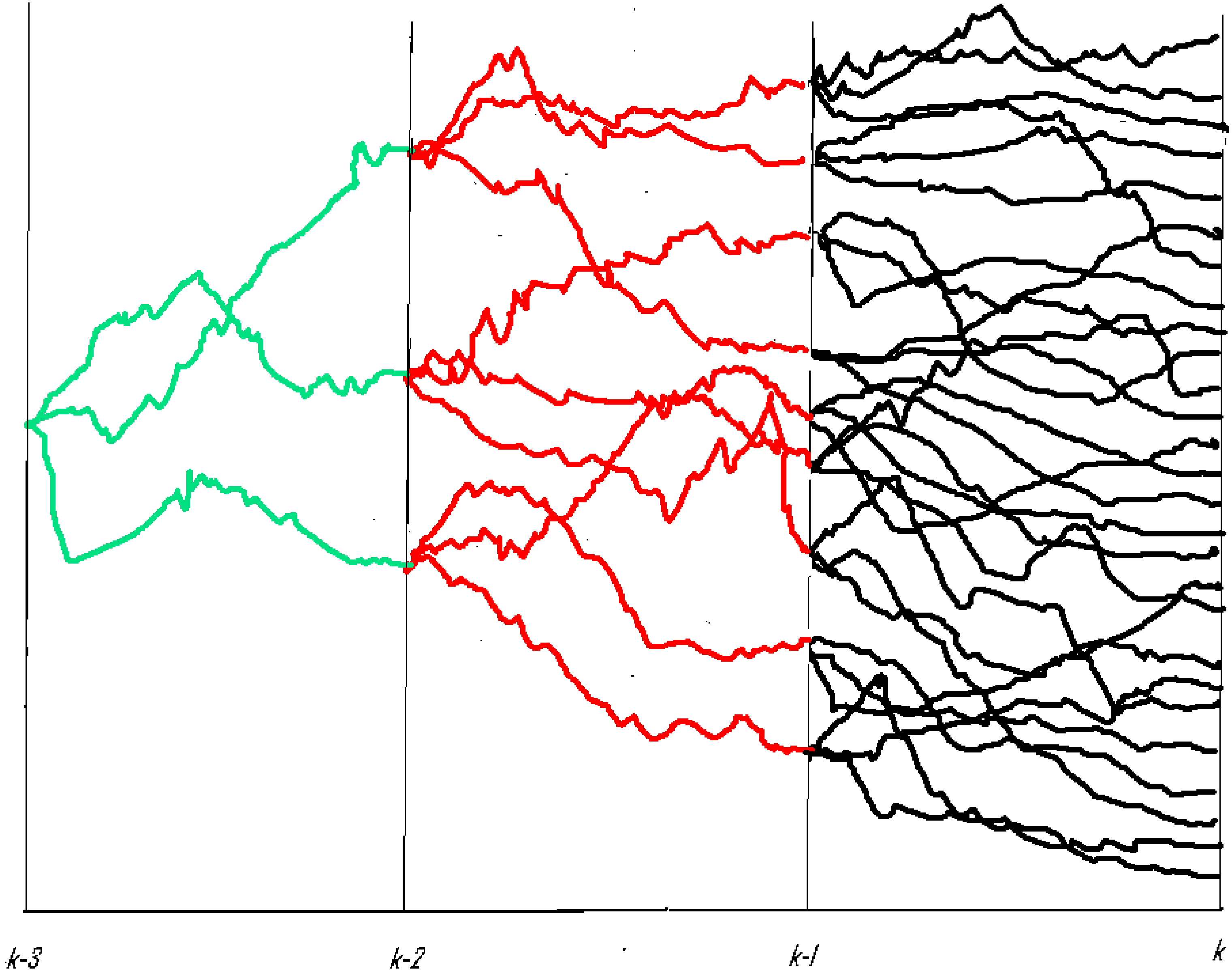
The first optimal stopping time (index) is then obtained by

$$\tau_i^* = \inf\left\{j, \; i \leq j \leq k : Y^{*(j)} \leq Z^{(j)}\right\}.$$

$\longrightarrow$ Nested Monte Carlo simulation of the price $Y_0^*$ would thus require $N^k$ samples when conditional expectations are computed with $N$ samples

Typically, $N$=10000, $k$=10 exercise opportunities, give $10^{40}$ samples !!

# Optimal stopping






W I A S

## New approaches:

I A path-by-path policy iteration methodology to improve upon standard methods (e.g. Longstaff-Schwartz, Piterbarg, Andersen)

II Application to complex exotic structures

III A linear Monte Carlo algorithm for price upper bounds via regression estimators of Doob martingale parts

# Iterative construction of the optimal stopping time

References:

Kolodko, A., Schoenmakers, J. (2006) Iterative construction of the optimal Bermudan stopping time. *Finance and Stochastics*, **10**(1), 27-49

## Improving upon given input stopping policies

We consider an input stopping family (policy) $(\tau_i)$, which satisfies the consistency conditions:

$$i \leq \tau_i \leq k, \; \tau_k = k, \quad \tau_i > i \Rightarrow \tau_i = \tau_{i+1}, \qquad 0 \leq i < k,$$

and the corresponding lower bound process $Y$ for the Snell envelope $Y^*$,

$$Y^{(i)} := E^{\mathcal{F}^{(i)}} Z^{(\tau_i)} \leq Y^{*(i)}$$

Example input policies:

- ▷ The policy, $\tau_i \equiv i.$ says: exercise immediately!
- ▷ The policy $\tau_i := \inf\{j \geq i : L(\mathcal{T}_j) \in G \subset \mathbb{R}^D\}$ exercises when the underlying process $L$ enters a certain region $G$
- ▷ The policy $\tau_i = \inf\{j : i \leq j \leq k, \; \max_{p:\, j \leq p \leq k} E^{\mathcal{F}^{(j)}} Z^{(p)} \leq Z^{(j)}\}$ waits until the cashflow is at least equal to the maximum of still-alive Europeans ahead

## One step improvement:

Introduce an intermediate process

$$\widetilde{Y}^{(i)} := \max_{p:\, i \le p \le k} E^{\mathcal{F}^{(i)}} Z^{(\tau_p)}$$

and use $\widetilde{Y}^{(i)}$ as a new exercise criterion to define a new exercise policy

$$\begin{aligned} \widehat{\tau}_i : &= \inf\{j : i \le j \le k,\ Z^{(j)} \ge \widetilde{Y}^{(j)}\} \\ &= \inf\{j : i \le j \le k,\ Z^{(j)} \ge \max_{p:\, j \le p \le k} E^{\mathcal{F}^{(j)}} Z^{(\tau_p)}\}, \quad 0 \le i \le k \end{aligned}$$

Then consider the process

$$\widehat{Y}^{(i)} := E^{\mathcal{F}^{(i)}} Z^{(\widehat{\tau}_i)}$$

as a next approximation of the Snell envelope

**Key Proposition** It holds

$$Y^{(i)} \le \widetilde{Y}^{(i)} \le \widehat{Y}^{(i)} \le Y^{*(i)}, \quad 0 \le i \le k$$

## Iterative construction of the optimal stopping time

Take an initial family of stopping times $(\tau_i^{(0)})$ satisfying the consistency conditions

$$i \leq \tau_i^{(0)} \leq k, \quad \tau_k^{(0)} = k, \quad \tau_i > i \Rightarrow \tau_i = \tau_{i+1},$$

and set $Y^{0(i)} := E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(0)})}$, $0 \leq i \leq k$. Suppose that for $m \geq 0$ the pair

$$\left( (\tau_i^{(m)}), (Y^{m(i)}) \right)$$

is constructed with $\tau_i^{(m)}$ being consistent and $Y^{m(i)} := E^{\mathcal{F}_i} Z^{(\tau_i^{(m)})}, \quad 0 \leq i \leq k.$ Then define

$$\begin{aligned} \tau_i^{(m+1)} &:= \inf\{j : i \leq j \leq k, \max_{p:\, j \leq p \leq k} E^{\mathcal{F}^{(j)}} Z^{(\tau_p^{(m)})} \leq Z^{(j)}\} \\ &=: \inf\{j : i \leq j \leq k, \widetilde{Y}^{m+1(j)} \leq Z^{(j)}\}, \quad 0 \leq i \leq k, \end{aligned}$$

and set

$$Y^{m+1(i)} := E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(m+1)})}$$

By the ‘key proposition’ we thus have

$$Y^{0(i)} \leq Y^{m(i)} \leq \widetilde{Y}^{m+1(i)} \leq Y^{m+1(i)} \leq Y^{*(i)}, \quad 0 \leq m < \infty, \quad 0 \leq i \leq k.$$

and it is shown that for $m \geq 1$,

$$\tau_i^{(m)} \leq \tau_i^{(m+1)} \leq \tau_i^*,$$

where $\tau_i^*$ is the first optimal stopping time.

We so may take limits and it holds,

$$Y^{\infty(i)} := \text{(a.s.)} \lim_{m \uparrow \infty} \uparrow Y^{m(i)} \quad \text{and} \quad \tau_i^{\infty} := \text{(a.s.)} \lim_{m \uparrow \infty} \uparrow \tau_i^{(m)}, \quad 0 \leq i \leq k, \text{ and,}$$

$$Y^{\infty(i)} = \text{(a.s.)} \lim_{m \uparrow \infty} \uparrow E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(m)})} = E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{\infty})}, \quad 0 \leq i \leq k$$

## Theorem

The constructed limit process $Y^\infty$ coincides with the Snell envelope process $Y^*$ and $(\tau_i^\infty)$ coincides with $(\tau_i^*)$; the family of first optimal stopping times. We have

$$Y^{*(i)} = Y^{\infty(i)} = E^{\mathcal{F}^{(i)}} Z^{(\tau_i^\infty)}, \quad 0 \le i \le k.$$

Moreover: It even holds

$$Y^{m(i)} = Y^{*(i)} \quad \text{for} \quad m \ge k - i$$

$\longrightarrow$ After $k = \#$exercise dates iterations the Snell Envelope is attained!

## Iteration procedure vs backward dynamic program

| Iteration level ↓ | — Exercise | date → | | | | |
|---|---|---|---|---|---|---|
| | $0$ | $1$ | $\cdots$ | $k-2$ | $k-1$ | $k$ |
| $0$ | $Y_0^{(0)}$ | $Y_1^{(0)}$ | $\cdots$ | $Y_{k-2}^{(0)}$ | $Y_{k-1}^{(0)}$ | $Y_k^*$ |
| $1$ | $Y_0^{(1)}$ | $Y_1^{(1)}$ | $\cdots$ | $Y_{k-2}^{(1)}$ | $Y_{k-1}^*$ | $Y_k^*$ |
| $2$ | $Y_0^{(2)}$ | $Y_1^{(2)}$ | | $Y_{k-2}^*$ | $Y_{k-1}^*$ | $Y_k^*$ |
| $\cdot$ | $\cdot$ | $\cdot$ | | $\cdot$ | $\cdot$ | $\cdot$ |
| $\cdot$ | $\cdot$ | $\cdot$ | | $\cdot$ | $\cdot$ | $\cdot$ |
| $\cdot$ | $\cdot$ | $\cdot$ | | $\cdot$ | $\cdot$ | $\cdot$ |
| $k-1$ | $Y_0^{(k-1)}$ | $Y_1^*$ | $\cdots$ | $Y_{k-2}^*$ | $Y_{k-1}^*$ | $Y_k^*$ |
| $k$ | $Y_0^*$ | $Y_1^*$ | $\cdots$ | $Y_{k-2}^*$ | $Y_{k-1}^*$ | $Y_k^*$ |

# Upper approximations of the Snell envelope by Duality

## The Dual Method

Consider a discrete martingale $(M_j)_{j=0,\ldots,k}$ with $M_0 = 0$ with respect to the filtration $\left(\mathcal{F}^{(j)}\right)_{j=0,\ldots,k}$. Following Rogers, Haugh and Kogan, we observe that

$$\begin{aligned} Y_0 &= \sup_{\tau \in \{0,\ldots,k\}} E^{\mathcal{F}_0} Z^{(\tau)} = \sup_{\tau \in \{0,,\ldots,k\}} E^{\mathcal{F}_0} \left[ Z^{(\tau)} - M_\tau \right] \\ &\leq E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \left[ Z^{(j)} - M_j \right] \end{aligned}$$

Hence the r.h.s. gives an upper bound for the Bermudan price $V_0 = Y_0$.

Theorem (Davis Karatzas (1994), Rogers (2001), Haugh & Kogan (2001))

Let $M^*$ be the (unique) Doob-Meyer martingale part of $\left(Y^{*(j)}\right)_{0\leq j\leq k}$, i.e. $M^*$ is an $\left(\mathcal{F}^{(j)}\right)$-martingale which satisfies

$$Y^{*(j)} = Y_0^* + M_j^* - F_j^*, \quad j = 0, ..., k,$$

with $M_0^* := F_0^* := 0$ and $F^*$ being such that $F_j^*$ is $\mathcal{F}^{(j-1)}$ measurable for $j = 1, ..., k$. Then we have

$$Y_0^* = E^{\mathcal{F}_0} \max_{0\leq j\leq k} \left[ Z^{(j)} - M_j^* \right].$$

## Convergent upper bounds from a convergent sequence of lower bounds

From our previously constructed sequence of lower bound processes $Y^{m(i)}$ with $Y^{m(i)} \uparrow Y^{*(i)}$, we deduce by duality a sequence of upper bound processes:

$$Y_{up}^{m(i)} := E^{\mathcal{F}_i} \max_{i \le j \le k} \left( Z^{(j)} - \sum_{l=i+1}^{j} Y^{m(l)} + \sum_{l=i+1}^{j} E^{\mathcal{F}^{(l-1)}} Y^{m(l)} \right) =: Y^{m(i)} + \Delta^{m(i)}.$$

Then, by a theorem of (Kolodko & Schoenmakers 2004),

$$0 \le \Delta^{m(i)} \le E^{\mathcal{F}_i} \sum_{j=i}^{k-1} \max \left( E^{\mathcal{F}_j} Y^{m(j+1)} - Y^{m(j)},\ 0 \right).$$

Thus, by letting $m \uparrow \infty$ on the r.h.s., (a.s.) $\lim_{m \to \infty} \Delta^{m(i)} = 0, \quad 0 \le i \le k$. Hence, the sequence $Y_{up}^{m}$ converges to the Snell envelope also, i.e.,

$$\text{(a.s.)} \lim_{m \to \infty} Y_{up}^{m(i)} = \text{(a.s.)} \lim_{m \to \infty} Y^{m(i)} = Y^{*(i)}, \quad 0 \le i \le k.$$

## A first numerical example: Bermudan swaptions in the LIBOR market model

Consider the Libor Market Model with respect to a tenor structure $0 < T_1 < T_2 < \ldots < T_n$, e.g. in the spot Libor measure $P^*$ induced by the numeraire

$$B^*(t) := \frac{B_{m(t)}(t)}{B_1(0)} \prod_{i=0}^{m(t)-1} (1 + \delta_i L_i(T_i))$$

with $m(t) := \min\{m : T_m \geq t\}$.
The dynamics of the forward Libor $L_i(t)$ is given by a system of SDE's

$$dL_i = \sum_{j=m(t)}^{i} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \, \gamma_i \cdot dW^*.$$

Here $\delta_i = T_{i+1} - T_i$ are day count fractions, and

$$t \to \gamma_i(t) = (\gamma_{i,1}(t), \ldots, \gamma_{i,d}(t))$$

are deterministic volatility vector functions defined in $[0, T_i]$, called factor loadings.

A (payer) Swaption over a period $[T_i, T_n]$, $1 \leq i \leq k$. A *swaption contract* with maturity $T_i$ and strike $\theta$ with principal $\$1$ gives the right to contract at $T_i$ for paying a fixed coupon $\theta$ and receiving floating Libor at the settlement dates $T_{i+1}$,...,$T_n$. So by this definition, its cashflow at maturity is

$$S_{i,n}(T_i) := \left( \sum_{j=i}^{n-1} B_{j+1}(T_i)\delta_j \left(L_j(T_i) - \theta\right) \right)^+ .$$

A Bermudan Swaption gives the the right to exercise a cashflow

$$C_{T_\tau} := S_{\tau,n}(T_\tau)$$

at an exercise date $T_\tau \in \{T_1, \ldots, T_n\}$ to be decided by the option holder.

10 yr. Bermudan swaption: (20 underlying LIBORS)

Comparison of $Y^1$, $Y^2$, $Y^{1,up}$, where $\tau_i^{(0)} \equiv i$ (trivial initial stopping family)

| $\theta$ | $d$ | $Y^1$ (SD) | $Y^2$ (SD) | $Y^{1,up}$ (SD) |
|---|---|---|---|---|
| 0.08 (ITM) | 1 | 1104.6(0.5) | 1108.9(2.4) | 1109.4(0.7) |
| | 2 | 1098.6(0.4) | 1100.5(2.4) | 1103.7(0.7) |
| | 10 | 1094.4(0.4) | 1096.9(2.1) | 1098.1(0.6) |
| | 40 | 1093.6(0.4) | 1096.1(2.0) | 1096.6(0.6) |
| 0.10 (ATM) | 1 | 374.3(0.4) | 381.2(1.6) | 382.9(0.8) |
| | 2 | 357.9(0.3) | 364.4(1.5) | 366.4(0.8) |
| | 10 | 337.8(0.3) | 343.5(1.3) | 345.6(0.7) |
| | 40 | 332.6(0.3) | 338.7(1.2) | 341.2(0.8) |
| 0.12 (OTM) | 1 | 119.0(0.2) | 121.0(0.6) | 121.3(0.4) |
| | 2 | 112.7(0.2) | 113.8(0.5) | 114.9(0.4) |
| | 10 | 100.2(0.2) | 100.7(0.4) | 101.5(0.3) |
| | 40 | 96.5(0.2) | 96.9(0.4) | 97.7(0.3) |

## Conclusions from the tables:

- ▷ The computed lower bound $Y^2$, hence the second iteration, is within 1% or less (relative to the price) of the Dual upper bound $Y^{1,up}$
- ▷ Computation times (order of minutes) may be considered low in view of the high-dimensionality of the problem!

## Some more general remarks

- ▷ The iterative approach provides a general method for improving upon any given input stopping policy obtained by other means (e.g. Andersen, Longstaff-Schwartz)
- ▷ Computation times may be reduced further by a scenario selection method by Bender, Kolodko, Schoenmakers

# Pricing of path-dependent cancellables

References:

C. Bender, A. Kolodko, and J. Schoenmakers. Iterating cancellable snowballs and related exotics. *Risk*, pages 126-130, September 2006.

Consider a path dependent cancelable contract which generates (possibly negative) cash-flows

$$C_1, \ldots, C_\tau$$

up to a cancelation date $\tau$. The cash-flows of this contract are equivalent to an aggregated cash-flow at cancellation date,

$$B_*(\mathcal{T}_\tau)\mathcal{Z}_\tau := B_*(\mathcal{T}_\tau)\sum_{j=1}^{\tau} Z_j,$$

with $Z_i := C_i/B_*(\mathcal{T}_i)$ being discounted cash-flows with respect to the numeraire $B_*$. Product price at time zero:

$$V_0^{cancel} := \sup_{\tau\in\{1,\ldots,k\}} E^{\mathcal{F}_0}\mathcal{Z}_\tau = \sup_{\tau\in\{1,\ldots,k\}} E^{\mathcal{F}_0}\sum_{j=1}^{\tau} Z_j,$$

where the supremum is taken over all stopping indices with values in the set $\{1, \ldots, k\}$.

## Path-dependent callables

A path dependent callable contract generates

$$C_{\tau+1}, \ldots, C_k$$

when called at $\tau$. It is equivalent to the sum of a non-callable and a cancelable one (and vice versa):

$$\begin{aligned} V_0^{call} &:= \sup_{\tau \in \{1,\ldots,k\}} E^{\mathcal{F}_0} \sum_{j=\tau+1}^{k} Z_j \\ &= E^{\mathcal{F}_0} \sum_{j=1}^{k} Z_j + \sup_{\tau \in \{1,\ldots,k\}} E^{\mathcal{F}_0} \sum_{j=1}^{\tau} (-Z_j) \end{aligned}$$

## Example: The cancelable snowball swap

Pays semi-annually a constant rate $I$ over the first year and in the forthcoming years $(\text{Previous coupon} + A - \text{Libor})^{+}$, semi-annually, where $A$ is given in the contract. For this case we take

$$\begin{aligned} K_i &:= I, \quad i = 0, 1, \\ K_i &:= (K_{i-1} + A_i - L_i(T_i))^{+}, \quad i = 2, ..., n-1, \end{aligned}$$

with $A_2 := 0.03$, $A_{i+1} = A_i$ for even $i$, $A_{i+1} = A_i + 0.005$ for odd $i$. Cancelation is allowed for $2 \leq \tau < n$, $n = 20$ (10 years)

Effective discounted cashflows at $T_j$:

$$Z_j := \frac{(L_{j-1}(T_{j-1}) - K_{j-1})\, \delta_{j-1}}{B_*(T_j)},$$

hence aggregated up to cancelation $\mathcal{Z}_\tau = \sum_{j=1}^{\tau} Z_j$.

## Iterating the snowball swap

Take an input policy satisfying

$$i \le \tau_i \le k, \ \tau_k = k,$$
$$\tau_i > i \Rightarrow \tau_i = \tau_{i+1}, \qquad 0 \le i < k,$$

construct the new policy

$$\begin{aligned} \widehat{\tau}_i &:= \inf\{j \ge i : \mathcal{Z}_j \ge \max_{p:\, j \le p \le k} E^{\mathcal{F}_j} \mathcal{Z}_{\tau_p}\} \\ &= \inf\{j \ge i : 0 \ge \max_{p:\, j \le p \le k} E^{\mathcal{F}_j} \sum_{q=j+1}^{\tau_p} Z_q\} \end{aligned}$$

and compute the iterated price

$$\widehat{Y}_0 := E^{\mathcal{F}_0} \mathcal{Z}_{\widehat{\tau}_0},$$

which is generally an improvement of $Y_0$ due to policy $\tau$.

## Numerical results for typical market data

### Improved Andersen

| $d$ | $Y(0;\tau_A)$ (SD) | $\widehat{Y}(0;\tau_A)$ (SD) | $Y_{up}(0;\tau_A)$ (SD) |
|---|---|---|---|
| 1 | 127.77(0.238) | 129.77(0.318) | 130.33(0.247) |
| 2 | 114.93(0.231) | 120.00(0.389) | 121.92(0.293) |
| 19 | 76.725(0.217) | 91.600(0.460) | 98.107(0.476) |

150 000 outer and 500 inner paths for $\widehat{Y}$ and 20 000 outer (with 500 inner) paths for $Y_{up}$.

### Improved least-squares regression method (Piterbarg)

| $d$ | $Y(0;\tau_{LS})$ (SD) | $\widehat{Y}(0;\tau_{LS})$ (SD) | $Y_{up}(0;\tau_{LS})$ (SD) |
|---|---|---|---|
| 1 | 117.73(0.243) | 128.81(0.632) | 132.28(0.313) |
| 2 | 103.70(0.238) | 120.73(0.466) | 123.54(0.346) |
| 19 | 74.913(0.224) | 93.515(0.469) | 97.479(0.379) |

200 000 outer and 500 inner paths for $\widehat{Y}$ and 20 000 outer (with 500 inner) paths for $Y_{up}$.

Improving an Andersen-like optimization of the LS exercise boundary

| $d$ | $Y(0;\tau_{LS\text{-}A})$ (SD) | $\widehat{Y}(0;\tau_{LS\text{-}A})$ (SD) | $Y_{up}(0;\tau_{LS\text{-}A})$ (SD) |
|---|---|---|---|
| 1 | 129.58(0.237) | 128.70(0.349) | 130.24(0.244) |
| 2 | 119.58(0.230) | 118.95(0.345) | 120.77(0.244) |
| 10 | 92.201(0.219) | 97.376(0.456) | 100.20(0.418) |
| 19 | 87.787(0.217) | 94.487(0.445) | 95.843(0.430) |

150 000 outer and 100 inner paths for $\widehat{Y}$ and 5000 outer (with 500 inner) paths for $Y_{up}$.

$$\tau_{LS\text{-}A,\, i} = \inf\{j \geq i : \mathcal{Z}_j \geq H_j + Y_{LS,\, j}\}$$

with optimized constants $H_j$.

Message:

(i) Price the callable using Pitterbarg's version of Longstaff-Schwartz;

(ii) Improve the obtained exercise boundary with an Andersen-like optimization;

(iii) Compute the Dual upperbound due to the stopping time $\tau_{LS\text{-}A,\ 0}$;

(iv) If there is still a significant gap between lower and upper bound, then improve the policy $\tau_{LS\text{-}A,\ i}$ by the iteration method.

## True upper bounds via non-nested Monte Carlo

Joint work with D. Belomestny and C. Bender

For any martingale $M_{T_j},\ 0 \le j \le k$ with respect to the filtration $(\mathcal{F}_{T_j};\ 0 \le j \le k)$ starting at $M_0 = 0$

$$Y_0^{up}(M) := E^{\mathcal{F}_0}\left[\max_{0 \le j \le k}(Z_{T_j} - M_{T_j})\right]$$

is an upper bound for the price of the Bermudan option with discounted cash-flow $Z_{T_j}$.

Exact Bermudan price is attained at the martingale part $M^*$ of the Snell envelope,

$$Y^*_{T_j} = Y^*_{T_0} + M^*_{T_j} + F^*_{T_j},$$

$M^*_{T_0} = F^*_{T_0} = 0$ and $F^*_{T_j}$ is $\mathcal{F}_{T_{j-1}}$ measurable

(I) Assume the underlying process $L$ to be Markovian, and the filtration $\mathcal{F}$ to be generated by a $d$-dimensional Brownian motion $W$.

(II) Assume $Y_{T_j} = u(T_j, L(T_j))$ is some approximation of the Snell envelope $Y^*_{T_j}$, $0 \leq j \leq k$, with Doob decomposition

$$Y_{T_j} = Y_{T_0} + M_{T_j} + F_{T_j},$$

$M_{T_0} = F_{T_0} = 0$ and $F_{T_j}$ is $\mathcal{F}_{T_{j-1}}$ measurable.

It then holds:

$$\begin{aligned} Y_{T_{j+1}} - Y_{T_j} &= M_{T_{j+1}} - M_{T_j} + F_{T_{j+1}} - F_{T_j} \\ M_{T_{j+1}} - M_{T_j} &= Y_{T_{j+1}} - E^{T_j}[Y_{T_{j+1}}], \end{aligned}$$

with

$$M_{T_j} =: \int_0^{T_j} H_t dW_t =: \int_0^{T_j} \mathfrak{h}(t, L(t)) dW_t, \; j = 0, \ldots, k.$$

We are going to estimate $\mathfrak{h}(\cdot,\cdot)$ (hence $H$) at the finite partition $\pi = \{t_0, \ldots, t_\mathcal{I}\}$ such that $t_0 = 0$, $t_\mathcal{I} = T$, and $\{T_0, \ldots, T_k\} \subset \pi$. We may write formally,

$$Y_{T_{j+1}} - Y_{T_j} \approx \sum_{t_l \in \pi; T_j \le t_l < T_{j+1}} H_{t_l}(W_{t_{l+1}} - W_{t_l}) + F_{T_{j+1}} - F_{T_j}.$$

By multiplying both sides with $(W^d_{t_{i+1}} - W^d_{t_i})$, $T_j \le t_i < T_{j+1}$, and taking $\mathcal{F}_{t_i}$-conditional expectations, we get by the $\mathcal{F}_{T_{j+1}}$-measurability of $F_{T_j}$,

$$H^d_{t_i} \approx \frac{1}{t_{i+1} - t_i} E^{\mathcal{F}_{t_i}} \left[ (W^d_{t_{i+1}} - W^d_{t_i}) Y_{T_{j+1}} \right],$$

and so define

$$H^\pi_{t_i} := \frac{1}{\Delta^\pi_i} E^{\mathcal{F}_{t_i}} \left[ (\Delta^\pi W_i)^\top Y_{T_{j+1}} \right], \; T_j \le t_i < T_{j+1},$$

with $\Delta^\pi_i := t_{i+1} - t_i$ and $\Delta^\pi W^d_i := W^d_{t_{i+1}} - W^d_{t_i}$.

The corresponding approximation of the martingale $M$ is

$$M^{\pi}_{T_j} := \sum_{t_i \in \pi; 0 \le t_i < T_j} H^{\pi}_{t_i}(\Delta^{\pi} W_i).$$

Theorem:

$$\lim_{|\pi| \to 0} E\left[\max_{0 \le j \le k} |M^{\pi}_{T_j} - M_{T_j}|^2\right] = 0$$

where $|\pi|$ denotes the mesh of $\pi$.

The conditional expectations in the definition of $H^\pi$ are, in fact, functions of $L(t_i)$. Precisely,

$$H_{t_i}^\pi = \mathfrak{h}^\pi(t, L(t_i)) = \frac{1}{\Delta_i^\pi} E^{(t_i, L(t_i))}\left[(\Delta^\pi W_i)^\top u(T_{j+1}, L(T_{j+1}))\right],\ T_j \le t_i < T_{j+1}.$$

which may be computed by regression: Take basis functions

$$\psi(t_i, \cdot) = (\psi_r(t_i, \cdot),\ r = 1, \ldots, R)$$

and $N$ independent samples $(t_i,\ {}_nL(t_i))$, $n = 1, \ldots, N$ of $L(t_i)$ constructed from the Brownian increments $\Delta_n^\pi W_i$, $n = 1, \ldots, N$.

Construct the regression matrix

$$A_{t_i}^\oplus := (A_{t_i}^\top A_{t_i})^{-1} A_{t_i}^\top,$$

where

$$A_{t_i} = (\psi_r(t_i,\ {}_nL(t_i)))_{n=1,\ldots,N, r=1,\ldots,R}$$

Result:

$$\widehat{\mathfrak{h}}^{\pi}(t_i, x) = \psi(t_i, x)\, A_{t_i}^{\oplus} \left( \frac{\Delta_{\cdot}^{\pi} W_i}{\Delta_i^{\pi}}\, {}_{\cdot}Y_{T_{j+1}} \right),\; T_j \leq t_i < T_{j+1}$$
$$=: \psi(t_i, x) \widehat{\beta}_{t_i},$$

where

$$\left( \frac{\Delta_{\cdot}^{\pi} W_i}{\Delta_i^{\pi}}\, {}_{\cdot}Y_{T_{j+1}} \right) = \left( \frac{\Delta_n^{\pi} W_i^d}{\Delta_i^{\pi}}\, {}_{n}Y_{T_{j+1}} \right)_{n=1,\ldots,N,\, d=1,\ldots,D},$$

${}_n\widetilde{Y}_{T_{j+1}} := u(T_{j+1},\, {}_nL(T_{j+1}))$, and $\widehat{\beta}_{t_i}$ is the $R \times D$ matrix of estimated regression coefficients at time $t_i$.

True linear Monte Carlo upperbound:

$$\widehat{Y}^{up}(\widehat{M^{\pi}}) = \frac{1}{\widetilde{N}} \sum_{n=1}^{\widetilde{N}} \max_{0 \le j \le k} \left[ z(T_j,\, {}_n\widetilde{L}(T_j)) - \underbrace{\sum_{t_i \in \pi; 0 \le t_i < T_j} \widehat{\mathfrak{h}}^{\pi}(t_i,\, {}_n\widetilde{L}(T_j))(\Delta^{\pi}\widetilde{W_i})}_{(*)} \right],$$

by doing a new simulation ${}_n\widetilde{L}(T_j)$, $\Delta^{\pi}_n\widetilde{W_i}$ $n = 1, \ldots, \widetilde{N}$.

$(*)$ is always a martingale, so the upper bound is true!

Black-Scholes model:

$$dX_t^d = (r - \delta)X_t^d dt + \sigma X_t^d dW_t^d, \quad d = 1, ..., D,$$

Pay-off:

$$Z_t := z(X_t) := (\max(X_t^1, ..., X_t^D) - \kappa)^+.$$

$T_k = 3$yr, $k = 9$ (ex. dates), $\kappa = 100$, $r = 0.05$, $\sigma = 0.2$, $\delta = 0.1$, different $D$ and $x_0$

| D | $x_0$ | Lower Bound $Y_0$ | Upper Bound $Y_0^{up}(\widehat{M}^\pi)$ | Upper Bound $Y_{10^4,200}^{up}(0)$ | Upper Bound $Y_{10^4,50}^{up}(0)$ |
|---|---|---|---|---|---|
| 2 | 90 | 7.9751±0.139 | 8.6963±0.052 | 8.231 | 8.70±0.06 |
| | 100 | 13.883±0.177 | 14.515±0.073 | 14.18 | 14.43±0.07 |
| | 110 | 21.291±0.205 | 21.972±0.095 | 21.68 | 22.00±0.11 |
| 5 | 90 | 16.523±0.194 | 18.134±0.069 | 17.46 | 18.21±0.06 |
| | 100 | 26.042±0.232 | 27.976±0.085 | 27.33 | 28.05±0.09 |
| | 110 | 36.526±0.263 | 38.882±0.098 | 38.27 | 39.0±0.12 |

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