KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

ACTUARIAL AND FINANCIAL MATHEMATICS CONFERENCE

Interplay between Finance and Insurance

February 6-7, 2014

Michèle Vanmaele, Griselda Deelstra, Ann De Schepper, Jan Dhaene, Wim Schoutens, Steven Vanduffel & David Vyncke (Eds.)

CONTACTFORUM

KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

ACTUARIAL AND FINANCIAL MATHEMATICS CONFERENCE

Interplay between Finance and Insurance

February 6-7, 2014

**Michèle Vanmaele, Griselda Deelstra, Ann De Schepper,
Jan Dhaene, Wim Schoutens, Steven Vanduffel & David Vyncke (Eds.)**

CONTACTFORUM

Handelingen van het contactforum "Actuarial and Financial Mathematics Conference. Interplay between Finance and Insurance" (6-7 februari 2014, hoofdaanvrager: Prof. M. Vanmaele, UGent) gesteund door de Koninklijke Vlaamse Academie van België voor Wetenschappen en Kunsten.

Afgezien van het afstemmen van het lettertype en de alinea's op de richtlijnen voor de publicatie van de handelingen heeft de Academie geen andere wijzigingen in de tekst aangebracht. De inhoud, de volgorde en de opbouw van de teksten zijn de verantwoordelijkheid van de hoofdaanvrager (of editors) van het contactforum.

KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

Paleis der Academiën
Hertogsstraat 1
1000 Brussel

D/2014/0455/05
ISBN 978 90 6569 135 4

KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

Actuarial and Financial Mathematics Conference
Interplay between finance and insurance

CONTENTS

KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

Actuarial and Financial Mathematics Conference
Interplay between finance and insurance

PREFACE

In 2014, our two-day international "Actuarial and Financial Mathematics Conference" was organized in Brussels for the seventh time. As for the previous editions, we could use the facilities of the Royal Flemish Academy of Belgium for Science and Arts. The organizing committee consisted of colleagues from 6 Belgian universities, i.e. the University of Antwerp, Ghent University, the KU Leuven and the Vrije Universiteit Brussel on the one hand, and the Université Libre de Bruxelles and the Université catholique de Louvain on the other hand. The conference included 8 invited lectures, 9 selected contributions and a poster session with 10 posters. As for the scientific committee, we were happy that we could rely on leading international researchers, and just as in the previous years, we could welcome renowned international speakers for the invited lectures.

There were 130 registrations in total, with 75 participants from Belgium, and 55 participants from 17 other countries from all continents. The population was mixed, with 70% of the participants associated with a university (PhD students, post doc researchers and professors), and with 30% working in the banking and insurance industry, from home and abroad.

On the first day, February 6, we had 9 speakers, with 4 international and eminent invited speakers, alternated with 5 interesting contributions selected by the scientific committee.
In de morning, the first speaker was *Prof.dr. Martijn Pistorius*, from *Imperial College London (U.K.)*, with a lecture entitled "Distance to default, inverse first-passage time problems & counterparty credit risk"; afterwards *Prof.dr. Tahir Choulli, University of Alberta (Canada)* gave a well-received talk about "Viability Structures under Additional Information & Uncertainty". These two lectures were followed by 2 presentations by researchers from Germany and France.
In the afternoon, we heard *Prof.dr. Christian Gouriéroux, University of Toronto (Canada) & CREST (France)*, who presented new research results about "Pricing default events: surprise, exogeneity and contagion", and *Prof.dr. Matthias Scherer, TU München (Germany)*, with his paper on "Consistent iterated simulation of multi-variate default times: a Markovian indicators characterization". As in the morning, these two lectures were alternated now by 3 presentations, with one speaker from France, one from Germany and one from Japan.

During the lunch break, we organized a poster session, preceded by a poster storm session, where the 10 different posters were introduced very briefly by the researchers. The posters remained in the main meeting room during the whole conference, so that they could be consulted and discussed during the lunches and coffee breaks. We were pleased with the lively interaction between the participants and the posters' authors, with very useful suggestions to the younger researchers.

Also on the second day, February 7, we had 8 lectures, again with 4 keynote speakers and 4 selected contributions. The first speaker was *Prof.dr. Pierre Devolder, Université Catholique de Louvain (Belgium)*, with a lecture on "Some actuarial questions around a possible reform of the Belgian pension system". Afterwards, *Prof.dr. Enrico Biffis, Imperial College London (U.K.)* presented his research on "Optimal collateralization with bilateral default risk". In the afternoon, we could listen to *Prof.dr. Marcus Christiansen, Universität Ulm (Germany)*, about "Deterministic optimal consumption and investment in a stochastic model with applications in insurance". *Prof.dr. Ralf Korn, TU Kaiserslautern (Germany)* was the last invited speaker, with a nice lecture entitled "Save for the bad times or consume as long as you have? Worst-case optimal lifetime consumption!". The other 4 presentations were again selected from a substantial number of submissions by the scientific committee; the speakers came from France, the Netherlands, Canada and Germany.

In these proceedings, you can find one paper of an invited speaker co-authored with a contributed speaker, four articles related to contributed talks, and six extended abstracts written by the poster presenters of the poster sessions, giving an overview of the topics and activities at the conference.

We are much indebted to the members of the scientific committee, *H. Albrecher (University of Lausanne, Switzerland), C. Bernard (University of Waterloo, Canada), J. Dhaene (Katholieke Universiteit Leuven, Belgium), E. Eberlein (University of Freiburg, Germany), M. Jeanblanc (Université d'Evry Val d'Essonne, France), R. Norberg (SAF, Université Lyon 1, France), Ludger Rüschendorf (University of Freiburg, Germany), S. Vanduffel (Vrije Universiteit Brussel, Belgium), M. Vellekoop (University of Amsterdam, the Netherlands) and the chair G. Deelstra (Université Libre de Bruxelles, Belgium).* We appreciate their excellent scientific support, their presence at the meeting and their chairing of sessions. We also thank Wouter Dewolf (Ghent University, Belgium), for the administrative work.
We are very grateful to our sponsors, namely the *Royal Flemish Academy of Belgium for Science and Arts, the Research Foundation — Flanders (FWO), the Scientific Research Network (WOG) "Stochastic modelling with applications in finance", le Fonds de la Recherche Scientifique (FNRS), KBC Bank en Verzekeringen, the BNP Paribas Fortis Chair in Banking at the Vrije Universiteit Brussel and Université Libre de Bruxelles, and exhibitors Cambridge, Springer and NAG.* Without them it would not have been possible to organize this event in this very enjoyable and inspiring environment. We are also grateful for the support by the *ESF Research Networking Programme Advanced Mathematical Methods for Finance (AMAMEF).*

The continuing success of the meeting encourages us to go on with the organization of this contact-forum, in order to create future opportunities for exchanging ideas and results in this fascinating research field of actuarial and financial mathematics.

The editors:
Griselda Deelstra, Ann De Schepper, Jan Dhaene, Wim Schoutens, Steven Vanduffel,
Michèle Vanmaele, David Vyncke

The other members of the organising committee:
Michel Denuit, Karel In 't Hout

INVITED TALK

MINIMIZATION OF HEDGING ERROR ON ORLICZ SPACE

Takuji Arai[†] **and Tahir Choulli**[§]

[†]*Department of Economics, Keio University, 2-15-45 Mita, Minato-ku, Tokyo, 108-8345, Japan*
[§]*Mathematical and Statistical Sciences Department, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada*
Email: arai@econ.keio.ac.jp, tchoulli@ualberta.ca

Abstract

Minimization problems on hedging error in the Orlicz space framework are discussed. In this paper, we deal with general forms of such problems as follows:

$$\inf_{v\in\mathcal{V}} E[\Phi(|H-v|)], \quad \inf_{v\in\mathcal{V}} N_\Phi(H-v), \quad \inf_{v\in\mathcal{V}} \|H-v\|_\Phi,$$

where Φ is a Young function, N_Φ and $\|\cdot\|_\Phi$ are norms on the Orlicz space L^Φ, H is a random variable, $\mathcal{V}$ is a convex subset of L^Φ. We aim to investigate relationships among the three problems. We focus on, firstly, properties of the first problem, and study its relationships to the others. Moreover, we prove that there exist solutions to the three when L^Φ is reflexive.

1. INTRODUCTION

In mathematical finance, it is very important to study pricing and hedging problem for contingent claims. If the underlying market is complete, any contingent claim H, given by a random variable, is represented as a stochastic integral with respect to underlying asset price process S, which is a semimartingale, that is, there exist a constant c and an $\mathbf{R}^d$-valued S-integrable predictable process ϑ such that

$$H = c + \int_0^T \vartheta_t dS_t, \tag{1}$$

where T is the maturity of our market. Under the no-arbitrage condition, the fair price of H must be given by the initial cost to replicate H, that is, the constant c in (1), and ϑ is regarded as a self-financing replicating strategy. On the other hand, in the case of incomplete markets, there is no pair (c, ϑ) satisfying (1), unfortunately. Instead of the replicating strategy, we should look for an optimal pair (c, ϑ) in an appropriate sense. There are, in fact, many ways to define optimality, say, mean-variance hedging (Schweizer (2001), Schweizer (2010)), risk minimizing hedging (Föllmer and Schweizer (2010), Schweizer (2001),), utility indifference valuation (Becherer (2010), Henderson and Hobson (2008)), and so forth. In this paper, we focus on problems finding a pair (c, ϑ) so that $c + \int_0^T \vartheta_t dS_t$ is as near to H as possible, that is, optimization problems on hedging error $\left| c + \int_0^T \vartheta_t dS_t - H \right|$. For example, mean-variance hedging is defined as the optimal strategy minimizing its hedging error in the L^2-sense, that is, a solution to the following:

$$\min_{c \in \mathbf{R}, \vartheta \in \Theta} E\left[\left(c + \int_0^T \vartheta_t dS_t - H \right)^2 \right], \tag{2}$$

where Θ is a set of $\mathbf{R}^d$-valued S-integrable predictable processes. In order to discuss various types of minimization problem on hedging error in a unified way, we try to extend mean-variance hedging to general Orlicz space setting, that is, we consider

$$\min_{c \in \mathbf{R}, \vartheta \in \Theta} E\left[\Phi\left(\left| H - c - \int_0^T \vartheta_t dS_t \right| \right) \right], \tag{3}$$

where Φ is a Young function, that is, a continuous increasing convex function defined on $[0, \infty)$ with starting at 0. Incidentally, we can rewrite (2) as

$$\min_{c \in \mathbf{R}, \vartheta \in \Theta} \left\| c + \int_0^T \vartheta_t dS_t - H \right\|_{L^2}.$$

Now, a question arises; we wonder if we can rewrite (3) similarly as follows:

$$\min_{c \in \mathbf{R}, \vartheta \in \Theta} N_\Phi\left(H - c - \int_0^T \vartheta_t dS_t \right), \tag{4}$$

and

$$\min_{c \in \mathbf{R}, \vartheta \in \Theta} \left\| H - c - \int_0^T \vartheta_t dS_t \right\|_\Phi, \tag{5}$$

where $N_\Phi(\cdot)$ and $\| \cdot \|_\Phi$ are norms on the Orlicz space induced by Φ, whose definitions will be introduced in the sequel.

Remark 1.1 *We can regard the three problems (3), (4) and (5) as purely mathematical problems. More precisely, these are projections of a random variable on a space of stochastic integrations. Thus, we can say that results obtained in this paper would be important not only for mathematical finance, but also for both stochastic analysis and functional analysis.*

The aim of this paper is to investigate relationships among the three problems (3), (4) and (5), and to give sufficient conditions under which all the three admit solutions. Note that we rewrite the three problems into general forms, and treat them throughout this paper. Model description and mathematical preliminaries are given in section 2. In section 3, we study the relationship among the three problems. In particular, we investigate properties of solutions to (3), and its relations to the two other problems. In section 4, we prove that, if the based Orlicz space is reflexive, then the existence of solutions to the three are guaranteed.

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, P; \mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]})$ be a filtered probability space with a right-continuous filtration $\mathbf{F}$ such that $\mathcal{F}_0$ is trivial and contains all null sets of $\mathcal{F}$, and $\mathcal{F}_T = \mathcal{F}$. Consider an incomplete financial market composed of one riskless asset and d risky assets. Suppose that the price of the riskless asset is 1 at all times, that is, the interest rate of our market is assumed to be 0. Note that $T > 0$ is the maturity. Let Φ be a continuous nondecreasing convex function defined on $[0, \infty)$ with starting at 0, which is called a Young function. Remark that Φ is differentiable a.e. and its left-derivative ϕ satisfies $\Phi(x) = \int_0^x \phi(u)du$. Note that ϕ is left continuous, and may have at most countably many jumps. Define $\psi(y) := \inf\{x \in (0, \infty)|\phi(x) \geq y\}$, which is called the generalized left-continuous inverse of ϕ. We define $\Psi(y) := \int_0^y \psi(v)dv$ for $y \geq 0$, which is a Young function and called the conjugate function of Φ. Now, we define the Orlicz space and the Orlicz heart for Φ, and norms on them as follows:

Definition 2.1 *We define two spaces of random variables for a Young function Φ:*
(Orlicz space) $L^\Phi := \{X \in L^0|E[\Phi(c|X|)] < \infty$ for some $c > 0\}$,
(Orlicz heart) $M^\Phi := \{X \in L^0|E[\Phi(c|X|)] < \infty$ for any $c > 0\}$,
where L^0 is the set of all $\mathcal{F}_T$-measurable random variables. In addition, we define two norms:
(Luxemburg norm) $\|X\|_\Phi := \inf\left\{\lambda > 0|E\left[\Phi\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\}$,
(Orlicz norm) $N_\Phi(X) := \sup\{E[XY]| \, \|Y\|_\Psi \leq 1\}$.

Remark that $M^\Phi \subset L^\Phi$ and both spaces L^Φ and M^Φ are linear. Moreover, the norm dual of $(M^\Phi, \|\cdot\|_\Phi)$ is given by $(L^\Psi, N_\Phi(\cdot))$, since Φ is finite. For more details on Orlicz space, see Edgar and Sucheston (1992) and Rao and Ren (1991). Henceforth, we fix arbitrarily a Young function Φ satisfying the following assumptions:

Assumption 2.1 *(1) $\Phi(x) > 0$ for any $x > 0$,*
(2) $\lim_{x\to\infty} \Phi(x)/x = +\infty$.

Example 2.1 *Typical examples of Φs satisfying all conditions mentioned are $\Phi(x) = e^x - 1$, $e^x - x - 1$, $(x+1)\log(x+1) - x$ and x^p/p for $p > 1$. On the other hand, $\Phi(x) = 0$ if $x < 1$; $= (x-1)^2$ if $x \geq 1$ and $\Phi(x) = ax$ for $a > 0$ are excluded in this paper.*

Letting S be an $\mathbf{R}^d$-valued semimartingale describing the fluctuation of risky assets, problems (3), (4) and (5) can be regarded as minimization problems on the space

$$\left\{c + \int_0^T \vartheta_t dS_t | c \in \mathbf{R}, \vartheta \in \Theta\right\} \text{ or } \left\{\int_0^T \vartheta_t dS_t | \vartheta \in \Theta\right\}, \tag{6}$$

where Θ is a set of $\mathbf{R}^d$-valued S-integrable predictable processes. Although we do not specify the definition of Θ, we assume the convexity of Θ, that is, the space (6) forms a convex set. Thus, we can rewrite problems (3), (4) and (5) as the following general forms:

Problem 2.2 $\min_{v\in\mathcal{V}} E[\Phi(|H - v|)]$ *or* $\inf_{v\in\mathcal{V}} E[\Phi(|H - v|)]$,

Problem 2.3 $\min_{v\in\mathcal{V}} N_\Phi(H - v)$ *or* $\inf_{v\in\mathcal{V}} N_\Phi(H - v)$,

Problem 2.4 $\min_{v\in\mathcal{V}} \|H-v\|_\Phi$ *or* $\inf_{v\in\mathcal{V}} \|H-v\|_\Phi$,

where $\mathcal{V}$ is a convex subset of L^Φ. As mentioned in section 1, we can regard these problems as the L^Φ-projections of a random variable H on a convex set $\mathcal{V}$. We shall investigate relationships among Problems 2.2–2.4, and the existence of solutions. We suppose, throughout the paper, that $H \in L^\Phi$. Since we are not interested in the case where $H \in \mathcal{V}$, we assume $H \notin \mathcal{V}$. For all unexplained notation, we refer to Dellacherie and Meyer (1982).

3. RELATIONSHIPS AMONG THE THREE PROBLEMS

3.1. Relationships on finiteness

First of all, we can see the following proposition:

Proposition 3.1 *(1)* $\inf_{v\in\mathcal{V}} \|H-v\|_\Phi = +\infty \Leftrightarrow \inf_{v\in\mathcal{V}} N_\Phi(H-v) = +\infty$.
(2) $\inf_{v\in\mathcal{V}} \|H-v\|_\Phi = +\infty \Rightarrow \inf_{v\in\mathcal{V}} E[\Phi(|H-v|)] = +\infty$.

Proof. These are clear by Theorem 2.2.9 of Edgar and Sucheston (1992). □

Actually, the relationship between Problems 2.2 and 2.4 (as well as 2.2 and 2.3) is not simple as the case of between Problems 2.3 and 2.4. The reverse assertion of (2) does not hold in general. We introduce a counterexample.

Example 3.1 *We consider a one period model. Let X and Y be two independent random variables following the exponential distribution with parameter 1 and $1/2$, respectively. The asset price process S is given by $S_0 = 0$ and $S_1 = X - 1$. Let Φ be $\Phi(x) = e^x - 1$ and H given by $X + Y$. $\mathcal{V}$ is assumed to be given by $\{\vartheta S_1 | \vartheta \in \mathbf{R}\}$. Then, we have, for any $\vartheta \geq 1$,*

$$
\begin{aligned}
E[\Phi(|H-\vartheta S_1|)] &= \int_0^\infty \int_0^\infty e^{|x+y-\vartheta x+\vartheta|} e^{-x} \frac{e^{-y/2}}{2} dy dx - 1 \\
&\geq \int_0^{\frac{\vartheta}{\vartheta-1}} \int_0^\infty e^{x+y-\vartheta x+\vartheta} e^{-x} \frac{e^{-y/2}}{2} dy dx - 1 \\
&\geq e^\vartheta \int_0^{\frac{\vartheta}{\vartheta-1}} \int_0^\infty \frac{e^{y/2}}{2} dy e^{-\vartheta x} dx - 1 \\
&= +\infty.
\end{aligned}
$$

Moreover, for any $\vartheta < 1$,

$$
\begin{aligned}
E[\Phi(|H-\vartheta S_1|)] &= \int_0^\infty \int_0^\infty e^{|x+y-\vartheta x+\vartheta|} e^{-x} \frac{e^{-y/2}}{2} dy dx - 1 \\
&\geq \int_{\frac{\vartheta}{\vartheta-1}\vee 0}^\infty \int_0^\infty e^{x+y-\vartheta x+\vartheta} e^{-x} \frac{e^{-y/2}}{2} dy dx - 1 \\
&\geq e^\vartheta \int_{\frac{\vartheta}{\vartheta-1}\vee 0}^\infty \int_0^\infty \frac{e^{y/2}}{2} dy e^{-\vartheta x} dx - 1 \\
&= +\infty.
\end{aligned}
$$

Thus, we obtain $\inf_{v\in\mathcal{V}} E[\Phi(|H-v|)] = +\infty$. *On the other hand, letting* $\vartheta = 0$ *and* $\lambda > 2$, *we have*

$$\begin{aligned} E\left[\Phi\left(\frac{|H-\vartheta S_1|}{\lambda}\right)\right] &= \int_0^\infty\int_0^\infty e^{\frac{|x+y-\vartheta x+\vartheta|}{\lambda}} e^{-x}\frac{e^{-y/2}}{2}dydx - 1 \\ &= \int_0^\infty\int_0^\infty e^{\frac{x+y}{\lambda}} e^{-x}\frac{e^{-y/2}}{2}dydx - 1 \\ &= \frac{1}{\left(1-\frac{2}{\lambda}\right)}\frac{1}{\left(1-\frac{1}{\lambda}\right)} - 1. \end{aligned}$$

Substituting $\lambda = 6$, $E\left[\Phi\left(\frac{|H-0S_1|}{6}\right)\right] = 4/5 \le 1$. *Hence, we have at least* $\|H-0S_1\|_\Phi \le 6$, *that is,* $\inf_{v\in\mathcal{V}}\|H-v\|_\Phi < +\infty$. □

3.2. Properties of solutions to Problem 2.2

Even though Problems 2.2–2.4 all have solutions, they do not necessarily coincide. Roughly speaking, if $\mathcal{V}$ is cone, and $v_3 \in \mathcal{V}$ is a solution to Problem 2.4, v_3/c is a solution to Problem 2.4 with respect to H/c for any $c > 0$, that is, we can say that Problem 2.4 has the conicality. On the other hand, when v_1 is a solution to Problem 2.2 with respect to H, v_1/c is not necessarily a solution to the problem with respect to H/c. We introduce such a counterexample.

Example 3.2 *We consider a simple one-period model with* $\Omega = \{\omega_1,\omega_2,\omega_3\}$, *and* $P(\{\omega_i\}) = 1/3$ *for* $i = 1,2,3$. *Moreover,* $S_0 = 0$, S_1 *is given by*

$$S_1(\omega_i) = \begin{cases} 2, & i=1, \\ 0, & i=2, \\ -1, & i=3. \end{cases}$$

Supposing that $H = 1_{\{\omega_1\}}$ *and* $\Phi(x) = e^x - 1$, *and* $\mathcal{V} = \{\vartheta S_1 | \vartheta \in \mathbf{R}\}$, *we have*

$$\begin{aligned} E[\Phi(|H-\vartheta S_1|)] &= \frac{1}{3}\left\{e^{|1-2\vartheta|} + 1 + e^{|\vartheta|}\right\} - 1 \\ &= \frac{1}{3}\left\{e^{|1-2\vartheta|} + e^{|\vartheta|}\right\} - \frac{2}{3}. \end{aligned}$$

Thus, the optimizer is given by $1/2$.

Next, letting $H = 21_{\{\omega_1\}}$, *we have*

$$\begin{aligned} E[\Phi(|H-\vartheta S_1|)] &= \frac{1}{3}\left\{e^{|2-2\vartheta|} + 1 + e^{|\vartheta|}\right\} - 1 \\ &= \frac{1}{3}\left\{e^{|2-2\vartheta|} + e^{|\vartheta|}\right\} - \frac{2}{3}. \end{aligned}$$

Thus, the optimizer is given by $\frac{\log 2+2}{3}$. *Hence, Problem 2.2 does not have the conicality.* □

Before stating relationships among solutions to the three problems, we study properties of solutions to Problem 2.2. In the rest of this section, we assume that

$$\Phi \text{ is differentiable},$$

for simplicity.

Proposition 3.2 *Assuming that there exists a $v^* \in \mathcal{V}$ such that*

$$\min_{v\in\mathcal{V}} E[\Phi(|H-v|)] = E[\Phi(|H-v^*|)], \tag{7}$$

and there exists a $c>1$ such that $E[\Phi(c|H-v^*|)]<+\infty$. We have the following two conditions:

1. *For any $v\in\mathcal{V}$, $E[\phi(|H-v^*|)|H-v^*|] \le E[\phi(|H-v^*|)|H-v|]$.*
2. *For any $v\in\mathcal{V}$, $E[\phi(|H-v^*|)\operatorname{sgn}(H-v^*)(v^*-v)] \ge 0$.*

Moreover, if $v^*\in\mathcal{V}$ satisfies $E[\phi(|H-v^*|)|H-v^*|]<+\infty$ and either the above condition, then v^* satisfies (7).

To prove Proposition 3.2, we need some preparations. We define the Gâteaux derivative DF as

$$DF(u_1,u) := \lim_{t\to 0}\frac{1}{t}E[\Phi(|u_1+tu|)-\Phi(|u_1|)], \text{ for any } u,u_1\in L^\Phi.$$

Proposition 3.3 *Let $v_1\in\mathcal{V}$ and $u\in L^\Phi$ such that $E[\Phi(c|H-v_1|)]<+\infty$ and $E\left[\Phi\left(\frac{2c}{c-1}|u|\right)\right]<+\infty$ for some $c>1$. Then, we have*

$$DF(H-v_1,u) = E[\phi(|H-v_1|)\operatorname{sgn}(H-v_1)u].$$

Proof. For any $t\in(0,1)$, we have

$$\begin{aligned}
&\frac{1}{t}\left\{\Phi(|H-v_1|+t|u|)-\Phi(|H-v_1|)\right\}\\
&= \frac{\Phi(|H-v_1|+t|u|)-\Phi(|H-v_1|)}{t|u|}|u|\\
&\le \phi(|H-v_1|+t|u|)|u|\\
&\le \Phi(|H-v_1|+(1+t)|u|)-\Phi(|H-v_1|+t|u|)\\
&\le \Phi(|H-v_1|+(1+t)|u|)\\
&\le \frac{1}{c}\Phi(c|H-v_1|)+\left(1-\frac{1}{c}\right)\Phi\left(\frac{1+t}{1-\frac{1}{c}}|u|\right)\\
&\le \frac{1}{c}\Phi(c|H-v_1|)+\left(1-\frac{1}{c}\right)\Phi\left(\frac{2c}{c-1}|u|\right)\in L^1
\end{aligned}$$

The dominated convergence theorem then implies that

$$\begin{aligned}
DF(H-v_1,u) &= \lim_{t\to 0}\frac{1}{t}E[\Phi(|H-v_1+tu|)-\Phi(|H-v_1|)]\\
&= E\left[\lim_{t\to 0}\frac{\Phi(|H-v_1+tu|)-\Phi(|H-v_1|)}{t}\right]\\
&= E[\phi(|H-v_1|)\operatorname{sgn}(H-v_1)u].
\end{aligned}$$

This completes the proof of Proposition 3.3. □

Proof of Proposition 3.2. *Condition 1:* Let $v \in \mathcal{V}$ be fixed arbitrarily. Denoting $X = |H - v|$ and $X^* = |H - v^*|$, we define $f(\alpha) := E[\Phi(\alpha X^* + (1-\alpha)X)]$ for any $\alpha \in [0,1]$. Under (7), we have

$$f(\alpha) \geq E[\Phi(|H - \alpha v^* - (1-\alpha)v|)] \geq f(1)$$

for any $\alpha \in [0,1]$, which implies that, for any $\alpha \in [0,1)$

$$0 \geq \frac{f(\alpha) - f(1)}{\alpha - 1} = \frac{1}{\alpha - 1} E\left[\Phi(\alpha X^* + (1-\alpha)X) - \Phi(X^*)\right].$$

In fact, we can prove that the right hand side converges to $E[\phi(X^*)(X^* - X)]$ as α tends to 1, which from Condition 1 follows.

Now, we shall prove the above convergence. To see it, we have only to prove the existence of $\alpha_0 \in [0,1)$ such that, for any $\alpha \in [\alpha_0, 1)$, there exists a random variable Z, independent of α, satisfying

$$\left| \frac{1}{\alpha - 1}(\Phi(\alpha X^* + (1-\alpha)X) - \Phi(X^*)) \right| \leq Z \in L^1,$$

since the rest is proved by the dominated convergence theorem. Note that α_0 depends on X. We have

$$\begin{aligned}
&\left| \frac{1}{\alpha - 1}(\Phi(\alpha X^* + (1-\alpha)X) - \Phi(X^*)) \right| \\
&= \frac{1}{1-\alpha}(\Phi(X^*) - \Phi(\alpha X^* + (1-\alpha)X))1_{\{X^* > X\}} \\
&\quad + \frac{1}{1-\alpha}(\Phi(\alpha X^* + (1-\alpha)X) - \Phi(X^*))1_{\{X > X^*\}} \\
&=: I_1 + I_2.
\end{aligned}$$

Now, we remark that there exists a $c > 1$ such that $\Phi(cX^*) \in L^1$ by the assumption. We have then, for any $\alpha \in (0,1)$,

$$\begin{aligned}
I_1 &= \frac{\Phi(X^*) - \Phi(X^* - (1-\alpha)(X^* - X))}{(1-\alpha)(X^* - X)}(X^* - X)1_{\{X^* > X\}} \\
&\leq \phi(X^*)(X^* - X)1_{\{X^* > X\}} \leq \phi(X^*)X^* \\
&\leq \frac{1}{c-1}(\Phi(cX^*) - \Phi(X^*)) \in L^1.
\end{aligned}$$

Next, we prove that there exists a random variable Z_2, which is independent of α, such that $I_2 \leq Z_2 \in L^1$. Since $X \in L^\Phi$, there exists an $\varepsilon > 0$ such that $\Phi(\varepsilon X) \in L^1$. We may assume that $\varepsilon < 1$. We take a sufficient small $\beta \in (0,1)$ to satisfy $\frac{1-\beta\varepsilon}{1-\beta} < c$. Set $\alpha_0 := 1 - \beta\varepsilon$. For any $\alpha \in [\alpha_0, 1)$, denoting $\alpha = 1 - \gamma\varepsilon$, which means $\gamma \leq \beta$, the convexity of Φ implies that

$$\begin{aligned}
I_2 &= \frac{\Phi(\alpha X^* + (1-\alpha)X) - \Phi(X^*)}{1-\alpha} 1_{\{X > X^*\}} \\
&= \frac{\Phi((1-\gamma\varepsilon)X^* + \gamma\varepsilon X) - \Phi(X^*)}{\gamma\varepsilon} 1_{\{X > X^*\}} \\
&\leq \frac{(1-\gamma)\Phi\left(\frac{1-\gamma\varepsilon}{1-\gamma}X^*\right) + \gamma\Phi(\varepsilon X) - \Phi(X^*)}{\gamma\varepsilon} 1_{\{X > X^*\}} \\
&\leq \frac{(1-\gamma)(\delta\Phi(cX^*) + (1-\delta)\Phi(X^*)) + \gamma\Phi(\varepsilon X) - \Phi(X^*)}{\gamma\varepsilon} 1_{\{X > X^*\}} \\
&= \frac{\frac{\gamma(1-\varepsilon)}{c-1}\Phi(cX^*) + \{(1-\delta)(1-\gamma) - 1\}\Phi(X^*) + \gamma\Phi(\varepsilon X)}{\gamma\varepsilon} 1_{\{X > X^*\}} \\
&\leq \frac{\frac{1-\varepsilon}{c-1}\Phi(cX^*) + \Phi(\varepsilon X)}{\varepsilon} \in L^1,
\end{aligned}$$

where $\delta = \frac{\gamma(1-\varepsilon)}{(c-1)(1-\gamma)}$. Note that, since $c-1 > \frac{\beta(1-\varepsilon)}{1-\beta}$, we have $\delta = \frac{\gamma(1-\varepsilon)}{(c-1)(1-\gamma)} < \frac{\gamma(1-\beta)}{\beta(1-\gamma)} \leq \frac{\gamma(1-\gamma)}{\gamma(1-\gamma)} = 1$, $\delta c + (1-\delta) = 1 + (c-1)\delta = 1 + \frac{\gamma(1-\varepsilon)}{1-\gamma} = \frac{1-\gamma\varepsilon}{1-\gamma}$, and $(1-\gamma)\delta = \frac{\gamma(1-\varepsilon)}{c-1}$. Thus, Condition 1 follows.

Condition 2: Suppose that there exists a $v \in \mathcal{V}$ such that $E[\phi(|H - v^*|)\operatorname{sgn}(H - v^*)(v^* - v)] < 0$. Now, we take a sufficient small constant $\varepsilon > 0$ satisfying $E\left[\Phi\left(\frac{2c\varepsilon}{c-1}|v^* - v|\right)\right] < +\infty$. Proposition 3.3 yields $DF(H - v^*, \varepsilon(v^* - v)) < 0$. On the other hand, we have $DF(H - v^*, \varepsilon(v^* - v)) \geq 0$ for any $v \in \mathcal{V}$ by the definition of the Gâteaux derivative and the optimality of v^*. This is a contradiction! Hence, Condition 2 holds.

The second assertion: First, we suppose Condition 1. By Young's inequality (Theorem 2.1.4 of Edgar and Sucheston (1992)) and Condition 1, we have

$$\begin{aligned}
E[\Phi(|H - v^*|)] &= E[\phi(|H - v^*|)|H - v^*|] - E[\Psi(\phi(|H - v^*|))] \\
&\leq E[\phi(|H - v^*|)|H - v|] - E[\Psi(\phi(|H - v^*|))] \\
&\leq E[\Phi(|H - v|)]
\end{aligned}$$

for any $v \in \mathcal{V}$, from which v^* satisfies (7). Note that Ψ is $\mathbf{R}_+$-valued.

Next, we suppose Condition 2. We can rewrite Condition 2 as follows:

$$E[\phi(|H - v^*|)\operatorname{sgn}(H - v^*)(H - v^*)] \leq E[\phi(|H - v^*|)\operatorname{sgn}(H - v^*)(H - v)]$$

for any $v \in \mathcal{V}$. Theorem 2.1.4 of Edgar and Sucheston (1992) implies that

$$E[\phi(|H - v^*|)\operatorname{sgn}(H - v^*)(H - v^*)] = E[\Psi(\phi(|H - v^*|))] + E[\Phi(|H - v^*|)].$$

and

$$E[\phi(|H - v^*|)\operatorname{sgn}(H - v^*)(H - v)] \leq E[\Psi(\phi(|H - v^*|))] + E[\Phi(|H - v|)]$$

for any $v \in \mathcal{V}$. Hence, v^* satisfies (7). □

Remark 3.1 *If $\mathcal{V} \subset M^\Phi$ and $H \in M^\Phi$, then we have $E[\Phi(c|H - v|)] < +\infty$ for any $c > 0$ and any $v \in \mathcal{V}$. Thus, in such a case, we can get rid of any condition on the existence of $c > 1$ from the statements in Propositions 3.2 and 3.3.*

3.3. Relationship on solutions

We investigate in this subsection relationships among Problems 2.2–2.4. The first is relationships between Problems 2.2 and 2.4.

Proposition 3.4 *We consider the following two conditions:*

1. *There exists a $v_1 \in \mathcal{V}$ such that*

$$\min_{v \in \mathcal{V}} E[\Phi(|H - v|)] = E[\Phi(|H - v_1|)] = 1. \tag{8}$$

2. *There exists a $v_2 \in \mathcal{V}$ such that*

$$\min_{v \in \mathcal{V}} \|H - v\|_\Phi = \|H - v_2\|_\Phi = 1. \tag{9}$$

Then, we have 1⇒2, and (9) holds for v_1. Moreover, if $\mathcal{V} \subset M^\Phi$ and $H \in M^\Phi$, then the reverse direction also holds and (8) holds for v_2.

Proof. We prove the first assertion. Under condition 1, two assertions (2) and (3) in Proposition 2.1.10 of Edgar and Sucheston (1992) provide $\|H - v_1\|_\Phi = 1$. Again, Proposition 2.1.10 of Edgar and Sucheston (1992) implies that $\|H - v\|_\Phi > 1$ whenever $E[\Phi(|H - v|)] > 1$. Thus, we have $\min_{v \in \mathcal{V}} \|H - v\|_\Phi = \|H - v_1\|_\Phi = 1$.

Next, we prove the second assertion. By Proposition 2.1.10 (4) of Edgar and Sucheston (1992), we have $E[\Phi(|H - v_2|)] = 1$. Moreover, we have $\|H - v\|_\Phi > 1 \Rightarrow E[\Phi(|H - v|)] > 1$ by Proposition 2.1.10 (3) of Edgar and Sucheston (1992). This completes the proof. □

In the above proposition, the inclusion 2⇒1 does not hold in general. We exemplify it as follows:

Example 3.3 *We consider a one-period model. Only one risky asset is tradable. Its price process $(S_t)_{t=0,1}$ is given as follows: $S_0 = 0$ and S_1 is expressed by $X - 2$, where X is a random variable whose probability density function f_X is given by:*

$$f_X(x) = \begin{cases} \frac{e^{-x}}{Dx^2}, & x \geq 1, \\ 0, & x < 1, \end{cases}$$

where $D := \int_1^\infty \frac{e^{-x}}{x^2} dx$. The underlying contingent claim H follows the same distribution as X, but is independent of X. Suppose that $\mathcal{V} = \{\vartheta S_1 | \vartheta \in \mathbf{R}\}$. Let $\Phi(x) := a(e^x - 1)$, where $0 < a < \frac{D}{1-D}$.

For any $\vartheta \in \mathbf{R}$, we have

$$E[\Phi(|H - \vartheta S_1|)] = a \left\{ \int_1^\infty \int_1^\infty e^{|y - \vartheta(x-2)|} \frac{e^{-y}}{Dy^2} \frac{e^{-x}}{Dx^2} dy dx - 1 \right\}.$$

In particular, when $\vartheta = 0$, we have

$$\begin{aligned} E[\Phi(|H - \vartheta S_1|)] &= E[\Phi(|H|)] = a \left\{ \int_1^\infty e^{|y|} \frac{e^{-y}}{Dy^2} dy - 1 \right\} \\ &= a \left\{ \int_1^\infty \frac{dy}{Dy^2} - 1 \right\} = a \frac{1 - D}{D}. \end{aligned}$$

Thus, $\inf_{\vartheta\in\mathbf{R}} E[\Phi(|H-\vartheta S_1|)] \leq a^{\frac{1-D}{D}}$. *Noting that* $a < \frac{D}{1-D}$, $\inf_{\vartheta\in\mathbf{R}} E[\Phi(|H-\vartheta S_1|)] < 1$.

Next, we consider, for any $\lambda > 0$, $E\left[\Phi\left(\frac{|H-\vartheta S_1|}{\lambda}\right)\right]$. *Note that*

$$E\left[\Phi\left(\frac{|H-\vartheta S_1|}{\lambda}\right)\right] = a\left\{\int_1^\infty\int_1^\infty e^{\frac{|y-\vartheta(x-2)|}{\lambda}}\frac{e^{-y}}{Dy^2}\frac{e^{-x}}{Dx^2}dydx - 1\right\}.$$

When $\lambda < 1$ *and* $\vartheta \leq 0$, *we have*

$$\begin{aligned}&\int_1^\infty\int_1^\infty e^{\frac{|y-\vartheta(x-2)|}{\lambda}}\frac{e^{-y}}{Dy^2}\frac{e^{-x}}{Dx^2}dxdy\\ \geq\ & e^{\frac{2\vartheta}{\lambda}}\int_1^\infty\frac{e^{\left(\frac{1}{\lambda}-1\right)y}}{Dy^2}dy\int_2^\infty\frac{e^{-\left(\frac{\vartheta}{\lambda}+1\right)x}}{Dx^2}dx = +\infty.\end{aligned}$$

Besides, when $\lambda < 1$ *and* $\vartheta > 0$, *we have*

$$\begin{aligned}&\int_1^\infty\int_1^\infty e^{\frac{|y-\vartheta(x-2)|}{\lambda}}\frac{e^{-y}}{Dy^2}\frac{e^{-x}}{Dx^2}dydx\\ \geq\ & \int_1^\infty\int_{\vartheta(x-2)\vee 1}^\infty e^{\frac{y-\vartheta(x-2)}{\lambda}}\frac{e^{-y}}{Dy^2}\frac{e^{-x}}{Dx^2}dydx = +\infty.\end{aligned}$$

Thus, for any $\vartheta\in\mathbf{R}$, $\|H-\vartheta S_1\|_\Phi \geq 1$. *On the other hand, if* $\vartheta = 0$, $E[\Phi(|H-\vartheta S_1|)] < 1$. *We can conclude that* $\min_{\vartheta\in\mathbf{R}}\|H-\vartheta S_1\|_\Phi = \|H-0S_1\|_\Phi = 1$. □

We state a result with respect to relationships between Problems 2.2 and 2.3.

Proposition 3.5 *Assume that there exists an element* $v^*\in\mathcal{V}$ *satisfying* $E[\Psi(\phi(|H-v^*|))] = 1$ *and* $E[\Phi(c|H-v^*|)] < +\infty$ *for some* $c > 1$. *The following are then equivalent:*

1. $E[\Phi(|H-v^*|)] = \min_{v\in\mathcal{V}} E[\Phi(|H-v|)]$.
2. $N_\Phi(H-v^*) = \min_{v\in\mathcal{V}} N_\Phi(H-v)$.

Proof. 1⇒2: By the definition of $N_\Phi(\cdot)$, Young's inequality and the assumption of this proposition, we have $N_\Phi(H-v^*) \geq E[\phi(|H-v^*|)|H-v^*|] = E[\Phi(|H-v^*|)]+1$. Moreover, Theorem 2.2.9 of Edgar and Sucheston (1992) yields $N_\Phi(H-v^*) \leq E[\Phi(|H-v^*|)]+1$. Thus, $N_\Phi(H-v^*) = E[\phi(|H-v^*|)|H-v^*|] = E[\Phi(|H-v^*|)]+1$. By Proposition 3.2, for any $v\in\mathcal{V}$, we have

$$\begin{aligned}N_\Phi(H-v^*) &= E[\phi(|H-v^*|)|H-v^*|]\\ &\leq E[\phi(|H-v^*|)|H-v|] \leq N_\Phi(H-v).\end{aligned}$$

Thus, v^* is also a solution to $\min_{v\in\mathcal{V}} N_\Phi(H-v)$.

2⇒1: Note that Ψ is finite by Assumption 2.1 (2). Thus, Proposition 2.2.8 (3) of Edgar and Sucheston (1992) implies that, for any $v\in\mathcal{V}$ and any $\varepsilon > 0$, there exists a $u\in M^\Psi$ with $\|u\|_\Psi \leq 1$

such that $N_\Phi(H-v) \leq E[|H-v||u|]+\varepsilon$. In addition, we have $E[\Phi(|H-v^*|)]+1 = N_\Phi(H-v^*)$ by the aforementioned argument. Hence, we obtain that

$$\begin{aligned} E[\Phi(|H-v^*|)]+1 &= N_\Phi(H-v^*) \leq N_\Phi(H-v) \\ &\leq E[|H-v||u|]+\varepsilon \\ &\leq E[\Phi(|H-v|)]+E[\Psi(|u|)]+\varepsilon \\ &\leq E[\Phi(|H-v|)]+1+\varepsilon, \end{aligned}$$

from which $E[\Phi(|H-v^*|)] \leq E[\Phi(|H-v|)]$ follows for any $v \in \mathcal{V}$, since $\varepsilon > 0$ is arbitrary. □

4. EXISTENCE OF SOLUTIONS TO MINIMIZATION PROBLEMS

Our aim of this section lies in obtaining sufficient conditions under which solutions to Problems 2.2–2.4 exist. Throughout this section, we assume

$$\mathcal{V} \text{ is a closed convex subset of } L^\Phi.$$

Although the closedness of $\mathcal{V}$ is essential, we need some additional conditions to ensure the existence of solutions. We consider the case where L^Φ is reflexive.

Remark 4.1 *Supposing that $\Phi(x) = x^p/p$ for $p > 1$, the space L^Φ is reflexive. More generally, the space L^Φ is reflexive, if both Φ and Ψ satisfy the following Δ_2-condition: there exist an $x_0 \in (0,\infty)$ and a $K > 0$ such that $\Phi(2x) < K\Phi(x)$ for any $x \geq x_0$. Thus, e^x-1, e^x-x-1 and $(x+1)\log(x+1)-x$ are not the case. For more details, see Corollary 2.2.12 of Edgar and Sucheston (1992) and Theorem IV.1.10 of Rao and Ren (1991).*

Proposition 4.1 *Let L^Φ be reflexive. Then, Problems 2.2–2.4 all have solutions.*

Proof. As for Problems 2.3 and 2.4, we can prove them easily by consulting with Proposition II.1.2 of Ekeland and Témam (1999). For example, as regards Problem 2.4, letting $F(v) := \|H-v\|_\Phi$, F is a lower semi-continuous convex function. Moreover, since $F(v) \geq \|v\|_\Phi - \|H\|_\Phi$ for any $v \in \mathcal{V}$, F is coercive. Thus, there exists a solution to Problem 2.4.

It remains to show the assertion with respect to Problem 2.2. Before proving it, we should remark that $L^\Phi = M^\Phi$ whenever L^Φ is reflexive. See Theorem IV.2.10 of Rao and Ren (1991). We denote $F(v) := E[\Phi(|H-v|)]$ and $d^* := \inf_{v\in\mathcal{V}} F(v)$. Remark that we are interested in only the case where $d^* < +\infty$; otherwise every $v \in \mathcal{V}$ becomes a solution. Let $(v_n)_{n\geq 1}$ be a minimizing sequence, that is, $F(v_n) \to d^*$ as $n \to \infty$.

For any $v \in \mathcal{V}$ satisfying $(d^* \leq) F(v) \leq d^*+1$, we have, by Theorem 2.2.9 of Edgar and Sucheston (1992),

$$\begin{aligned} \|v\|_\Phi &\leq N_\Phi(v) \leq N_\Phi(H)+N_\Phi(H-v) \leq N_\Phi(H)+F(v)+1 \\ &\leq N_\Phi(H)+d^*+2. \end{aligned}$$

Consequently, we have $\inf_{v\in\mathcal{V}} F(v) = \inf_{v\in\mathcal{A}} F(v)$, where $\mathcal{A} := \{v \in \mathcal{V} | \|v\|_\Phi \leq N_\Phi(H)+d^*+2\}$. We can then extract a minimizing sequence (v_n) within $\mathcal{A}$. Since L^Φ is reflexive, (v_n) converges to some $v^* \in \mathcal{A}$ in the weak topology $\sigma(M^\Phi, L^\Psi)$ by extracting a subsequence if need be.

Letting $w^* := \mathrm{sgn}(H - v^*)\phi(|H - v^*|)$, we have

$$\begin{aligned} E[\Psi(|w^*|)] &= E[\Psi(\phi(|H - v^*|))] \\ &= E[|H - v^*|\phi(|H - v^*|)] - E[\Phi(|H - v^*|)] \\ &\leq E[\Phi(2|H - v^*|)] - 2E[\Phi(|H - v^*|)] < +\infty, \end{aligned}$$

since $H - v^* \in M^\Phi$. Thus, $w^* \in L^\Psi$ follows. Moreover, we have

$$\begin{aligned} F(v^*) &= E[\Phi(|H - v^*|)] = E[(H - v^*)w^*] - E[\Psi(|w^*|)] \\ &= \lim_{n\to\infty} E[(H - v_n)w^*] - E[\Psi(|w^*|)] \\ &\leq \liminf_{n\to\infty} E[\Phi(|H - v_n|)] + E[\Psi(|w^*|)] - E[\Psi(|w^*|)] \\ &= \liminf_{n\to\infty} F(v_n). \end{aligned}$$

Consequently, v^* is a solution to Problem 2.2. □

ACKNOWLEDGEMENTS
The research of Tahir Choulli is supported financially by the Natural Sciences and Engineering Research Council of Canada, through Grant G121210818.

References

B. Becherer. Utility indifference valuation. In R. Cont, editor, *Encyclopedia of Quantitative Finance*, pages 1854–1860. Wiley, 2010.

C. Dellacherie and P.A. Meyer. *Probabilities and Potential B*. North-Holland, Amsterdam, 1982.

G.A. Edgar and L. Sucheston. *Stopping times and directed processes*. Cambridge University Press, 1992.

I. Ekeland and R. Témam. *Convex analysis and variational problems*. Society for Industrial & Applied Mathematics, Philadelphia, 1999.

H. Föllmer and M. Schweizer. The minimal martingale measure. In R. Cont, editor, *Encyclopedia of Quantitative Finance*, pages 1200–1204. Wiley, 2010.

V. Henderson and D. Hobson. Utility indifference pricing – an overview. In R. Carmona, editor, *Indifference pricing*, pages 44–74. Princeton University Press, 2008.

M.M. Rao and Z.D. Ren. *Theory of Orlicz spaces*. Marcel Dekker, New York, 1991.

M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini et al., editor, *Option Pricing*, pages 538–574. Cambridge Univ. Press, 2001.

M. Schweizer. Mean-variance hedging. In R. Cont, editor, *Encyclopedia of Quantitative Finance*, pages 1177–1181. Wiley, 2010.

CONTRIBUTED TALKS

GREEKS WITHOUT RESIMULATION IN SPATIALLY HOMOGENEOUS MARKOV CHAIN MODELS[1]

Stéphane Crépey[‡] and Tuyet Mai Nguyen[‡]

[‡]*Université d'Evry, Laboratoire de Mathématiques et Modélisation d'Evry, France*
Email: `stephane.crepey@univ-evry.fr, tuyetmai.nguyen@univ-evry.fr`

Abstract

In this paper, we model credit portfolios by continuous-time Markov chains with some form of spatial homogeneity, so that direct Monte Carlo Greeks estimates, without resimulation, can be derived. We implement our results in two specific credit models: the shock model of Bielecki et al. (2012), where the spatial homogeneity is straightforward, and the group model of section 11.2 in Crépey (2013), where spatial homogeneity can be recovered by a change of measure and tools of Malliavin calculus. The direct Monte Carlo Greek estimates are competitive with previously developed simulation/regression estimates, but they are also unbiased, and there is some evidence that they would be less impacted by the curse of dimensionality.

Keywords: Markov chains, portfolio credit risk, Greeks, Monte Carlo simulation, Malliavin calculus, Clark-Ocone formula.

1. INTRODUCTION

Though CDO issuances have become quite rare since the crisis, there is still a huge amount of outstanding CDO contracts which need to be marked to market and hedged up to their maturity dates. Moreover, the issue of valuation and hedging of counterparty risk on credit portfolios is very topical since the crisis. With these motivations in mind, we develop in this paper Monte Carlo Greeking schemes without resimulation for continuous-time models of portfolio credit risk. Without resimulation means that all the Greeks (and there are many of them in the case of large

[1]This research benefited from the support of the "Chaire Marchés en Mutation", Fédération Bancaire Française.

portfolios) are estimated based on a single set of model trajectories. Simulation/regression estimates were proposed in the section 11.2 of Crépey (2013) (cf. also Crépey and Rahal (2013) for a short version in article form focusing on CVA applications), but these are biased by construction. Here we propose unbiased estimates under a suitable spatial homogeneity condition on the Markov chain. As in Crépey (2013) (for comparison purposes), we illustrate our approach in the shock model of Bielecki et al. (2012), where spatial homogeneity is straightforward, and in a group model where spatial homogeneity does not hold in the first place but can be recovered under a changed probability measure, using tools of Malliavin calculus for jump processes. The practical performances of our estimates are competitive with those of Crépey (2013) (but, again, our estimates are unbiased, as opposed to those of Crépey (2013)). Moreover, in the shock model, where exact formulas can be used for benchmarking our results, the performance of our estimates doesn't deteriorate with the dimension. This yields one more example of the abilities of simulation schemes to deal with high-dimensional problems , by exploiting the degeneracies of the underlying factor processes, when deterministic schemes are banned by the curse of dimensionality (see also, e.g., Crépey and Rahal (2012)).

Sect. 2 presents the approach in general. Sect. 3 and 4 study its applicability in the shock and in the group model, respectively.

2. GENERAL SETUP

We consider a risk neutral pricing model $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ where $T \geq 0$ is a fixed time horizon and $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration of a continuous-time d-variate Markov chain $\mathcal{N} = (N^1, \cdots, N^d)$ with components in $\mathbb{N}_\nu = \{0, 1, \cdots, \nu - 1\}$, for some fixed integer ν. So $\mathcal{N}$ lives in the state space $\mathcal{I} = \mathbb{N}_\nu^d$. The cumulative default process N_t on a credit risk portfolio is modeled as $N_t = \varphi(\mathcal{N}_t)$, for some integer valued loss function φ, e.g. $\varphi(\imath) = \sum_{k=1}^d i_k$, for $\imath = (i_1, \cdots, i_d) \in \mathcal{I}$. Given a credit derivative payoff $\xi = \pi(N_T) = \pi(\varphi(\mathcal{N}_T)) = \phi(\mathcal{N}_T)$, where $\phi = \pi \circ \varphi$, we have the corresponding price process, by the Markov property of $\mathcal{N}$ assuming zero risk-free rate for simplicity:

$$\Pi_t = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\phi(\mathcal{N}_T)|\mathcal{F}_t] = \mathbb{E}[\phi(\mathcal{N}_T)|\mathcal{N}_t] = u(t, \mathcal{N}_t), \text{ for } t \in [0, T], \tag{1}$$

for some pricing function $u(t, \imath), t \in [0, T], \imath \in \mathcal{I}$. We are interested in the sensitivity of the pricing function with respect to events Y (specified in later sections), which can be represented as

$$\delta u^Y(t, \imath) = u(t, \imath^Y) - u(t, \imath),$$

where $\imath$ and $\imath^Y$ represent the state of the chain right before and after the event Y. Since

$$u(t, \imath) = \mathbb{E}[\phi(\mathcal{N}_T)|\mathcal{N}_t = \imath] \text{ and } u(t, \imath^Y) = \mathbb{E}[\phi(\mathcal{N}_T)|\mathcal{N}_t = \imath^Y],$$

in general, computing $\delta u^Y(t, \imath)$ by Monte Carlo implies resimulation (conditionally given $\mathcal{N}_t = \imath$ and then given $\mathcal{N}_t = \imath^Y$, and this for each Y). However, this is unnecessary under the following spatial homogeneity condition on $\mathcal{N}$ (since then $\mathbb{E}[\phi(\mathcal{N}_T)|\mathcal{N}_t = \imath^Y] = \mathbb{E}[\phi(\varphi_Y(\mathcal{N}_T))|\mathcal{N}_t = \imath]$ above).

Definition 2.1 *The Markov chain $\mathcal{N}$ is said to be spatially homogeneous if for every event Y, there exists a deterministic function φ_Y such that*

$$(\mathcal{N}_T|\mathcal{N}_t = \imath^Y) \stackrel{\mathcal{L}}{=} (\varphi_Y(\mathcal{N}_T)|\mathcal{N}_t = \imath).$$

Example: If $\mathcal{N}_t = N_t$ is a Poisson process capped at n and Y represents a Poisson jump, then $\mathcal{N}$ is spatially homogeneous with $\varphi_Y(i) = \min(i+1, n)$.

In the following sections we use the above results to greek CDO contracts in two specific Markov chain models of credit portfolio. For the prerequisites of the CDO pricing problem, see e.g. Crépey and Rahal (2013). The nominal on each credit name is set to 100 and all the recovery rates are set to 40%.

3. SHOCK MODEL

First we describe briefly the shock model of Bielecki et al. (2012) (or, in extended book form, Chapter 8 in Crépey et al. (2014)). We consider n reference credit names, indexed from 1 to n. With respect to the general setup, this corresponds to a case where $d = n$ and $N^l, l = 1, \cdots, n$ stands for the default indicator process of name l. The state space $\mathcal{I}$ is equal to $\{0,1\}^n$. First, we define a family $\mathcal{Y}$ of "shocks", i.e. subsets Y of obligors, typically the singletons (or "idiosyncratic shocks") $\{1\}, \ldots, \{n\}$ and a small number of "common (or systemic) shocks" $I_1, \ldots, I_m$ representing simultaneous defaults. For every $Y \in \mathcal{Y}$, we define

$$\tau_Y = \frac{\mathcal{E}_Y}{\lambda_Y},$$

where the $\mathcal{E}_Y$ are i.i.d. standard exponential random variables and the $\lambda_Y > 0$ are constant shock intensities. At last, we define for each obligor l

$$\tau_l = \bigwedge_{\{Y \in \mathcal{Y}; l \in Y\}} \tau_Y, \;\; N^l_t = \mathbb{I}_{t \geq \tau_l}.$$

The idea is that the advent of the shock I_j at time t triggers the default of all the surviving names in I_j at t, which corresponds to a kind of "instantaneous" credit contagion in the form of simultaneous defaults. As shown in Chapter 8 of Crépey et al. (2014), $\mathcal{N} = (N^l)_{1 \leq l \leq n}$ is a Markov process and the greeks needed for hedging are the $\delta u^Y(t, \imath) = u(t, \imath^Y) - u(t, \imath)$, where, for $\imath \in \mathcal{I}$ and $Y \in \mathcal{Y}$, $\imath^Y$ represents $\imath$ with coordinates in Y replaced by one (when not already so).

Proposition 3.1 *(Spatial homogeneity in the shock model) For every $\imath \in \mathcal{I}$ and $Y \in \mathcal{Y}$,*

$$(\mathcal{N}_T|\mathcal{N}_t = \imath^Y) \stackrel{\mathcal{L}}{=} (\mathcal{N}^Y_T|\mathcal{N}_t = \imath). \tag{2}$$

Proof. This can be verified on the explicit formula that is available for the conditional joint survival probability in the shock model (see Proposition 2.1 in Bielecki et al. (2012)). ■

As a consequence,

$$\delta u^Y(t, \imath) = \mathbb{E}\big[\phi(\mathcal{N}^Y_T) - \phi(\mathcal{N}_T)\big|\mathcal{N}_t = \imath\big]$$

and, in particular,

$$\delta u^Y(0, 0) = \mathbb{E}[\phi(\mathcal{N}^Y_T)] - \mathbb{E}[\phi(\mathcal{N}_T)].$$

3.1. Numerical Results

In this section we use the above results to greek CDO contracts on n=30, 60, 90 or 120 underlying names. We use a nested structure of common shocks I_j, $j = 1, \ldots, m = 4$, so that I_1, I_2 and I_3 respectively correspond to the 8%, 16% and 32% riskiest names (riskiest in the sense of the corresponding CDS spreads at time 0) and I_4 is the "Armageddon" shock corresponding to all names. We consider equity and junior mezzanine CDO tranches (the tranches the most important to hedge) insuring the buyer of protection against the first 5% of underlying credit losses and against losses from 5% to 10%, respectively. We use $m = 5 \times 10^5$ simulations. Figures 1 and 2 illustrate the absolute and relative errors of the simulated deltas as compared with the exact values, visible in Table 1, computed by recursive algorithms described in Chapter 8 of Crépey et al. (2014).

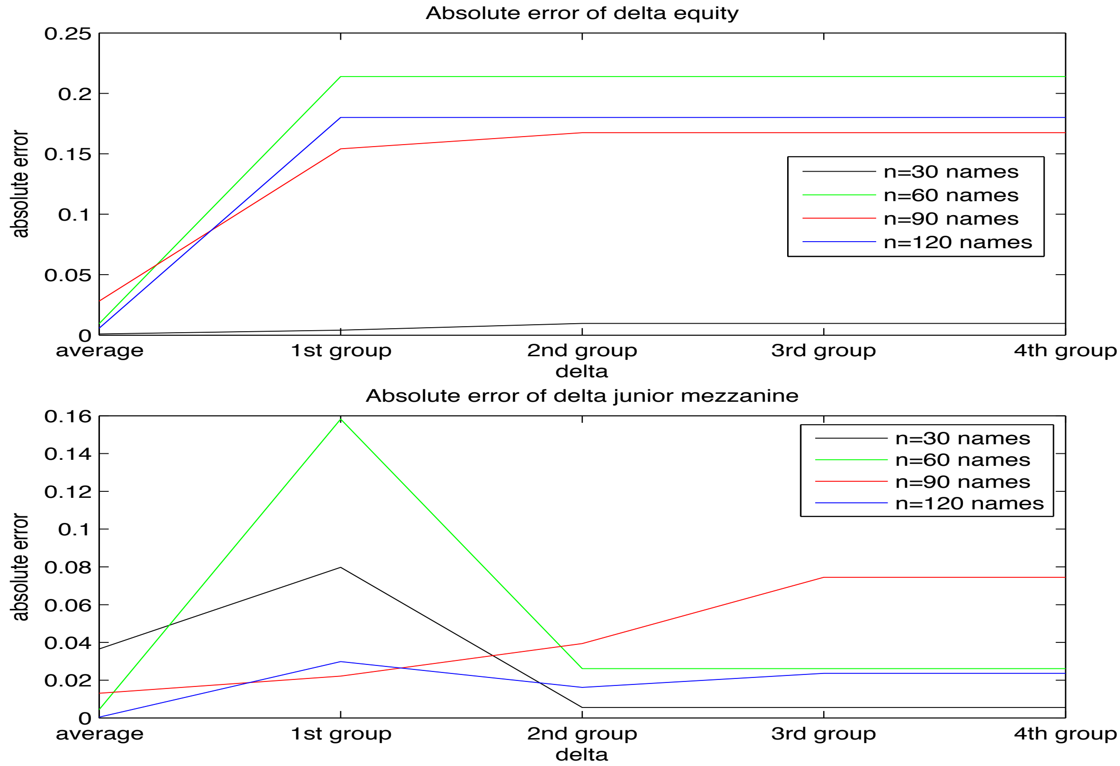


Figure 1: Absolute errors of the simulated deltas in the shock model.

The results show that our method can efficiently deal with high dimensional problems. The errors don't explode with the dimension (number of names).

4. GROUP MODEL

In the group model, the n names of the pool are shared into d groups of $\nu - 1 = \frac{n}{d}$ obligors (taking n a multiple of d). The cumulative default processes $N^k, k = 1, \cdots, d$ in the different

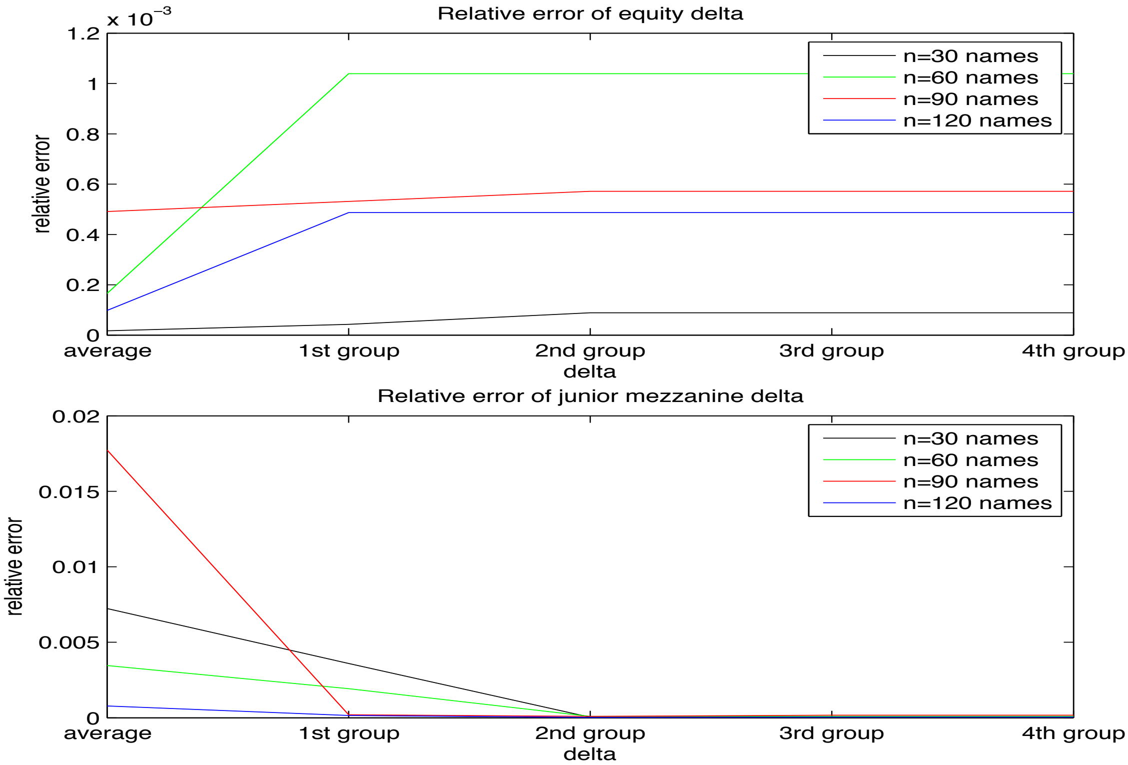


Figure 2: Relative errors of the simulated deltas in the shock model.

Equity	$\bar{\delta}$	δ_1	δ_2	δ_3	δ_4
n = 30	53.27896	93.54673	109.27821	109.27821	109.27821
n = 60	56.92980	205.88311	205.88311	205.88311	205.88311
n = 90	57.27941	290.14565	292.97392	292.97392	292.97392
n = 120	57.36067	369.74917	369.74917	369.74917	369.74917
JMezz	$\bar{\delta}$	δ_1	δ_2	δ_3	δ_4
n = 30	5.05002	22.18655	148.04106	148.04106	148.04106
n = 60	1.25017	82.18569	298.04551	298.04551	298.04551
n = 90	0.74252	111.03935	440.11343	447.40568	447.40568
n = 120	0.47938	200.12579	593.64953	596.48387	596.48387

Table 1: Exact values of the idiosyncratic deltas averaged over all names (column 2) and of the four systemic deltas (columns 3 to 6).

groups are jointly modeled as a continuous-time d-variate Markov chain $\mathcal{N} = (N^1, \cdots, N^d)$ with components in $\mathbb{N}_\nu = \{0, 1, \cdots, \nu - 1\}$. The state space $\mathcal{I} = \mathbb{N}_\nu^d$. We assume no simultaneous default, so the cumulative default processes N^k never jump together. The intensity of jump in the group k is given in the form

$$\lambda^k(t, \imath) = (\nu - 1 - i_k)\tilde{\lambda}^k(t, \imath),$$

where $\imath = (i_1, \cdots, i_d) \in \mathcal{I}$ represents the current state of $\mathcal{N}$, i_k is the number of defaults in group k and $\tilde{\lambda}^k : [0, T] \times \mathcal{I} \to \mathbb{R}^+$ is a (measurable and bounded) pre-default individual intensity function for an obligor in group k. The compensated process

$$M_t^k = N_t^k - \int_0^t \lambda^k(s, \mathcal{N}_s)ds$$

is an $\mathcal{F}$-martingale under $\mathbb{P}$. Since the intensity processes depend on the state of the Markov chain, we do not have homogeneity under $\mathbb{P}$. But we can always view a Markov chain with intensities $\lambda^k(t, \mathcal{N}_t)$ under $\mathbb{P}$ as a measure-changed homogeneous Markov chain. More precisely, let us consider a Markov chain $\mathcal{N}$ under a probability measure $\hat{\mathbb{P}}$ where all the counting processes N^k have intensity 1. We define the process $(\Gamma_t)_{t\in[0,T]}$ such that

$$\frac{d\Gamma_t}{\Gamma_{t-}} = \sum_{k=1}^{d}(\lambda^k(t, \mathcal{N}_{t-}) - 1)d\hat{M}_t^k,\ \Gamma_0 = 1, \tag{3}$$

where $\hat{M}_t^k = N_t^k - t$ is the compensated martingale of the process N_t^k under $\hat{\mathbb{P}}$. Hence Γ_t is a $\hat{\mathbb{P}}$-martingale, playing the role of a Radon-Nikodym density in the change of measure, explicitly given by Doléan-Dade exponentials as

$$\begin{aligned}
\Gamma_t &= \prod_{k=1}^{d} \mathcal{E}\left(\int_0^t (\lambda^k(s, \mathcal{N}_{s-}) - 1)d\hat{M}_s^k\right) \\
&= \prod_{k=1}^{d} e^{\int_0^t \left(1-\lambda^k(s,\mathcal{N}_s)\right)ds} \prod_{\tau^k \le t, N^k_{\tau^k} \neq N^k_{\tau^k-}} \lambda^k(\tau^k, \mathcal{N}_{\tau^k-}) &\quad (4)\\
&= e^{\int_0^t (d-\lambda(s,\mathcal{N}_s))ds} \prod_{k=1}^{d} \prod_{\tau^k \le t, N^k_{\tau^k} \neq N^k_{\tau^k-}} \lambda^k(\tau^k, \mathcal{N}_{\tau^k-}) &\quad (5)
\end{aligned}$$

where $\lambda(s, \mathcal{N}_s) = \sum_{k=1}^d \lambda^k(s, \mathcal{N}_s)$ is the intensity of jump of $\mathcal{N}$ at time s. In (4), for each k, the second product runs over all jump times of the process N^k up to t. In (5), the double product runs over all jump times of the process N up to t. By defining a change of measure

$$\frac{d\mathbb{P}}{d\hat{\mathbb{P}}} = \Gamma_T,$$

we obtain processes N_t^k with intensity $\lambda^k(t, \mathcal{N}_t)$ under $\mathbb{P}$:

Lemma 4.1 *For every $k = 1, \cdots, d$, M_t^k is a $\mathbb{P}$-martingale.*

Proof. We have

$$\begin{aligned} d(M_t^k \Gamma_t) &= M_{t-}^k d\Gamma_t + \Gamma_{t-} dM_t^k + d[M^k, \Gamma]_t \\ &= M_{t-}^k d\Gamma_t + \Gamma_{t-}(dN_t^k - \lambda^k(t, \mathcal{N}_t)dt) + \Gamma_{t-}(\lambda^k(t, \mathcal{N}_{t-}) - 1)dN_t^k \\ &= M_{t-}^k d\Gamma_t + \Gamma_{t-}\lambda^k(t, \mathcal{N}_{t-})d\hat{M}_t^k, \end{aligned}$$

where M^k and Γ are bounded, so $M^k\Gamma$ is a $\hat{\mathbb{P}}$-martingale, hence M^k is a $\mathbb{P}$-martingale. ■

In the group model, the martingale representation has the form

$$\Pi_t = \Pi_0 + \sum_{k=1}^{d} \int_0^t \delta u^k(s, \mathcal{N}_{s-}) dM_s^k, \tag{6}$$

where $\delta u^k(t, \imath) = u(t, \imath^k) - u(t, \imath)$, in which $\imath^k$ represents the state $\imath$ with component k increased by one.

Proposition 4.2 *For every $t \in [0, T]$ such that $\Gamma_{t-} \neq 0$ and $\lambda^k(t, \mathcal{N}_{t-}) \neq 0$,*

$$\delta u^k(t, \mathcal{N}_{t-}) = \frac{1}{\lambda^k(t, \mathcal{N}_{t-})} \mathbb{E}\left[\frac{\epsilon_{t,0^k}^+(\Gamma_T \xi)}{\Gamma_T} | \mathcal{F}_t\right] - \mathbb{E}[\xi | \mathcal{F}_t], \tag{7}$$

where $\epsilon_{t,z}^+$, so-called creation operator (see lemma III.3 of Bouleau and Denis (2013)), adds a jump of size z at t in the process $\mathcal{N}$. In particular,

$$\delta u^k(0, \mathcal{N}_0) = \frac{1}{\lambda^k(0, \mathcal{N}_0)} \mathbb{E}\left[\frac{\epsilon_{0,0^k}^+(\Gamma_T \xi)}{\Gamma_T}\right] - \mathbb{E}[\xi]. \tag{8}$$

Proof. The group model $\mathcal{N}$ can be represented as

$$\mathcal{N}_t = \sum_{i=1}^{N_t} Z_i, \tag{9}$$

where $N_t = \sum_{k=1}^d N_t^k$ is the cumulative default process and the Z_i are the successive jump sizes of $\mathcal{N}$ in

$$\mathcal{Z} = \{0^1 := (1, 0, \cdots, 0), 0^2 := (0, 1, 0, \cdots, 0), \cdots, 0^d := (0, \cdots, 0, 1)\} \subset \mathbb{N}^d.$$

Under the probability $\hat{\mathbb{P}}$, $\mathcal{N}$ has the form (9), where N_t is a homogeneous Poisson process of intensity d and $(Z_i)_{i \geq 0}$ are i.i.d. with uniform distribution $\mathcal{U}$ on $\mathcal{Z}$. Hence, $\mathcal{N}$ is a compound Poisson process under $\hat{\mathbb{P}}$. The jump counting measure ν of $\mathcal{N}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathcal{Z}$ with intensity measure $\mu(dt, dz) = d dt \otimes \mathcal{U}(dz)$ and with compensated random measure $\tilde{\nu}(dt, dz) = \nu(dt, dz) - \mu(dt, dz)$. The Clark-Ocone formula for the random variable $\Gamma_T \xi$ under $\hat{\mathbb{P}}$ yields (see Di Nunno et al. (2008)):

$$\Gamma_T \xi = \hat{\mathbb{E}}[\Gamma_T \xi] + \int_0^T \int_{\mathcal{Z}} \hat{\mathbb{E}}[D_{s,z}(\Gamma_T \xi) | \mathcal{F}_s] \tilde{\nu}(ds, dz),$$

where $D_{s,z}(\Gamma_T\xi)$ is the Malliavin derivative of $\Gamma_T\xi$ at (s,z) (and for a predictable version of the conditional expectation process $\hat{\mathbb{E}}[D_{s,z}(\Gamma_T\xi)|\mathcal{F}_s], s \geq 0$). Hence,

$$\Gamma_t\Pi_t = \Gamma_t\mathbb{E}[\xi|\mathcal{F}_t] = \hat{\mathbb{E}}[\Gamma_T\xi|\mathcal{F}_t] = \hat{\mathbb{E}}[\Gamma_T\xi] + \int_0^t\int_{\mathcal{Z}} \hat{\mathbb{E}}[D_{s,z}(\Gamma_T\xi)|\mathcal{F}_s]\tilde{\nu}(ds,dz)$$

and

$$d(\Gamma_t\Pi_t) = \hat{\mathbb{E}}[D_{t,z}(\Gamma_T\xi)|\mathcal{F}_t]\tilde{\nu}(dt,dz) = \sum_{k=1}^d \hat{\mathbb{E}}[D_{t,0^k}(\Gamma_T\xi)|\mathcal{F}_t]d\hat{M}_t^k. \tag{10}$$

Moreover, from (3) and (6), we obtain

$$\begin{aligned}
d(\Gamma_t\Pi_t) &= \Gamma_{t-}d\Pi_t + \Pi_{t-}d\Gamma_t + d[\Pi,\Gamma]_t \\
&= \Gamma_{t-}\sum_{k=1}^d \delta u^k(t,\mathcal{N}_{t-})dM_t^k + \Pi_{t-}\Gamma_{t-}\sum_{k=1}^d(\lambda^k(t,\mathcal{N}_{t-})-1)d\hat{M}_t^k \\
&\quad +\Gamma_{t-}\sum_{k=1}^d \delta u^k(t,\mathcal{N}_{t-})(\lambda^k(t,\mathcal{N}_{t-})-1)dN_t^k \\
&= \Gamma_{t-}\sum_{k=1}^d[\delta u^k(t,\mathcal{N}_{t-})\lambda^k(t,\mathcal{N}_{t-}) + \Pi_{t-}(\lambda^k(t,\mathcal{N}_{t-})-1)]d\hat{M}_t^k.
\end{aligned} \tag{11}$$

By identifying (10) and (11) we get

$$\Gamma_{t-}[\delta u^k(t,\mathcal{N}_{t-})\lambda^k(t,\mathcal{N}_{t-}) + \Pi_{t-}(\lambda^k(t,\mathcal{N}_{t-})-1)] = \hat{\mathbb{E}}[D_{t,0^k}(\Gamma_T\xi)|\mathcal{F}_t].$$

But by properties of the Malliavin derivative and of the creation operator ϵ^+ (see lemma III.3 of Bouleau and Denis (2013)), we have $D_{t,0^k}(\Gamma_T\xi) = \epsilon^+_{t,0^k}(\Gamma_T\xi) - \Gamma_T\xi$, and

$$\hat{\mathbb{E}}[D_{t,0^k}(\Gamma_T\xi)|\mathcal{F}_t] = \hat{\mathbb{E}}[\epsilon^+_{t,0^k}(\Gamma_T\xi) - \Gamma_T\xi|\mathcal{F}_t] = \hat{\mathbb{E}}[\epsilon^+_{t,0^k}(\Gamma_T\xi)|\mathcal{F}_t] - \Gamma_{t-}\Pi_{t-}$$

(for a predictable version of the conditional expectation $\hat{\mathbb{E}}[\epsilon^+_{t,0^k}(\Gamma_T\xi)|\mathcal{F}_t]$). Therefore,

$$\Gamma_{t-}\lambda^k(t,\mathcal{N}_{t-})[\delta u^k(t,\mathcal{N}_{t-}) + \Pi_{t-}] = \hat{\mathbb{E}}[\epsilon^+_{t,0^k}(\Gamma_T\xi)|\mathcal{F}_t] = \Gamma_{t-}\mathbb{E}\left[\frac{\epsilon^+_{t,0^k}(\Gamma_T\xi)}{\Gamma_T}|\mathcal{F}_t\right], \tag{12}$$

with the convention that the ratio equals to 0 when $\Gamma_T = 0$, hence also $\epsilon^+_{t,0^k}(\Gamma_T\xi) = 0$, in the right hand side. In the case where $\Gamma_{t-} \neq 0$ and $\lambda^k(t,\mathcal{N}_{t-}) \neq 0$, we deduce

$$\delta u^k(t,\mathcal{N}_{t-}) + \Pi_{t-} = \frac{1}{\lambda^k(t,\mathcal{N}_{t-})}\mathbb{E}\left[\frac{\epsilon^+_{t,0^k}(\Gamma_T\xi)}{\Gamma_T}|\mathcal{F}_t\right].$$

■

Now we consider the problem of min-variance hedging an equity or senior CDO tranche by the underlying credit index. Let Π and P (resp. u and v) denote the price processes (resp. pricing functions) of a tranche and of the index. By application of the formula (11.14) in Crépey (2013), we can min-variance hedge a tranche by the index and the riskless (constant) asset by using the strategy ζ in the index defined by

$$\zeta_t = \frac{\sum_{l=1}^{d} \lambda^l (\delta u^l)(\delta v^l)}{\sum_{l=1}^{d} \lambda^l (\delta v^l)^2}(t, \mathcal{N}_{t-}) = \sum_{l=1}^{d} w^l \left(\frac{\delta u^l}{\delta v^l}\right) \text{ with } w^l = \frac{(\delta v^l)^2}{\sum_{j=1}^{d} \lambda^j (\delta v^j)^2}, \text{ for } t \in [0, T], \tag{13}$$

where δu^l and δv^l can be represented in the form (7) (or, at time 0, (8)). In case of a local intensity model ($d = 1$), the martingale representation (6) yields

$$d\Pi_t = \delta u(t, \mathcal{N}_{t-}) dM_t, \; dP_t = \delta v(t, \mathcal{N}_{t-}) dM_t.$$

Therefore,

$$d\Pi_t = \delta_t dP_t, \text{ where } \delta_t = \delta(t, \mathcal{N}_{t-}) = \frac{u(t, \mathcal{N}_t) - u(t, \mathcal{N}_{t-})}{v(t, \mathcal{N}_t) - v(t, \mathcal{N}_{t-})}. \tag{14}$$

In this case, it is thus possible to replicate the tranche by the index using the strategy δ_t defined by (14), which coincides with the min-variance hedging strategy ζ_t in (13).

4.1. Numerical Results

We estimate, by Monte Carlo based on (8) using $m = 10^4$ or $m = 10^6$ simulations, the deltas of the equity tranche and of the senior tranche with maturity $T = 5$ and "strike" $k = 45\%$ (equity tranche $[0, 45\%]$ and senior tranche $[45\%, 100\%]$ with pricing functions denoted by u^+ and u^-, respectively). The nominal is set to 1. The results are compared with the exact values computed by matrix exponentiation and with the simulation/regression estimates of section 11.2 in Crépey (2013) (note that these are based on $m = 4 \times 10^4$ simulations).

One group This is the special case where $d = 1$. For tractability of the matrix exponentiation method that is used for validating our simulation results, we consider a small portfolio of $n = 8$ obligors. The pre-default individual intensity function is taken as

$$\tilde{\lambda}(i) = \frac{1 + i}{n}.$$

The results are displayed in Table 2.

$k = 45\%$	val δ	err $\hat{\delta}_1^1$	err $\hat{\delta}_s^1$	err $\hat{\delta}_s^2$
Eq	0.41513	0.29196	-7.97263	-0.08968
Sen	0.58487	-0.20723	5.65883	0.06365

Table 2: One group: Exact values (column 2) and percentage relative errors for $\delta = \frac{\delta u_0^{\pm}(0)}{\delta v_0(0)}$ estimated by simulation/regression with $m = 10^4$ (column 3) or by simulation based on spatial homogeneity with $m = 10^4$ (column 4) or $m = 10^6$ (column 5).

Two groups This time the $n = 8$ names are divided into $d = 2$ groups. The pre-default individual intensity function in each group is given by

$$\tilde{\lambda}^k(\imath) = \frac{k(1 + i_k)}{n}.$$

We keep the other parameters as in the local intensity model. The results are displayed in Table 3.

$k = 45\%$	val δ	err $\hat{\delta}_1^1$	err $\hat{\delta}_s^1$	err $\hat{\delta}_s^2$
Eq1	0.39453	0.05174	3.22293	0.45355
Eq2	0.53172	-6.3304	14.88279	0.34204
Sen1	0.60547	-0.03372	-2.10011	-0.29554
Sen2	0.46828	7.18803	-16.89909	-0.38838

Table 3: Two groups: Exact values (column 2) and percentage relative errors for $\delta = \frac{\delta^1 u^\pm}{\delta^1 v}(0,0,0)$ or $\frac{\delta^2 u^\pm}{\delta^2 v}(0,0,0)$ estimated by simulation/regression with $m = 10^4$ (column 3) or by simulation based on spatial homogeneity with $m = 10^4$ (column 4) or $m = 10^6$ (column 5).

Note that $\hat{\delta}_1^1$ in tables 2 and 3 is the best simulation/regression estimate of Crépey and Rahal (2013) (the indices mean that the regression is affine in time and restricted to the scenarios where the first default takes place before $T_1 = 1$ year). The error of this estimate, as of simulation/regression estimates in general, varies a lot with the parameters of the simulation, whereas our estimates $\hat{\delta}_s$ seem more robust. Moreover, unlike $\hat{\delta}_1^1$, our estimates are unbiased and have a guaranteed convergence rate in $\frac{1}{\sqrt{m}}$ (compare the errors of $\hat{\delta}_s^1$ and $\hat{\delta}_s^2$).

References

T. R. Bielecki, A. Cousin, S. Crépey, and A. Herbertsson. Dynamic hedging of portfolio credit risk in a Markov copula model. *Journal of Optimization Theory and Applications*, 2012. Forthcoming.

N. Bouleau and L. Denis. *Dirichlet Forms Methods for Poisson Point Measures and Lévy Processes*. Springer, 2013. Preprint.

S. Crépey. *Financial Modeling: A Backward Stochastic Differential Equations Perspective*. Springer Finance Textbooks. Springer, 2013.

S. Crépey and A. Rahal. Pricing convertible bonds with call protection. *Journal of Computational Finance*, 15(2):37–75, 2012.

S. Crépey and A. Rahal. Simulation/regression pricing schemes for CVA computations on CDO tranches. *Communications in Statistics - Theory and Methods*, 2013. forthcoming.

S. Crépey, T. R. Bielecki, and D. Brigo. *Counterparty Risk and Funding–A Tale of Two Puzzles*. Taylor & Francis, 2014. Forthcoming Spring 2014.

G. Di Nunno, B.K. Øksendal, and F. Proske. *Malliavin Calculus for Lévy Processes with Applications to Finance*. Universitext (En ligne). Springer-Verlag Berlin Heidelberg, 2008. ISBN 9783540785729. URL `http://books.google.fr/books?id=G9EvB\_HZCVwC`.

WORST-CASE OPTIMIZATION FOR AN INVESTMENT CONSUMPTION PROBLEM

Tina Engler

Department of Mathematics, Martin Luther University Halle-Wittenberg
Theodor-Lieser-Str. 5, 06120 Halle (Saale), Germany
Email: tina.engler@mathematik.uni-halle.de

Abstract

We investigate a Merton-type investment-consumption problem under the threat of a market crash, where the interest rate of the savings account is stochastic. Inspired by the recent work of Desmettre et al. (2013), we model the market crash as an uncertain event (τ, l). While the stock price is driven by a geometric Brownian motion at times $t \in [0, \tau) \cup (\tau, \infty]$, it loses a fraction l of its value at the crash time τ. We maximize the expected discounted logarithmic utility of consumption over an infinite time horizon in the worst-case scenario, and solve the problem by separating it into a post- and a pre-crash problem. We determine the optimal post-crash strategy by means of classical stochastic optimal control theory. Finally, based on the martingale approach, developed by Seifried (2010), we characterize the optimal pre-crash strategy.

1. INTRODUCTION AND MOTIVATION

The classical Merton-type model for determining optimal rules for investment and consumption on a complete market with constant market parameters was solved by Merton (1969) using Dynamic Programming. Since then, several generalizations, such as stochastic volatilities of the stock price, transaction costs or acting on an incomplete market were considered in a wide-ranging body of literature. Moreover, in contrast to the classical work of Merton, for example, Fleming and Pang (2004) and Pang (2006) considered a model where the market parameter r, which represents the interest rate, is an ergodic Markov diffusion process. The authors motivated this by the fact that even for money in the bank, the interest rate may fluctuate over time. On the other hand, the fluctuations of the stock price were generalized to model market crashes. The standard approach often used in the literature is to replace the geometric Brownian motion by a jump diffusion process, which requires distributional assumptions on the jumps. However, Korn and Wilmott (2002) proposed modeling a market crash as an uncertain event and optimized the expected discounted utility of consumption in the worst-case scenario.

This paper combines both of these aspects in a model with a stochastic interest rate and the threat of a market crash modeled as an uncertain event. We are interested in finding the infinite horizon optimal investment and consumption behavior of an investor with a logarithmic utility function in the worst-case scenario with respect to a market crash. As in Desmettre et al. (2013), we model the market crash as an uncertain once-in-a-lifetime event (τ, l), where τ denotes the random crash time and l indicates the crash size. The advantage of this method is that no distributional assumptions about price jumps are needed.

After explaining the investment-consumption model in Section 2, we apply the worst-case optimization theory to our model with a stochastic interest rate. In Section 3 we solve the worst-case optimization problem for two different models of interest rates. Therein, we proceed in three steps. First, we can solve the post-crash problem by standard stochastic optimal control theory (Section 3.1) for both a Vasicek interest rate model and a Cox-Ingersoll-Ross (CIR) model. Then, in Section 3.2, we reformulate the worst-case problem into a pre-crash problem that we reduce to a controller-vs-stopper game. Finally, we can determine the optimal pre-crash strategy by applying a martingale approach by Seifried (2010).

2. THE WORST-CASE OPTIMIZATION PROBLEM

Let us consider a financial market with one risky asset and a savings account with a stochastic interest rate. Throughout the paper, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$. As in Desmettre et al. (2013), we are interested in finding the optimal investment and consumption behavior of an investor under the threat of a market crash (τ, l), which is defined as follows. The event (τ, l) consists of the crash time τ and the crash size l. The crash time τ is a $[0, \infty]$-valued stopping time. At time τ, the risky asset loses a fraction l of its value, where l is an $\mathcal{F}_\tau$-adapted random variable with $0 \leq l \leq l^*$ and $l^* < 1$ denotes the maximal crash size. We abbreviate the set of all crash scenarios briefly by

$$\mathcal{C} := \{(\tau, l) : \tau \in [0, \infty],\ \text{stopping time}, l \in [0, l^*]\, \mathcal{F}_\tau\text{- measurable random variable}\}.$$

Moreover, we assume at normal times $t \in [0, \tau) \cup (\tau, \infty]$ that the asset price P_t follows a geometric Brownian motion

$$dP_t = P_t[\mu\, dt + \sigma_1\, dw_{1,t}], \qquad P_0 = p^0,$$

where $\mu, \sigma_1 > 0$ are constant, and $w_1 = (w_{1,t})_{t\geq 0}$ is a standard Wiener process. At the crash time τ, we have

$$P_\tau = (1 - l)P_{\tau_-}.$$

Our model and the model considered in Desmettre et al. (2013) differ in the interest rate modeling. Here, we assume that the interest rate, denoted by $r = (r_t)_{t\geq 0}$, follows a stochastic process. We consider two different interest rate models in this paper. On the one hand, we consider an interest rate $r = r^V$ that follows a Vasicek process after the market crash

$$r_t^V = \begin{cases} r_c & : t \leq \tau \\ r_c e^{-a(t-\tau)} + r_M\left(1 - e^{-a(t-\tau)}\right) + \sigma_2 e^{-at}\int_\tau^t e^{as} d\tilde{w}_s & : t > \tau \end{cases}, \tag{1}$$

and on the other hand, we consider an interest rate $r = r^C$ that follows a CIR process after time τ

$$r_t^C = \begin{cases} r_c & : t \leq \tau \\ r_c e^{-a(t-\tau)} + r_M \left(1 - e^{-a(t-\tau)}\right) + \sigma_2 e^{-at} \int_\tau^t \sqrt{r_s^C} e^{as} d\tilde{w}_s & : t > \tau \end{cases}, \tag{2}$$

where $a, r_M, \sigma_2 > 0$ and $\tilde{w} = (\tilde{w}_t)_{t \geq 0}$ denotes a Wiener process, correlated with w_1 by a correlation coefficient $\rho \in [-1, 1]$. Assuming model (1) or (2), the interest rate before the market crash is given by a positive constant r_c with $\mu - r_c > 0$. After the market crash, the interest rate follows an affine linear stochastic process, of either Vasicek- or CIR-type, with a speed of reversion a to the long-term mean level r_M. If we require $2ar_M > \sigma_2^2$, then we have $r_t^C > 0$ for all $t \geq 0$. This property is an advantage of the CIR model over the Vasicek interest rate. In the text below, we use the universal notation r_t for the interest rate if it makes no difference which model is considered.
We denote the ratio of investor's wealth invested in the risky asset by k_t, while c_t is the ratio of wealth consumed at time t. Below, we separate the problem into a pre- and a post-crash problem. Thus, we denote the pre-crash strategy, valid for $t \leq \tau$, by $(\underline{k}_t, \underline{c}_t)$, and the post-crash strategy, valid for $t > \tau$, by $(\overline{k}_t, \overline{c}_t)$.
Now, the investor's wealth at time $t \geq 0$ is denoted by X_t and it is defined by the following stochastic differential equations:

$$\begin{aligned} X_0 &= x^0 > 0, \\ dX_t &= X_t \left[r_c + (\mu - r_c)\underline{k}_t - \underline{c}_t\right] dt + X_t \sigma_1 \underline{k}_t \, dw_{1,t}, \qquad \text{on } [0, \tau), \\ X_\tau &= (1 - l\underline{k}_\tau) X_{\tau_-}, \\ dX_t &= X_t \left[\overline{r}_t + (\mu - \overline{r}_t)\overline{k}_t - \overline{c}_t\right] dt + X_t \sigma_1 \overline{k}_t \, dw_{1,t}, \qquad \text{on } (\tau, \infty], \end{aligned}$$

where, as mentioned above, we can write the post-crash interest rate for model (1), denoted by $\overline{r}_t$, in the form

$$\begin{aligned} d\overline{r}_t &= a(r_M - \overline{r}_t) + \sigma_2(\rho \, dw_{1,t} + \sqrt{1 - \rho^2} \, dw_{2,t}), \qquad \text{on } (\tau, \infty], \\ \overline{r}_\tau &= r_c. \end{aligned} \tag{3}$$

If we consider (2), we find that:

$$\begin{aligned} d\overline{r}_t &= a(r_M - \overline{r}_t) + \sigma_2 \sqrt{\overline{r}_t}(\rho \, dw_{1,t} + \sqrt{1 - \rho^2} \, dw_{2,t}), \qquad \text{on } (\tau, \infty], \\ \overline{r}_\tau &= r_c. \end{aligned} \tag{4}$$

Given these assumptions, the investor aims to maximize the expected discounted logarithmic utility of consumption over an infinite time horizon in the worst-case crash scenario. Thus, we formulate the following worst-case optimization problem:

$$\sup_{(k,c) \in \Pi} \inf_{(\tau,l) \in \mathcal{C}} \mathbb{E} \left(\int_0^\infty e^{-\varepsilon t} \ln(c_t X_t) \, dt \right), \tag{5}$$

where $\varepsilon > 0$ denotes the discount factor and Π is the admissible control space defined below.

Definition 2.1 (Admissible control space Π) *An investment and consumption portfolio $(k, c) := (\underline{k}, \underline{c}, \overline{k}, \overline{c})$ belongs to the admissible control space Π, if the following conditions hold:*

1. $(\underline{k}_t, \underline{c}_t)$ *and* $(\overline{k}_t, \overline{c}_t)$ *are* $\mathcal{F}_t$*-adapted for all* $t \geq 0$*,*

2. $\mathbb{E}\left(\int_0^t k_s^2 ds\right) < \infty, \quad \forall\, t \geq 0$*,*

3. $0 \leq \overline{c}_t \leq C < \infty$ *for all* $t \geq 0$*, where* $C > 0$ *is a sufficiently large constant,*

4. $\lim_{T\to\infty} e^{-\varepsilon T}\mathbb{E}\int_0^T \overline{k}_t^2\, dt = 0$*,*

5. $\underline{k}_t < \frac{1}{l^*}$ *for all* $t \geq 0$ *and* $\underline{k}$ *is right continuous.*

Remark 2.1 *Condition 2 in Definition 2.1 has to be fulfilled for both the pre-crash strategy* $(\underline{k}, \underline{c})$ *and the post-crash strategy* $(\overline{k}, \overline{c})$*, respectively. Conditions 3 and 4 are assumed in order to apply a verification theorem when identifying the optimal post-crash strategy (see Section 3.1 below). Note that the admissible control space contains strategies* k *with values in* $(-\infty, \infty)$*. Negative values of* k *correspond to short-selling. Condition 5 ensures that the wealth at the crash time* τ *stays positive.*

The aim of the next section is to determine the optimal worst-case strategy (k^*, c^*) for problem (5). It turns out that we can apply the same main steps as in Desmettre et al. (2013) to solve the worst-case optimization problem under a stochastic interest rate.

3. THE SOLUTION BY A MARTINGALE APPROACH

First, in Section 3.1 we can find an optimal post-crash strategy $(\overline{k}^*, \overline{c}^*)$ by solving a classical stochastic optimal control problem. Using the special structure of the resulting post-crash value function, we can reformulate problem (5) into a pre-crash problem. This will be done in Section 3.2. Finally, in Section 3.3 we identify the optimal pre-crash strategy $(\underline{k}^*, \underline{c}^*)$ by solving a constrained stochastic optimal control problem.

3.1. The optimal post-crash strategy

In this section we consider the optimization problem that the investor faces at the crash time τ. In fact, the investor is faced with a classical stochastic optimal control problem over an infinite time horizon because, at the crash time, he knows that no further crash can occur. Equipped with a wealth x and an observed interest rate r at the crash time, the investor has to maximize the expected discounted utility of consumption. Because the interest rate after the crash is stochastic, we have to consider a two-dimensional state process $(\overline{X}_t, \overline{r}_t)$. Let us define the *post-crash value function*:

$$\overline{V}(x, r) \quad = \sup_{(\overline{k},\overline{c})\in\Pi} \mathbb{E}^{x,r}\left(\int_0^\infty e^{-\varepsilon t}\ln(\overline{c}_t \overline{X}_t)\, dt\right) \tag{6}$$

with respect to the post-crash dynamics:

$$d\overline{X}_t = \overline{X}_t\left[\overline{r}_t + (\mu - \overline{r}_t)\overline{k}_t - \overline{c}_t\right]\, dt + \overline{X}_t \sigma_1 \overline{k}_t\, dw_{1,t}, \qquad \overline{X}_0 = x, \tag{7}$$

where the post-crash interest rate $\overline{r}_t$ in the Vasicek and the CIR model is given by (3) and (4), respectively.

Remark 3.1 *The post-crash value function $\overline{V}(x,r)$ depends on the initial values of the post-crash dynamics, given by arbitrary $x \in \mathbb{R}_+$ and $r \in \mathbb{R}$, that will represent the wealth and the interest rate at the crash time, respectively. Note that the starting point 0 takes the role of the crash time τ.*

Vasicek model. We can use the result in (Pang 2006, Chp.5) to obtain the optimal post-crash strategy for (6) with the post-crash interest rate of Vasicek-type (see (3)). Pang solved this infinite horizon stochastic control problem by Dynamic Programming Principle. Thus, we obtain the optimal post-crash strategy

$$\overline{k}_t^* = \overline{k}^*(\overline{r}_t) = \frac{\mu - \overline{r}_t}{\sigma_1^2}, \qquad \overline{c}_t^* \equiv \varepsilon$$

and an explicit form of the post-crash value function:

$$\overline{V}(x,r) = \frac{1}{\varepsilon}\ln(x) + f(r), \qquad f(r) = \alpha_2 r^2 + \alpha_1 r + \alpha_0 \tag{8}$$

where α_i $(i = 1,2,3)$ are given by

$$\begin{aligned}
\alpha_2 &= \frac{1}{2\varepsilon\sigma_1^2(\varepsilon + 2a)},\\
\alpha_1 &= \frac{1}{\varepsilon(\varepsilon + a)}\left[\frac{ar_M + (\varepsilon + 2a)(\sigma_1^2 - \mu)}{\sigma_1^2(\varepsilon + 2a)}\right],\\
\alpha_0 &= \frac{1}{\varepsilon}\left[\frac{\sigma_2^2}{2\sigma_1^2\varepsilon(\varepsilon + 2a)} + \frac{ar_M}{\varepsilon(\varepsilon + a)}\left[\frac{ar_M + (\varepsilon + 2a)(\sigma_1^2 - \mu)}{\sigma_1^2(\varepsilon + 2a)}\right] + \frac{\mu^2}{2\sigma_1^2\varepsilon} + \ln(\varepsilon) - 1\right].
\end{aligned}$$

Moreover, by reducing the Hamilton-Jacobi-Bellman (HJB) equation, we know that $f \in C^2(\mathbb{R})$ solves the differential equation

$$\frac{\sigma_2^2}{2} f_{rr} + a(r_M - r) f_r - \varepsilon f + \ln(\varepsilon) - 1 + \frac{1}{\varepsilon}\left[\frac{(\mu - r)^2}{2\sigma_1^2} + r\right] = 0, \quad \forall\, r \in \mathbb{R}. \tag{9}$$

In order to prove that the solution of the HJB equation $\overline{V}(x,r)$ is indeed equal to the post-crash value function, Pang also required conditions 1-4 in Definition 2.1. Hence, we also included these requirements. The same requirements are needed for the solution of problem (6) under the post-crash interest of CIR-type (see (4)).

CIR model. In contrast to the Vasicek model, as far as we know, no previous work exists that solves problem (6). Here, we can also determine the optimal post-crash strategy by applying the Dynamic Programming Principle. In this case, the HJB equation for the value function $\overline{V}(x,r)$ is given by

$$\begin{aligned}
\varepsilon\overline{V} &= \sup_{\overline{k}}\left[(\mu - r)\overline{k}x\overline{V}_x + \frac{1}{2}\sigma_1^2\overline{k}^2 x^2 \overline{V}_{xx} + \rho\sigma_1\sigma_2\overline{k}\sqrt{r}x\overline{V}_{xr}\right] + rx\overline{V}_x\\
&\quad + a(r_M - r)\overline{V}_r + \frac{1}{2}\sigma_2^2 r\overline{V}_{rr} + \sup_{\overline{c}\geq 0}\left[-\overline{c}x\overline{V}_x + \ln(\overline{c}x)\right].
\end{aligned}$$

Using the standard approach $\overline{V}(x,r) = A\ln(x) + g(r)$ with $A = \frac{1}{\varepsilon}$ and $g \in C^2(\mathbb{R})$, we can reduce the HJB equation to

$$\varepsilon g = \frac{1}{\varepsilon}\sup_{\overline{k}\in\Pi}\left[(\mu - r)\overline{k} - \frac{1}{2}\sigma_1^2\overline{k}^2\right] + \frac{r}{\varepsilon} + a(r_M - r)g_r + \frac{1}{2}\sigma_2^2 r g_{rr} + \sup_{\overline{c}\in\Pi}\left[-\frac{\overline{c}}{\varepsilon} + \ln(\overline{c})\right].$$

The optimal post-crash strategy is then given by

$$\overline{k}_t^* = \overline{k}^*(\overline{r}_t) = \frac{\mu - \overline{r}_t}{\sigma_1^2}, \qquad \overline{c}^* = \varepsilon, \tag{10}$$

where $\overline{r}_t$ is given by (4). We verify this result in the verification theorem below. Inserting these optimal candidates, we obtain the differential equation for $g \in C^2(\mathbb{R})$

$$\frac{\sigma_2^2}{2} r\, g_{rr} + a(r_M - r)g_r - \varepsilon g + \ln(\varepsilon) - 1 + \frac{1}{\varepsilon}\left[\frac{(\mu - r)^2}{2\sigma_1^2} + r\right] = 0, \quad \forall\, r \in \mathbb{R}. \tag{11}$$

In contrast to equation (9), the coefficient of g_{rr} is linear in r. Nevertheless, since the last term is quadratic in r, we suppose that $g(r) = \beta_2 r^2 + \beta_1 r + \beta_0$. Comparing the coefficients, we obtain

$$\begin{aligned}
\beta_2 &= \frac{1}{2\varepsilon\sigma_1^2(\varepsilon + 2a)},\\
\beta_1 &= \frac{1}{\varepsilon(\varepsilon + a)}\left[\frac{ar_M + (\varepsilon + 2a)(\sigma_1^2 - \mu) + \frac{\sigma_2^2}{2}}{\sigma_1^2(\varepsilon + 2a)}\right],\\
\beta_0 &= \frac{1}{\varepsilon}\left[\frac{ar_M}{\varepsilon(\varepsilon + a)}\left[\frac{ar_M + (\varepsilon + 2a)(\sigma_1^2 - \mu) + \frac{\sigma_2^2}{2}}{\sigma_1^2(\varepsilon + 2a)}\right] + \frac{\mu^2}{2\sigma_1^2\varepsilon} + \ln(\varepsilon) - 1\right].
\end{aligned}$$

In order to show that the candidates in (10) are in fact optimal for the stochastic control problem (6), we can prove the following verification theorem.

Theorem 3.1 (Verification theorem) *Suppose $g(r) = \beta_2 r^2 + \beta_1 r + \beta_0$ is a classical solution of* (11) *and define*

$$\tilde{V}(x,r) := \frac{1}{\varepsilon}\ln(x) + g(r). \tag{12}$$

If

$$\overline{k}^*(\overline{r}_t) = \frac{\mu - \overline{r}_t}{\sigma_1^2}, \qquad \overline{c}^*(\overline{r}_t) \equiv \varepsilon,$$

where $\overline{r}_t$ is given by (4)*, then $(\overline{k}^*, \overline{c}^*) \in \Pi$ and*

$$\tilde{V}(x,r) = \mathbb{E}^{x,r}\left(\int_0^\infty e^{-\varepsilon t}\ln(\overline{c}_t^*\overline{X}_t^*)\,dt\right),$$

where $\overline{X}_t^*$ denotes the process that solves (7) *corresponding to (k^*, c^*). That means, $\tilde{V}(x,r) = \overline{V}(x,r)$, where $\overline{V}(x,r)$ is defined by* (6) *under the CIR interest rate.*

Proof. We prove the result by rather standard arguments. By the definition of $\tilde{V}$ and by applying Ito's formula, we obtain for arbitrary $(\overline{k}, \overline{c}) \in \Pi$ that $\tilde{V}(x,r) \geq \mathbb{E}^{x,r}\left(\int_0^\infty e^{-\varepsilon t}\ln(\overline{c}_t \overline{X}_t)\,dt\right)$. Afterwards, we get $(\overline{k}^*, \overline{c}^*) \in \Pi$, and by the above calculation we have

$$\begin{aligned}\overline{k}^* &\in \arg\max_{\overline{k}}\left[(\mu - r)\overline{k}x\tilde{V}_x + \frac{1}{2}\sigma_1^2\overline{k}^2 x^2\tilde{V}_{xx} + \rho\sigma_1\sigma_2\overline{k}x\sqrt{r}\tilde{V}_{xr}\right],\\ \overline{c}^* &\in \arg\max_{\overline{c}\geq 0}\left[-\overline{c}x\tilde{V}_x + \ln(\overline{c}x)\right].\end{aligned}$$

Using Ito's formula and the explicit form of the first and second moment of $\overline{r}_t$, we are able to show that

$$\tilde{V}(x,r) \leq \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\ln(\overline{c}_t^*\overline{X}_t^*)\,dt\right) = \overline{V}(x,r).$$

Thus, the assertion holds. ■

Remark 3.2 *Due to the stochastic interest rate $\overline{r}_t$ after the market crash, the optimal post-crash strategy is a feedback control depending on the stochastic interest rate $\overline{r}_t$, given by a Vasicek process and a CIR process, respectively.*

At the crash time, the investor has an amount of wealth of $x = (1 - l\underline{k}_\tau)X_\tau$ and the interest rate at the crash time is $r = r_\tau$. These values are the initial values of the post-crash problem and can be inserted in the post-crash value function $\overline{V}(x,r)$. From now on, we write for the post-crash value function:

$$\overline{V}(x,r) = \frac{1}{\varepsilon}\ln(x) + W(r),$$

where $W(r)$ stands for $f(r)$ in the Vasicek case and for $g(r)$ in the CIR case. Thus, we can reformulate the worst-case optimization problem into a pre-crash problem.

3.2. Reformulation of the worst-case optimization problem

From the post-crash analysis in the previous section we know that the performance of the optimal post-crash strategy at time τ is given by the post-crash value function at $x = (1 - l\underline{k}_\tau)X_\tau$ and $r = r_\tau$, namely $\overline{V}((1 - l\underline{k}_\tau)X_\tau, r_\tau)$. Since $\overline{V}(x,r)$, given by (8) and (12), is monotone increasing in x, we obtain

$$\overline{V}((1 - l\underline{k}_\tau)X_\tau, r_\tau) \geq \overline{V}((1 - l^*\underline{k}_\tau^+)X_\tau, r_\tau),$$

where $k^+ := \max\{0, k\}$. Thus, we can conclude that the worst-case crash size is realized for $l = l^*$. Because we assumed a constant interest rate r_c before and including the crash time, we have $r_\tau = r_c$. Now, we discount $\overline{V}((1 - l^*\underline{k}_\tau^+)X_\tau, r_c)$ to the starting time 0 by $e^{-\varepsilon\tau}$ and we reformulate the worst-case problem (5) into the following pre-crash problem:

$$\sup_{(\underline{k},\underline{c})\in\Pi}\inf_{\tau\in\mathcal{C}}\mathbb{E}\left(\int_0^\tau e^{-\varepsilon t}\ln(\underline{c}_t X_t)\,dt + e^{-\varepsilon\tau}\overline{V}((1 - l^*\underline{k}_\tau^+)X_\tau, r_c)\right) \tag{13}$$

with respect to the pre-crash dynamics

$$dX_t = X_t\left[r_c + (\mu - r_c)\underline{k}_t - \underline{c}_t\right]dt + X_t\sigma_1\underline{k}_t\,dw_{1,t}, \qquad X_0 = x^0 > 0.$$

Note that the pre-crash problem is considered with respect to the pre-crash dynamics. Because of constant interest rates before the crash, we have to consider only the state equation for the pre-crash wealth. In the pre-crash problem (13) the infimum is only taken over the crash time τ, because we already identified the worst-case crash size by l^*. From now on, we write (k, c) instead of $(\underline{k}, \underline{c})$ for the pre-crash strategy and therefore, by (13), the worst-case problem (5) reduces to a controller vs. stopper game of the form

$$\sup_{(k,c)\in\Pi} \inf_{\tau} \mathbb{E}\left(M_\tau^{k,c}\right), \tag{14}$$

where

$$M_t^{k,c} := \int_0^t e^{-\varepsilon s} \ln(c_s X_s)\, ds + e^{-\varepsilon t}\overline{V}((1 - l^* k_t^+)X_t, r_c), \qquad t \geq 0.$$

Such a controller vs. stopper game is also explained in Seifried (2010). Here, we also try to solve this kind of a stochastic game, where the investor controls $M^{k,c}$ by choosing (k, c) and the stopper, namely the market, decides on the duration of the game $\tau \in \mathcal{C}$. In the text below, we see that we can apply the concept of *Indifference* and *Indifference Optimality Principle*, developed in Seifried (2010) and Desmettre et al. (2013) to identify the optimal pre-crash strategy for the stochastic game (14). Analogously, we define an indifference strategy as follows.

Definition 3.1 (Indifference Strategy, cf. (Seifried 2010, p.566)) *A pre-crash strategy $(\hat{k}, \hat{c})$ is called indifference strategy if*

$$\mathbb{E}(M_{\tau_1}^{\hat{k},\hat{c}}) = \mathbb{E}(M_{\tau_2}^{\hat{k},\hat{c}}) \tag{15}$$

for stopping times $\tau_1 \neq \tau_2$.

The idea here is that the investor chooses an indifference strategy before the crash, such that the performance of this choice does not depend on the crash time τ. After formulating a sufficient condition for a strategy to be an indifference strategy, we can use the notion of an *Indifference Frontier* and an *Indifference Optimality Principle* to identify the worst-case optimal pre-crash strategy.

Proposition 3.2 (Indifference Condition) *Let $(\hat{k}, \hat{c})$ be a constant pre-crash strategy such that $H(\hat{k}, \hat{c}) = 0$, where*

$$H(k, c) := \ln(c) - \ln(1 - l^* k^+) + \frac{1}{\varepsilon}[r_c + (\mu - r_c)k - c] - \frac{\sigma_1^2}{2\varepsilon}k^2 - \varepsilon W(r_c). \tag{16}$$

Then $(\hat{k}, \hat{c}) \in \Pi$ is an indifference strategy, which means $\mathbb{E}(M_{\tau_1}^{\hat{k},\hat{c}}) = \mathbb{E}(M_{\tau_2}^{\hat{k},\hat{c}})$.

Proof. The proof is similar to that in Desmettre et al. (2013) and it is divided into two steps. First, we show that $M^{\hat{k},\hat{c}}$ is a uniformly integrable martingale. In the second step we apply Doob's Optional Sampling theorem and the assertion follows.
By the definition of $M^{k,c}$ and for arbitrary (k, c), we have

$$dM_t^{k,c} = e^{-\varepsilon t} \ln(c_t X_t)\, dt + d\left[e^{-\varepsilon t}\overline{V}((1 - l^* k_t^+)X_t, r_c)\right].$$

Now, we restrict ourselves to constant pre-crash strategies. With $\overline{V}(x,r) = \frac{1}{\varepsilon}\ln(x) + W(r)$ (see section 3.1) and by applying Ito's formula, we obtain

$$\begin{aligned} dM_t^{k,c} &= e^{-\varepsilon t}\left\{ \ln(c) - \ln(1-l^* k^+) + \frac{1}{\varepsilon}[r_c + (\mu - r_c)k - c] - \frac{\sigma_1^2}{2\varepsilon}k^2 - \varepsilon W(r_c)\right\} dt \\ &\quad + e^{-\varepsilon t}\frac{1}{\varepsilon}\sigma_1 k\, dw_{1,t}. \end{aligned} \tag{17}$$

Let $(\hat{k},\hat{c})$ be a constant pre-crash strategy such that $H(\hat{k},\hat{c}) = 0$, then

$$dM_t^{\hat{k},\hat{c}} = e^{-\varepsilon t}\frac{1}{\varepsilon}\sigma_1 \hat{k}\, dw_{1,t}.$$

Since $\hat{k}$ is assumed to be constant, it is easy to check that $M_t^{\hat{k},\hat{c}} = \int_0^t e^{-\varepsilon s}\frac{1}{\varepsilon}\sigma_1 \hat{k}\, dw_{1,s}$ is a uniformly integrable martingale. By (Protter 1990, Thm. 12), we find that the uniformly integrable martingale $M^{\hat{k},\hat{c}}$ is closed by the random variable $M_\infty^{\hat{k},\hat{c}} := \lim_{t\to\infty} M_t^{\hat{k},\hat{c}}$. Then, by applying Doob's Optional Sampling Theorem (for example, see (Protter 1990, Thm.16)), we obtain (15). By Definition 3.1, it follows that $(\hat{k},\hat{c})$ is an indifference strategy. Finally, it follows that a constant $(\hat{k},\hat{c})$ with $\hat{c} \geq 0$ is an admissible strategy. ■

Remark 3.3 *In our model it is essential that the interest rate before the crash is constant. This leads to the fact that the dt-coefficient in* (17) *does not depend on $\omega \in \Omega$. Moreover, due to the infinite time horizon, the indifference strategy does not depend on time t. By these arguments it is sufficient to consider constant pre-crash strategies.*

Now, having a sufficient condition for a pre-crash strategy to be an indifference strategy, we can apply the *Indifference Optimality Principle* stated in Seifried (2010) and Desmettre et al. (2013). Let $(\hat{k},\hat{c})$ be an indifference strategy and $(k,c) \in \Pi$ be an arbitrary admissible pre-crash strategy. Then, by (Desmettre et al. 2013, Lemma 4.3), we can improve the worst-case performance for (k,c) by cutting off at the strategy $(\hat{k},\hat{c})$. In detail, they showed that

$$\inf_\tau \mathbb{E}(M_\tau^{\tilde{k},\tilde{c}}) \geq \inf_\tau \mathbb{E}(M_\tau^{k,c}), \tag{18}$$

where

$$\tilde{k}_t = \begin{cases} k_t & : t < \eta \\ \hat{k} & : t \geq \eta \end{cases} \qquad \tilde{c}_t = \begin{cases} c_t & : t < \eta \\ \hat{c} & : t \geq \eta \end{cases},$$

and $\eta := \inf\{t \geq 0 : k_t > \hat{k}\}$. The application of (Desmettre et al. 2013, Lemma 4.3) is possible for our model, because of the required right continuity of the pre-crash strategy k_t and the fact that the post-crash value function $\overline{V}(x,r)$ is also monotone increasing in x for the log-utility function. Thus, for details of the proof of (18) we refer to the literature.

By (18), we can restrict our considerations on strategies that are dominated by an indifference strategy because all other strategies would provide worse performances. In order to do this, we abbreviate the set of such strategies by

$$\mathcal{A}(\hat{k}) := \left\{(k,c) \in \Pi : k_t \leq \hat{k}, \quad \forall t \geq 0\right\}.$$

Next, we apply the *Indifference Optimality Principle* (see (Desmettre et al. 2013, Proposition 5.1)) to identify a worst-case optimal pre-crash strategy for our model with a stochastic interest rate. This principle provides us a sufficient condition for a pre-crash strategy (k^*, c^*) to be optimal in the worst-case scenario: An indifference strategy $(\hat{k}, \hat{c}) = (k^*, c^*)$ is the worst-case optimal investment consumption strategy for (5), if it is optimal in the no-crash scenario $\{\tau = \infty\}$ in the class of all strategies respecting the associated indifference frontier. Thus, our task in the next Section is to identify an indifference strategy (k^*, c^*), which fulfills the condition:

$$\mathbb{E}(M_\infty^{k,c}) \le \mathbb{E}(M_\infty^{k^*,c^*}) \qquad \forall\, (k,c) \in \mathcal{A}(k^*). \tag{M}$$

3.3. Identification of the optimal pre-crash strategy

The aim of this Section is to identify the optimal pre-crash strategy by applying the Indifference Optimality Principle. We are done if we can find $(k^*, c^*) \in \Pi$ which fulfills the Indifference Condition of Proposition 3.2 and the Indifference Optimality Condition (M).

Theorem 3.3 *Let $k^* < m := \min\left\{\frac{1}{l^*}, \frac{\mu - r_c}{\sigma_1^2}\right\}$ be the unique root of the function $H(k,\varepsilon)$, where H is defined in* (16)*. Then, the optimal pre-crash strategy for the worst-case problem* (5) *is given by*

$$k_t^* \equiv k^*, \qquad c_t^* \equiv \varepsilon.$$

Proof. We divide the proof into two steps. First, we show that there exists a uniquely determined k^*, such that $H(k^*,\varepsilon) = 0$. In a second step we show that (k^*,ε) is optimal in the no-crash scenario in the class of strategies that respect the Indifference Frontier (k^*,ε). Finally, we can conclude that (k^*,ε) is the optimal pre-crash strategy.

Step 1.
By definition of H in (16), we have

$$\frac{\partial}{\partial k} H(k,\varepsilon) = \begin{cases} -\frac{1}{(1-l^*k^+)} \cdot (-l^*) + \frac{1}{\varepsilon}\left((\mu - r_c) - \sigma_1^2 k\right) & : 0 \le k < m \\ \frac{1}{\varepsilon}\left((\mu - r_c) - \sigma_1^2 k\right) & : k < 0 \end{cases}.$$

It follows that $H(k,\varepsilon)$ is strictly monotone increasing for $k < m$. If $m = \frac{1}{l^*}$, it holds

$$\lim_{k \nearrow 1/l^*} H(k,\varepsilon) = \ln(\varepsilon) - \lim_{k \nearrow 1/l^*} \ln(1 - l^*k^+) + \frac{1}{\varepsilon}[r_c + (\mu - r_c)\frac{1}{l^*} - \varepsilon] - \frac{\sigma_1^2}{2\varepsilon}\frac{1}{l^{*2}} - \varepsilon W(r_c) = +\infty$$

and if $m = \frac{\mu - r_c}{\sigma_1^2}$ we obtain

$$H\left(\frac{\mu - r_c}{\sigma_1^2}, \varepsilon\right) = \ln(\varepsilon) - \ln\left(1 - l^* \cdot \frac{\mu - r_c}{\sigma_1^2}\right) + \frac{r_c}{\varepsilon} - 1 + \frac{1}{2\varepsilon}\frac{(\mu - r_c)^2}{\sigma_1^2} - \varepsilon W(r_c) > 0$$

because $W(r_c)$ is given in Section 3.1 and we can choose $\varepsilon > 0$ large enough such that the last inequality holds.

Moreover, we get

$$\lim_{k\to-\infty} H(k,\varepsilon)$$
$$= \ln(\varepsilon) - \lim_{k\to-\infty} \ln\left(1 - l^* k^+\right) + \frac{r_c}{\varepsilon} + \frac{1}{\varepsilon}\lim_{k\to-\infty}\left((\mu - r_c)k - \frac{\sigma_1^2}{2}k^2\right) - 1 - \varepsilon W(r_c) = -\infty.$$

Because of the fact that $H(k,\varepsilon)$ is strictly monotone increasing for $k < m$, $H(m,\varepsilon) > 0$ and $\lim_{k\to-\infty} H(k,\varepsilon) < 0$, we know that there exists a uniquely determined k^* such that $H(k^*,\varepsilon) = 0$. Thus (k^*,ε) is an indifference strategy, and the first step of the proof is complete. Now, it remains to show the second step.

Step 2.
Here, we consider the constrained optimization problem

$$\sup_{(k,c):k\le k^*} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\ln(c_t X_t)\,dt\right)$$

with respect to the pre-crash dynamics

$$dX_t = X_t\left[r_c + (\mu - r_c)k_t - c_t\right]dt + X_t\sigma_1 k_t\,dw_{1,t}, \qquad X_0 = x^0 > 0. \tag{19}$$

We solve this problem by separating the investment and consumption decisions. This separation is possible due to the logarithmic utility function (see Korn and Seifried (2013) for more details). This separation leads, on the one hand, to an optimal consumption strategy $c^* = \varepsilon$, and on the other hand to the remaining problem for determining k^*

$$\sup_{k\le k^*} \mathbb{E}\left(\int_0^\infty e^{-\varepsilon t}\ln(\varepsilon X_t^\varepsilon)\,dt\right), \tag{20}$$

where X_t^ε denotes the solution of (19) controlled by $c_t = \varepsilon$. By Ito's formula, we have

$$\mathbb{E}\left(\ln(X_t^\varepsilon)\right) = \ln(x^0) + \mathbb{E}\int_0^t \phi(k_s)\,ds - \varepsilon t, \qquad \phi(k_s) := (\mu - r_c)k_s - \frac{\sigma_1^2}{2}k_s^2 + r_c.$$

Thus, the maximization of (20) is reduced to the maximization of ϕ over $k \le k^*$. Obviously, ϕ is strictly monotone increasing for $k < \frac{\mu - r_c}{\sigma_1^2}$. Since $k^* < m$, the maximum is attained for $k = k^*$. Thus, (k^*,ε) fulfills (M). Finally, the application the Infifference Optimality Principle proves the assertion. ■

4. CONCLUSIONS

The optimal investment and consumption behavior consists of determining a constant investment control $\underline{k}^*$ such that $H(\underline{k}^*,\varepsilon) = 0$ and $\underline{c}^* = \varepsilon$. It is important to note that determining $\underline{k}^*$ depends on the interest rate model being considered. In contrast to the pre-crash strategy, the optimal post-crash investment depends on the stochastic interest rate, whereas the optimal post-crash consumption strategy is again equal to the discount factor ε. The latter point is not surprising for the case of a logarithmic utility function because the investment and consumption decisions can be separated. This will certainly not be the case if we consider, for example, a power utility function. Thus, the case of more general utility functions is left for future research.

References

S. Desmettre, R. Korn, and F.T. Seifried. Worst-case consumption-portfolio optimization. Technical Report 227, Fraunhofer ITWM, Kaiserslautern, 2013. URL `http://publica.fraunhofer.de/documents/N-241439.html`.

W. H. Fleming and T. Pang. An application of stochastic control theory to financial economics. *SIAM Journal on Control and Optimization*, 43(2):502–531, 2004.

R. Korn and F.T. Seifried. A concise characterization of optimal consumption with logarithmic preferences. *International Journal of Theoretical and Applied Finance*, 16(06):1–7, 2013.

R. Korn and Paul Wilmott. Optimal portfolios under the threat of a crash. *International Journal of Theoretical and Applied Finance*, 5(2):171–187, 2002.

R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.

T. Pang. Stochastic portfolio optimization with log utility. *International Journal of Theoretical and Applied Finance*, 9(6):869–887, 2006.

P. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990.

F.T. Seifried. Optimal investment for worst-case crash scenarios: a martingale approach. *Mathematics of Operations Research*, 35(3):559–579, 2010.

FIRST-PASSAGE TIME PROBLEMS UNDER REGIME SWITCHING: APPLICATIONS IN FINANCE AND INSURANCE

Peter Hieber[†]

[†]*Lehrstuhl für Finanzmathematik, Technische Universität München, Parkring 11, 85748 Garching-Hochbrück, Germany*
Email: `hieber@tum.de`

Abstract

One common approach in stochastic modeling of (rare) events (for example company defaults or natural catastrophes) is to assume that those events are triggered as soon as a stochastic risk process falls below a constant barrier. In this paper, we model this risk process as a regime switching Brownian motion, a process that allows to incorporate many of the cyclical patterns that either naturally arise (i.e. seasons, rainfall) or result from peoples' interactions (i.e. business cycles). Two numerical examples demonstrate the flexibility of a regime switching risk process: Stress testing option prices on exchange rates and the pricing of CAT bonds.

1. INTRODUCTION

Recently, first-passage time problems for regime switching models have earned considerable attention. One reason for this increased interest is that first-passage time problems occur in many different areas of science, for example:

- In Biology, they can be used as a tool in population modeling to, for example, estimate extinction probabilities for species. Food-supply is usually regime-dependent.
- In Insurance, companies have to provide enough capital to pay any claim that results out of their insurance policies. The claim arrival intensity is often regime-dependent; it might – for example – be coupled to weather (sunny, rain) or season.
- In Finance, economic cycles (boom, recession) influence many financial time series. Regime switching models reflect the tendency of financial markets to often change their behavior abruptly and persistently. First-passage time problems have to be solved if one wants to price certain exotic options, i.e. American, barrier, lookback, or digital options (see, e.g., Guo (2001b), Buffington and Elliott (2002), Elliott et al. (2005), Jobert and Rogers (2006), Boyle and Draviam (2007), Jiang and Pistorius (2008), Hieber and Scherer (2010), and many others).

Another reason for the popularity of regime switching models is the fact that – conceptionally – they are rather simple (conditional on the regimes, the distribution is normal) and thus analytically tractable. Nevertheless, they can generate many non-linear effects like heavy tails or volatility clusters.

In this paper, we want to focus on applications of first-passage time problems under regime switching. In two numerical examples on the pricing of first-touch options and of CAT bonds, we demonstrate that this rather simple (but very tractable) model has several nice features that cannot be captured by a Lévy process.

The paper is organized as follows: In Section 2, we introduce notation and regime switching Brownian motion. Section 3 recalls some of the results on the first-passage time of regime switching models (see also Guo (2001a), Jiang and Pistorius (2008), Hieber (2013a)). Then – the main part of this paper – is the application of those results to Finance and Insurance (see Section 4).

2. MODEL DESCRIPTION

On the filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$, we consider the process $B = \{B_t\}_{t\geq 0}$ described by the stochastic differential equation (sde)

$$dB_t = \mu_{Z_t} dt + \sigma_{Z_t} dW_t, \quad B_0 = x, \tag{1}$$

where $\mu_{Z_t} \in \mathbb{R}$, $\sigma_{Z_t} > 0$, $Z = \{Z_t\}_{t\geq 0} \in \{1, 2, \ldots, M\}$ is a time-homogeneous Markov chain with intensity matrix[1] Q_0, and $W = \{W_t\}_{t\geq 0}$ an independent Brownian motion. The initial value is $B_0 = x \in \mathbb{R}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by the pair (W, Z), i.e. $\mathcal{F}_t = \sigma\{W_s, Z_s : 0 \leq s \leq t\}$. The time to a state change from the current state i is an exponential random variable with intensity parameter $Q_0(i,i)$. The probability of moving to state $j \neq i$ is $-Q_0(i,j)/Q_0(i,i)$. The model is fully determined if an initial state (or, more generally, an initial distribution $\pi_0 := \big(\mathbb{P}(Z_0 = 1), \mathbb{P}(Z_0 = 2), \ldots, \mathbb{P}(Z_0 = M)\big)$ on the states) is defined.

The first-passage time on a lower barrier $b < B_0 = x$ is defined as

$$T_b := \begin{cases} \inf\{t \geq 0 \,:\, B_t \leq b\}\,, & \text{if such a } t \text{ exists,} \\ \infty, & \text{else.} \end{cases} \tag{2}$$

The one-sided Fourier transform of the first-passage time is given by

$$\Psi_b(u) := \frac{1}{\pi} \int_0^\infty \exp\big(iuT_b\big)\, dT_b\,. \tag{3}$$

3. FIRST-PASSAGE TIME RESULTS

This section recalls some of the results on the first-passage time of regime switching models. For proofs and a more detailed review, we refer to Hieber (2013a). First, Theorem 3.1 presents the Fourier transform of the first-passage time contingent on the solution of a matrix equation.

[1] An intensity matrix has negative diagonal and non-negative off-diagonal entries. Each row sums up to zero.

Theorem 3.1 (First-passage times) *We denote the class of irreducible $M \times M$ generator matrices (non-negative off-diagonal entries and non-positive row sums) by $\mathcal{Q}_M$. The Fourier transform of the first-passage time on a lower barrier $b < B_0 = x$ is given by*

$$\Psi_b(u,x) = \pi_0 \, \exp\big(Q_-(x-b)\big)\, \mathbf{1}, \tag{4}$$

where $\exp(\,\cdot\,)$ *denotes the matrix exponential, $\pi_0 \in \mathbb{R}^{1\times M}$ is the initial distribution on the states, and $\mathbf{1}$ a vector of ones of appropriate size. The tuple (Q_+, Q_-), Q_+, $Q_- \in \mathcal{Q}_M$, is for $u > 0$ defined via the two unique solutions $\Xi(-Q_+) = \Xi(Q_-) = 0$ of*

$$\Xi(Q) := \begin{pmatrix} \sigma_1^2 & 0 & \dots \\ 0 & \dots & 0 \\ \dots & 0 & \sigma_M^2 \end{pmatrix} \frac{Q^2}{2} + \begin{pmatrix} \mu_1 & 0 & \dots \\ 0 & \dots & 0 \\ \dots & 0 & \mu_M \end{pmatrix} Q + Q_0 + \begin{pmatrix} iu & 0 & \dots \\ \dots & iu & 0 \\ \dots & 0 & iu \end{pmatrix}. \tag{5}$$

Proof. For a proof, we refer to Rogers (1994), Jiang and Pistorius (2008), and the references therein. ■

Similar results have been derived in the more general case of regime switching exponential jump-diffusion models and in the case of two barriers, see, e.g., Jiang and Pistorius (2008). The first-passage time problems then rest on solving matrix equations similar to Equation (5).

For 2 and 3 regimes and in the case where $\mu_{Z_t}/\sigma_{Z_t}^2$ is constant over time, closed-form solutions for Equation (5) are available, see, e.g., Hieber (2013a). Theorem 3.2 recalls the 2-regime case.

Theorem 3.2 (2-state model: Matrix Wiener–Hopf factorization) *Consider the regime switching model as defined in Equation (1) with $M = 2$ states and $q_{11}q_{22} \neq 0$. The tuple (Q_+, Q_-) that solves Equation (5) is given by*

$$Q_+ = \begin{pmatrix} \frac{-\beta_{3,u}\beta_{4,u} + \frac{2(q_{11}-u)}{\sigma_1^2}}{\beta_{3,u}+\beta_{4,u}+\frac{2\mu_1}{\sigma_1^2}} & \frac{-\frac{2q_{11}}{\sigma_1^2}}{\beta_{3,u}+\beta_{4,u}+\frac{2\mu_1}{\sigma_1^2}} \\ \frac{-\frac{2q_{22}}{\sigma_2^2}}{\beta_{3,u}+\beta_{4,u}+\frac{2\mu_2}{\sigma_2^2}} & \frac{-\beta_{3,u}\beta_{4,u} + \frac{2(q_{22}-u)}{\sigma_2^2}}{\beta_{3,u}+\beta_{4,u}+\frac{2\mu_2}{\sigma_2^2}} \end{pmatrix}, \quad Q_- = \begin{pmatrix} \frac{\beta_{1,u}\beta_{2,u} - \frac{2(q_{11}-u)}{\sigma_1^2}}{\beta_{1,u}+\beta_{2,u}+\frac{2\mu_1}{\sigma_1^2}} & \frac{\frac{2q_{11}}{\sigma_1^2}}{\beta_{1,u}+\beta_{2,u}+\frac{2\mu_1}{\sigma_1^2}} \\ \frac{\frac{2q_{22}}{\sigma_2^2}}{\beta_{1,u}+\beta_{2,u}+\frac{2\mu_2}{\sigma_2^2}} & \frac{\beta_{1,u}\beta_{2,u} - \frac{2(q_{22}-u)}{\sigma_2^2}}{\beta_{1,u}+\beta_{2,u}+\frac{2\mu_2}{\sigma_2^2}} \end{pmatrix},$$

where $q_{11} := Q_0(1,1)$, $q_{22} := Q_0(2,2)$, and $\beta_{1,u}$, $\beta_{2,u}$, $\beta_{3,u}$, and $\beta_{4,u}$ are the unique roots of the Cramér–Lundberg equation given by

$$\Big(\frac{1}{2}\sigma_1^2\beta^2 + \mu_1\beta + q_{11} + iu\Big)\Big(\frac{1}{2}\sigma_2^2\beta^2 + \mu_2\beta + q_{22} + iu\Big) - q_{11}q_{22} = 0\,, \tag{6}$$

where $-\infty < \Re(\beta_{1,u}) < \Re(\beta_{2,u}) < 0 < \Re(\beta_{3,u}) < \Re(\beta_{4,u}) < \infty$.

Proof. For a detailed proof, see Hieber (2013a). ■

The Fourier transform of the first-passage times is then a straightforward implication of Theorem 3.1. Exploiting the fact that the matrix exponential for 2×2 matrices can be derived explicitly, the one-sided first-passage time result can be further simplified, see Theorem 3.3.

Theorem 3.3 (2-state model: First-passage time) *Consider the regime switching model as defined in Equation (1) with $M = 2$ states and $q_{11}q_{22} \neq 0$. The Fourier transform of the first passage time on a lower barrier $b < B_0 = x$, is given by*

$$\Psi_b(u) = \frac{\beta_{1,u}e^{\beta_{2,u}(x-b)} - \beta_{2,u}e^{\beta_{1,u}(x-b)}}{\beta_{1,u} - \beta_{2,u}} + \frac{e^{\beta_{1,u}(x-b)} - e^{\beta_{2,u}(x-b)}}{\beta_{1,u} - \beta_{2,u}} \left(\overline{\pi} \frac{\beta_{1,u}\beta_{2,u} + \frac{2u}{\sigma_1^2}}{\beta_{1,u} + \beta_{2,u} + \frac{2\mu_1}{\sigma_1^2}} + (1 - \overline{\pi}) \frac{\beta_{1,u}\beta_{2,u} + \frac{2u}{\sigma_2^2}}{\beta_{1,u} + \beta_{2,u} + \frac{2\mu_2}{\sigma_2^2}} \right),$$

where $-\infty < \Re(\beta_{1,u}) < \Re(\beta_{2,u}) < 0 < \Re(\beta_{3,u}) < \Re(\beta_{4,u}) < \infty$ are the roots of Equation (6) and $\pi_0 = (\overline{\pi}, 1 - \overline{\pi}) := (\mathbb{P}(Z_0 = 1), \mathbb{P}(Z_0 = 2))$ is the initial distribution on the states.

Proof. See, e.g., Guo (2001a), Hieber (2013a), Hieber (2013b). ■

Similarly, one can treat $M = 3$ regimes and the case of two barriers. In both cases there are still closed-form expressions for the Fourier transform of the first-passage time, see Hieber (2013a). For more than three states one has to rely on numerical schemes, see Rogers and Shi (1994) for a comparison of several approaches.

However, as discussed above, more than three states translate into a large number of parameters. That is why, sometimes a reduction in the number of parameters might be useful. In the following, we assume that the quotient $\mu_{Z_t}/\sigma^2_{Z_t}$ is constant over all states (see, e.g., Eloe et al. (2009)). One can then easily solve Equation (5), see Theorem 3.4.

Theorem 3.4 ($\mu_{Z_t}/\sigma^2_{Z_t}$ constant: Matrix Wiener–Hopf factorization) *Consider the regime switching model (1) with $\mu_1/\sigma_1^2 = \mu_2/\sigma_2^2 = \ldots = \mu_M/\sigma_M^2 =: c \in \mathbb{R}$. Equation (5) can then be solved to*

$$Q_\pm = \begin{pmatrix} \pm c & \ldots & \ldots \\ 0 & \ldots & \ldots \\ \ldots & 0 & \pm c \end{pmatrix} - \sqrt{ \begin{pmatrix} c^2 & \ldots & \ldots \\ 0 & c^2 & \ldots \\ \ldots & 0 & c^2 \end{pmatrix} - \begin{pmatrix} \frac{2}{\sigma_1^2} & 0 & \ldots \\ 0 & \ldots & 0 \\ \ldots & 0 & \frac{2}{\sigma_M^2} \end{pmatrix} \left(Q_0 - \begin{pmatrix} u & 0 & \ldots \\ 0 & \ldots & 0 \\ \ldots & 0 & u \end{pmatrix} \right) }.$$

Proof. See Hieber (2013a). ■

In this special case, one can represent model (1) as a time-changed Brownian motion. In the case of two barriers this allows one to avoid Fourier inversion and to compute the first-passage time probability by a rapidly converging infinite series, see, e.g., Hieber and Scherer (2012), Hieber (2013a).

4. APPLICATIONS IN FINANCE AND INSURANCE

The main goal of this paper is to apply the theoretical results on the first-passage time probabilities to demonstrate the flexibility of regime switching models. First, we deal with the pricing of CAT bonds whose payoff depends on whether a risk index (for example coupled to temperatures or rainfall) falls below a threshold. We observe that a state change has a similar effect as a jump in the risk index: It leads to a sudden jump in the first-passage time probabilities. Secondly, we use the first-passage time results to price first-touch options. Here, we exploit another advantage of regime switching models: The inclusion of additional (probably distressed) regimes allows us to stress test our option prices.

For an implementation of the first-passage time probabilities, the Fourier transforms in Theorem 3.1 have to be inverted to yield the first-passage time probability $\mathbb{P}(T_b \leq T)$. One possibility to implement this inversion is to numerically evaluate

$$\mathbb{P}\big(T_b \leq T\big) = \frac{\Psi_b(0)}{2} - \frac{1}{\pi}\int_0^\infty e^{-iuT}\,\frac{\Psi_b(u)}{iu}\,du\,. \tag{7}$$

4.1. PRICING CAT BONDS

Pricing CAT bonds is similar to pricing corporate bonds; the difference being that instead of default risk, we now deal with insurance risk (see, e.g., Vaugirard (2003)). The bondholders accept to lose part of their investment if a risk index I_t – for example coupled to temperatures or the accumulated rainfall – falls below a constant threshold b. More specifically – at a fixed maturity T – the CAT bondholders receive the face value F if this threshold is not hit, and $(1-R)F$, for $0 \leq R \leq 1$, otherwise. The risk index is modeled as a regime switching Brownian motion, i.e. – under the risk-neutral measure $\mathbb{Q}$

$$dI_t = \kappa_{Z_t} dt + \sigma_{Z_t} dW_t, \quad I_0 = x\,, \tag{8}$$

where $\kappa_{Z_t} \in \mathbb{R}$ and $\sigma_{Z_t} > 0$. Independent of $\{I_t\}_{t\geq 0}$, we assume that the risk-free interest rate follows an Ornstein-Uhlenbeck process, i.e.

$$dr_t = \xi(\varkappa - r_t)dt + k\,d\tilde{W}_t, \quad r_0 > 0, \tag{9}$$

where ξ, $\varkappa$, and k are positive constants; $\{\tilde{W}_t\}_{t\geq 0}$ a one-dimensional Brownian motion. We can now price CAT bonds as

$$\begin{aligned} B(0) &= \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_0^T r_t dt\Big)\big(\mathbb{1}_{\{T_b\leq T\}}\,(1-R)F + \mathbb{1}_{\{T_b>T\}}\,F\big)\Big] \\ &= D_T\big(F - R\,F\,\mathbb{Q}(T_b \leq T)\big)\,, \end{aligned} \tag{10}$$

where the bond can be priced as

$$D_T := \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_0^T r_t dt\Big)\Big] = \exp\big(A_T - B_T\,r_0\big), \tag{11}$$

with

$$\begin{aligned} B_T &:= \frac{1}{\xi}\big(1 - \exp(-\xi T)\big), \\ A_T &:= \Big(\varkappa - \frac{k^2}{2\xi^2}\Big)(B_T - T) - \frac{k^2}{4\xi}B_T^2\,. \end{aligned}$$

Table 1 gives a numerical example of a 2-state regime switching risk process. The parameters of the risk process are chosen as

$$I_0 = 0, \quad \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} 10\% \\ -10\% \end{pmatrix}, \quad \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 10\% \\ 20\% \end{pmatrix}, \quad Q_0 = \begin{pmatrix} -1.0 & 1.0 \\ 0.5 & -0.5 \end{pmatrix}. \tag{12}$$

	$Z_0 = 1$	$Z_0 = 2$
$b = -0.05$	0.7204	0.5597
$b = -0.10$	0.8225	0.6373
$b = -0.20$	0.9131	0.7766
$b = -0.25$	0.9370	0.8329

Table 1: Prices $B(0)$ of CAT bonds in a 2-state regime switching model for several thresholds b. The remaining parameters are set as follows: $F = 1$, $R = 0.5$, $\xi = 2$, $\varkappa = 3\%$, $k = 0.02$, $r_0 = 2\%$, $T = 1$ (year).

We observe large price differences if the current state is $Z_0 = 1$, respectively $Z_0 = 2$. Thus, if there is a state change in the risk process, we observe a sudden jump in CAT bond prices, a very convenient feature in stochastic modeling.

4.2. OPTION PRICING

We now turn to financial mathematics and work on the pricing of options on exchange rates. Empirically, the existence of regimes in exchange rates is strongly confirmed (see, e.g., Bollen et al. (2000), Cheung and Erlandsson (2005)). We model the Canadian Dollar (CAD) – Euro (EUR) exchange rate as a regime switching geometric Brownian motion $\{S_t\}_{t\geq 0}$

$$\frac{dS_t}{S_t} = \sigma_{Z_t}\, dW_t, \quad S_0 > 0. \tag{13}$$

A 2-state model is chosen using the following set of parameters[2]

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 9.5\% \\ 6.3\% \end{pmatrix}, \quad Q_0 = \begin{pmatrix} -1.34 & 1.34 \\ 0.56 & -0.56 \end{pmatrix}. \tag{14}$$

The steady state of this parameter set is 30% (state 1) and 70% (state 2). Regime-switching models can easily be used for stress-testing by adding additional regimes. To demonstrate this capability, a third regime $\sigma_3 = 16.0\%$ is included in the ongoing analysis to assess the effect of a (possibly appearing) volatility increase that cannot be observed historically. Therefore, a second parameter set is introduced:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 9.5\% \\ 6.3\% \\ 16.0\% \end{pmatrix}, \quad Q_0 = \begin{pmatrix} -1.34 & 1.20 & 0.14 \\ 0.50 & -0.56 & 0.06 \\ 1.45 & 1.24 & -2.69 \end{pmatrix}. \tag{15}$$

[2]It is not the goal of this paper to discuss calibration, we therefore refer to the literature (e.g. Henriksen (2011) and the references therein). The given parameter set fits to the annualized historical volatility of the CAD-EUR exchange rate in the period 2000-2012.

The steady state of this parameter set is (29%, 68%, 3%). The first two states are the same as in parameter set (14).

In the following, we price digital first-touch options on the CAD-EUR exchange rate. According to Carr and Crosby (2010), those kind of options are the "most liquid and actively traded" exotic options on foreign exchange markets. We price a first-touch option that pays 1 at maturity T if the exchange rate stays above a barrier $B < S_0$. Under the risk-neutral measure $\mathbb{Q}$ with risk-less interest rate r, this contract can be priced as

$$FT(0) = \exp(-rT)\left(1 - \mathbb{P}(T_b \leq T)\right), \tag{16}$$

where we set $b = \log(B/S_0)$, $S_0 = 1$, and $\mu_{Z_t} = -\sigma^2_{Z_t}/2$.

	no third state	$\sigma_3 = 12.0\%$	$\sigma_3 = 16.0\%$	$\sigma_3 = 20.0\%$
$B = 0.6$	0.9913	0.9892	0.9861	0.9813
$B = 0.7$	0.9353	0.9262	0.9177	0.9074
$B = 0.8$	0.7455	0.7336	0.7218	0.7091
$B = 0.9$	0.4022	0.3932	0.3851	0.3767

Table 2: Prices $FT(0)$ of digital first-touch options for different magnitudes of risk in the third state (crisis state). The left column shows the result if there are only two states (parameter set (14)); the other three columns add a third state with different volatilities σ_3 (parameter set (15)). Several thresholds B are chosen. The remaining parameters are set as follows: $S_0 = 1$, $b = \log(S_0/B)$, $r = 0\%$, $x = \log(S_0) = 0$, $T = 6$ (years), and $\mathbb{P}(Z_0 = 1) = 1$.

Table 2 examines the effect of the turbulent third regime to first-touch option prices. Although this regime is very unlikely (on average 3% of the time is spent in this third regime), there is an apparent effect on option prices. Being aware of unforeseeable turbulent periods thus seems to be important for the pricing and risk management of digital options.

5. CONCLUSION

In this paper, we discussed applications of first-passage time problems under regime switching. The possibility of changing the state space is a possibility to stress test option prices by adding possibly distressed economic regimes. Similar to jump models, state changes can lead to sudden and significant changes in, for example, bond prices.

References

N.P.D. Bollen, S.F. Gray, and R.E. Whaley. Regime switching in foreign exchange rates. *Journal of Econometrics*, 84:239–276, 2000.

P. Boyle and T. Draviam. Pricing exotic options under regime switching. *Insurance: Mathematics and Economics*, 40:267–282, 2007.

J. Buffington and R. J. Elliott. American options with regime switching. *International Journal of Theoretical and Applied Finance*, 5:497–514, 2002.

P. Carr and J. Crosby. A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options. *Journal of Quantitative Finance*, 10(10):1115–1136, 2010.

J.-W. Cheung and U.G. Erlandsson. Exchange rates and Markov switching dynamics. *Journal of Business & Econcomic Statistics*, 23(3):314–320, 2005.

R. J. Elliott, L. Chan, and T. K. Siu. Option pricing and Esscher transform under regime switching. *Annals of Finance*, 1(4):423–432, 2005.

P. Eloe, R.H. Liu, and J.Y. Sun. Double barrier option under regime-switching exponential mean-reverting process. *International Journal of Computer Mathematics*, 86(6):964–981, 2009.

X. Guo. When the "Bull" meets the "Bear" – A first passage time problem for a hidden Markov process. *Methodology and Computing in Applied Probability*, 3(2):135–143, 2001a.

X. Guo. An explicit solution to an optimal stopping problem with regime switching. *Journal of Applied Probability*, 38:464–481, 2001b.

P. N. Henriksen. Pricing barrier options by a regime switching model. *Journal of Quantitative Finance*, 11(8):1221–1231, 2011.

P. Hieber. First-passage times of regime switching models. *working paper*, 2013a.

P. Hieber. A correction note on: When the "Bull" meets the "Bear" – A first passage time problem for a hidden Markov process. *Methodology and Computing in Applied Probability*, in press, 2013b.

P. Hieber and M. Scherer. Efficiently pricing barrier options in a Markov-switching framework. *Journal of Computational and Applied Mathematics*, 235:679–685, 2010.

P. Hieber and M. Scherer. A note on first-passage times of continuously time-changed Brownian motion. *Statistics & Probability Letters*, 82(1):165–172, 2012.

Z. Jiang and M.R. Pistorius. On perpetual American put valuation and first-passage in a regime-switching model with jumps. *Finance and Stochastics*, 12(3):331–355, 2008.

A. Jobert and L.C.G. Rogers. Option pricing with Markov modulated dynamics. *SIAM Journal on Control and Optimization*, 44:2063–2078, 2006.

L.C.G. Rogers. Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Annals of Applied Probability*, 4(2):390–413, 1994.

L.C.G. Rogers and Z. Shi. Computing the invariant law of a fluid model. *Journal of Applied Probability*, 31(4):885–896, 1994.

V. E. Vaugirard. Pricing catastrophe bonds by an arbitrage approach. *The Quarterly Review of Economics and Finance*, 43:119–132, 2003.

CONSTRUCTION OF COST-EFFICIENT SELF-QUANTO CALLS AND PUTS IN EXPONENTIAL LÉVY MODELS

Ernst August v. Hammerstein[†], Eva Lütkebohmert[†], Ludger Rüschendorf[§] and Viktor Wolf[§]

[†]*Department of Financial Mathematics, University of Freiburg, Platz der Alten Synagoge, D-79098 Freiburg, Germany*
[§]*Department of Mathematical Stochastics, University of Freiburg, Eckerstrasse 1, D-79104 Freiburg, Germany*
Email: ernst.august.hammerstein@finance.uni-freiburg.de, eva.luetkebohmert@finance.uni-freiburg.de, ruschen@stochastik.uni-freiburg.de, wolf@stochastik.uni-freiburg.de

Abstract

In this paper we derive explicit representations for cost-efficient puts and calls in financial markets which are driven by a Lévy process and where the pricing of derivatives is based on the Esscher martingale measure. Whereas the construction and evaluation of the efficient self-quanto call is a straightforward application of the general theory, the pricing of an efficient self-quanto put is more involved due to the lack of monotonicity of the standard payoff function. We show how to circumvent these difficulties and arrive at numerically tractable expressions. The potential savings of the cost-efficient strategies are illustrated in market models driven by NIG- and VG-processes using estimated parameters from German stock market data.

1. INTRODUCTION

The task of determining cost-efficient strategies is to construct resp. derive a payoff function which provides a predetermined payoff distribution at minimal costs. In other words, a cost-efficient strategy should provide the same chances of gaining or losing money as a given asset or derivative, but has a lower price than the latter one. This problem was first introduced by Dybvig (1988a,b) in the case of a discrete and arbitrage-free binomial model. Bernard and Boyle (2010), Bernard et al. (2014) give a solution of the efficient claim problem in a fairly general setting. They calculate in explicit form efficient strategies for several options in Black–Scholes markets.

In v. Hammerstein et al. (2014), their results are applied to certain classes of exponential Lévy models driven by Variance Gamma and Normal inverse Gaussian distributions. Under the assumption that the Esscher martingale measure is used for risk-neutral pricing, they investigate the impact of the risk-neutral Esscher parameter on the cost-efficient strategies and associated efficiency losses and derive concrete formulas for a variety of efficient options such as puts, calls, forwards, and spreads. Moreover, they consider the problem of hedging and provide explicit formulas for the deltas of cost-efficient calls and puts. Built on these results, we show in this paper how to obtain and price cost-efficient versions of self-quanto calls and puts and illustrate the theoretical results with a practical example using German stock market data.

The paper is structured as follows: Section 2 summarizes some basic definitions and results on cost-efficient payoffs in Lévy models. The self-quanto call and its efficient counterpart are discussed in Section 3, and formulas for the efficient self-quanto put are derived in Section 4. Explicit results based on real data from the German stock market are presented in Section 5, and Section 6 concludes.

2. GENERAL SETUP, BASIC NOTATION AND RESULTS

We assume to be given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions with finite trading horizon $[0,T], T \in \mathbb{R}_+$, on which the risky asset price process $(S_t)_{0\le t\le T}$ is defined and adapted to the filtration. Further, we suppose that there exists a constant risk-free interest rate r and a risk-neutral measure Q with $\frac{dQ}{dP}\big|_{\mathcal{F}_t} = Z_t$. A European style option with terminal payoff $X_T = h(S_T)$ for some payoff function h then has the initial price (or cost)

$$c(X_T) = e^{-rT} E\big[Z_T X_T\big]$$

where we denote here and in the following with $E[\cdot] = E_P[\cdot]$ the expectation w.r.t. P.

Definition 2.1 (Cost-efficient and most-expensive strategies)

a) *A strategy (or payoff)* $\underline{X}_T \sim G$ *is called* cost-efficient *w.r.t. the payoff-distribution G if any other strategy* X_T *that generates the same payoff-distribution G costs at least as much, i.e.*

$$c(\underline{X}_T) = e^{-rT} E[Z_T \underline{X}_T] = \min_{\{X_T\sim G\}} e^{-rT} E[Z_T X_T]. \tag{1}$$

b) *A strategy (or payoff)* $\overline{X}_T \sim G$ *is called* most-expensive *w.r.t. the payoff-distribution G if any other strategy* X_T *that generates the same payoff-distribution G costs at most as much:*

$$c(\overline{X}_T) = e^{-rT} E[Z_T \overline{X}_T] = \max_{\{X_T\sim G\}} e^{-rT} E[Z_T X_T]. \tag{2}$$

c) *The* efficiency loss *of a strategy with payoff* $X_T \sim G$ *at maturity* T *is defined as*

$$c(X_T) - c(\underline{X}_T).$$

Since the distribution $F_{Z_T}$of Z_T and the payoff distribution G have to be kept fixed, it can easily be seen that the problem of minimizing the cost is equivalent to finding a strategy $\underline{X}_T \sim G$ such that the covariance $\mathrm{Cov}(\underline{X}_T, Z_T)$ is minimized which can be achieved by constructing $\underline{X}_T$ in such a way that it is countermonotonic to Z_T. Analogously, the most-expensive payoff $\overline{X}_T$ has to be chosen comonotonic to Z_T. This general result was first obtained in Bernard and Boyle (2010).

To obtain a more explicit representation of cost-efficient resp. most-expensive payoffs, we further suppose that the asset price process $(S_t)_{0\le t\le T} = (S_0 e^{L_t})_{0\le t\le T}$ is of exponential Lévy type and that the risk-neutral measure Q is the Esscher martingale measure. This approach is widespread and has been well established since the last two decades. Further information on the use of exponential Lévy processes in financial modeling can be found in the books of Schoutens (2003), Cont and Tankov (2004), and Rachev et al. (2011). For a more detailed description of Lévy processes themselves, we refer the reader to the book of Barndorff-Nielsen et al. (2001) and the monographs of Sato (1999), and Applebaum (2009). The Esscher transform of a probability measure has originally been introduced in actuarial sciences by Esscher (1932) and was first suggested as a useful tool for option pricing in the seminal paper of Gerber and Shiu (1994). A more precise analysis of the Esscher transform for exponential Lévy models is given in (Raible 2000, Chapter 1) and Hubalek and Sgarra (2006). For the Esscher martingale measure to be well-defined in our setting, the Lévy process $(L_t)_{t\ge 0}$ has to fulfill the

Assumption 2.1 *The random variable L_1 is nondegenerate and possesses a moment generating function (mgf) $M_{L_1}(u) = E[e^{uL_1}]$ on some open interval (a, b) with $a < 0 < b$ and $b - a > 1$.*

This condition is necessary (but not always sufficient) for the existence of the risk-neutral Esscher measure. Sufficient conditions were first given in (Raible 2000, Proposition 2.8).

Definition 2.2 *We call an Esscher transform any change of P to a locally equivalent measure Q^θ with a density process $Z_t^\theta = \frac{dQ^\theta}{dP}|_{\mathcal{F}_t}$ of the form*

$$Z_t^\theta = \frac{e^{\theta L_t}}{M_{L_t}(\theta)}, \tag{3}$$

where M_{L_t} is the mgf of L_t as before, and $\theta \in (a, b)$.

It can easily be shown that $(Z_t^\theta)_{t\ge 0}$ indeed is a density process for all $\theta \in (a, b)$, and $(L_t)_{t\ge 0}$ also is a Lévy process under Q^θ for all these θ (see, for example, (Raible 2000, Proposition 1.8)). However, there will be at most one parameter $\bar\theta$ for which the discounted asset price process $(e^{-rt}S_t)_{t\ge 0}$ is a martingale under the so-called *risk-neutral Esscher measure* or *Esscher martingale measure* $Q^{\bar\theta}$. This $\bar\theta$ has to solve the equation

$$e^r = \frac{M_{L_1}(\bar\theta + 1)}{M_{L_1}(\bar\theta)}. \tag{4}$$

With these preliminaries, the general results of (Bernard et al. 2014, Proposition 3) can be reformulated in the present framework as follows (see (v. Hammerstein et al. 2014, Proposition 2.1)):

Proposition 2.1 *Let $(L_t)_{t\ge 0}$ be a Lévy process with continuous distribution function F_{L_T} at maturity $T > 0$, and assume that a solution $\bar\theta$ of (4) exists.*

a) If $\bar\theta < 0$, then the cost-efficient payoff $\underline{X}_T$ and the most-expensive payoff $\overline{X}_T$ with distribution function G are a.s. unique and are given by

$$\underline{X}_T = G^{-1}(F_{L_T}(L_T)) \quad \textit{and} \quad \overline{X}_T = G^{-1}(1 - F_{L_T}(L_T)). \tag{5}$$

Further, the following bounds for the cost of any strategy with terminal payoff $X_T \sim G$ hold:

$$
\begin{aligned}
c(X_T) &\geq E\big[e^{-rT} Z_T^{\bar\theta} \underline{X}_T\big] = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y)-rT} G^{-1}(1-y)\,dy,\\
c(X_T) &\leq E\big[e^{-rT} Z_T^{\bar\theta} \overline{X}_T\big] = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y)-rT} G^{-1}(y)\,dy.
\end{aligned}
$$

b) If $\bar\theta > 0$, then the cost-efficient and the most-expensive payoffs are a.s. unique and given by

$$
\underline{X}_T = G^{-1}(1 - F_{L_T}(L_T)) \quad \textit{and} \quad \overline{X}_T = G^{-1}(F_{L_T}(L_T)). \tag{6}
$$

The bounds in a) hold true with $F_{L_T}^{-1}(1-y)$ replaced by $F_{L_T}^{-1}(y)$.

From the previous proposition one can easily deduce the following characterization of cost-efficiency in exponential Lévy models where the notions increasing and decreasing have to be understood in the weak sense.

Corollary 2.2 *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution F_{L_T} at maturity $T > 0$, and assume that a solution $\bar\theta$ of* (4) *exists.*

a) If $\bar\theta < 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is increasing in L_T.

b) If $\bar\theta > 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is decreasing in L_T.

For the most-expensive strategy, the reverse holds true.

Let us remark that the sign of the risk-neutral Esscher parameter $\bar\theta$ not only plays an essential role for the construction of cost-efficient strategies, but also characterizes the current market scenario. More specifically, a negative $\bar\theta < 0$ corresponds to a bullish market, and in case of $\bar\theta > 0$ we have a bearish market behaviour. A more detailed formulation and proof of this fact can be found in (v. Hammerstein et al. 2014, Proposition 2.2).

For the practical applications in Section 5 we shall consider two specific exponential Lévy models which we shortly describe in the following. Both are based on special sub- resp. limiting classes of the more general family of generalized hyperbolic (GH) distributions which was introduced in Barndorff-Nielsen (1977). A detailed description of uni- and multivariate GH distributions as well as their weak limits is provided in (v. Hammerstein 2011, Chapters 1 and 2).

Normal inverse Gaussian model. The Normal inverse Gaussian distribution (NIG) has been introduced to finance in Barndorff-Nielsen (1998). It can be obtained as a normal mean-variance mixture with an inverse Gaussian mixing distribution. This in particular entails that the infinite divisibility of the mixing inverse Gaussian distribution transfers to the NIG mixture distribution, thus there exists a Lévy process $(L_t)_{t\geq 0}$ with $\mathcal{L}(L_1) = NIG(\alpha, \beta, \delta, \mu)$. The density and mgf of an NIG distribution are given by

$$
d_{NIG}(x) = \frac{\alpha\delta e^{\delta\sqrt{\alpha^2-\beta^2}}}{\pi} \frac{K_1\big(\alpha\sqrt{\delta^2 + (x-\mu)^2}\big)}{\sqrt{\delta^2 + (x-\mu)^2}}\, e^{\beta(x-\mu)}, \qquad M_{NIG}(u) = \frac{e^{u\mu+\delta\sqrt{\alpha^2-\beta^2}}}{e^{\delta\sqrt{\alpha^2-(\beta+u)^2}}}. \tag{7}
$$

The parameter $\overline{\theta}$ of the risk-neutral Esscher martingale measure $Q^{\overline{\theta}}$, i.e., the solution of (4) (if it exists) is given by

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta + \frac{r-\mu}{\delta}\sqrt{\frac{\alpha^2}{1+(\frac{r-\mu}{\delta})^2} - \frac{1}{4}}. \tag{8}$$

We have

$$d^{\bar{\theta}}_{L_t}(x) = \frac{e^{\bar{\theta}x}}{M_{NIG(\alpha,\beta,\delta t,\mu t)}(\bar{\theta})}\, d_{NIG(\alpha,\beta,\delta t,\mu t)}(x) = d_{NIG(\alpha,\beta+\bar{\theta},\delta t,\mu t)}(x) \tag{9}$$

which implies that $(L_t)_{t\geq 0}$ remains a NIG Lévy process under the risk-neutral Esscher measure $Q^{\bar{\theta}}$, but with skewness parameter β replaced by $\beta + \bar{\theta}$.

Variance Gamma model. Similar to the NIG distributions, a Variance Gamma distribution (VG) can be represented as a normal mean-variance mixture with a mixing Gamma distribution. Symmetric VG distributions were first defined (with a different parametrization) in Madan and Seneta (1990), the general case with skewness was considered in Madan et al. (1998). Again, the infinite divisibility of the Gamma distribution transfers to the Variance Gamma distribution $VG(\lambda, \alpha, \beta, \mu)$ whose density and mgf are given by

$$d_{VG}(x) = \frac{(\alpha^2-\beta^2)^{\lambda}|x-\mu|^{\lambda-\frac{1}{2}}}{\sqrt{\pi}(2\alpha)^{\lambda-\frac{1}{2}}\Gamma(\lambda)}K_{\lambda}(\alpha|x-\mu|)e^{\beta(x-\mu)},\ M_{VG}(u) = e^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda}. \tag{10}$$

Here the condition $2\alpha > 1$ is sufficient to guarantee a unique solution $\bar{\theta}$ of equation (4) which is given by

$$\bar{\theta}_{VG} = \begin{cases} -\frac{1}{2} - \beta, & r = \mu, \\ -\frac{1}{1-e^{-\frac{r-\mu}{\lambda}}} - \beta + \operatorname{sign}(r-\mu)\sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1-e^{-\frac{r-\mu}{\lambda}}\right)^2} + \alpha^2}, & r \neq \mu. \end{cases} \tag{11}$$

Similar as above, we have

$$d^{\bar{\theta}}_{L_t}(x) = \frac{e^{\bar{\theta}x}}{M_{VG(\lambda t,\alpha,\beta,\mu t)}(\bar{\theta})}\, d_{VG(\lambda t,\alpha,\beta,\mu t)}(x) = d_{VG(\lambda t,\alpha,\beta+\bar{\theta},\mu t)}(x), \tag{12}$$

hence under $Q^{\bar{\theta}}$ $(L_t)_{t\geq 0}$ again is a VG process, but with skewness parameter $\beta + \bar{\theta}$ instead of β.

3. STANDARD AND EFFICIENT SELF-QUANTO CALLS

A quanto option is a (typically European) option whose payoff is converted into a different currency or numeraire at maturity at a pre-specified rate, called the quanto-factor. In the special case of a self-quanto option the numeraire is the underlying asset price at maturity itself. The payoff of a long self-quanto call with maturity T and strike price K therefore is

$$X^{sqC}_T = S_T \cdot (S_T - K)_+ = S_0 e^{L_T}(S_0 e^{L_T} - K)_+$$

Applying the risk-neutral pricing rule, together with equation (4), we obtain the following formula for the time-0-price of a self-quanto call:

$$\begin{aligned} c(X_T^{sqC}) &= e^{-rT}E\big[Z_T^{\bar\theta}S_T\cdot(S_T-K)_+\big] \\ &= \frac{M_{L_T}(\bar\theta)}{M_{L_T}(\bar\theta+1)}\,E\left[\frac{e^{\bar\theta L_T}}{M_{L_T}(\bar\theta)}S_0e^{L_T}\big(S_0e^{L_T}-K\big)\mathbb{1}_{(\ln(K/S_0),\infty)}(L_T)\right] \\ &= S_0^2\frac{M_{L_T}(\bar\theta+2)}{M_{L_T}(\bar\theta+1)}\,E\big[Z_T^{\bar\theta+2}\mathbb{1}_{(\ln(K/S_0),\infty)}(L_T)\big]-KS_0\,E\big[Z_T^{\bar\theta+1}\mathbb{1}_{(\ln(K/S_0),\infty)}(L_T)\big] \end{aligned}$$

From equations (7) and (9) resp. (10) and (12) we can derive more explicit formulas for the NIG and VG models:

$$c(X_T^{sqC}) = \begin{cases} S_0^2\frac{e^{\mu T+\delta T\sqrt{\alpha^2-(\beta+\bar\theta+1)^2}}}{e^{\delta T\sqrt{\alpha^2-(\beta+\bar\theta+2)^2}}}\bar F_{NIG(\alpha,\beta+\bar\theta+2,\delta T,\mu T)}\big(\ln(K/S_0)\big)-KS_0\bar F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln(K/S_0)\big) \\ S_0^2e^{\mu T}\left(\frac{\alpha^2-(\beta+\bar\theta+1)^2}{\alpha^2-(\beta+\bar\theta+2)^2}\right)^{\lambda T}\bar F_{VG(\lambda T,\alpha,\beta+\bar\theta+2,\mu T)}\big(\ln(K/S_0)\big)-KS_0\bar F_{VG(\lambda T,\alpha,\beta+\bar\theta+1,\mu T)}\big(\ln(K/S_0)\big) \end{cases}$$

where $\bar F(x)=1-F(x)$ denotes the survival function of the corresponding distribution. For $0\le t\le T$, the time-t-price $c(X_{T,t}^{sqC})$ of the self-quanto call is obtained from the preceding formulas by replacing S_0 by S_t and T by $T-t$.

The payoff X_T^{sqC} of a self-quanto call obviously is increasing in L_T and therefore not cost-efficient if $\bar\theta>0$ by Corollary 2.2. According to Proposition 2.1 b), its efficient counterpart $\underline{X}_T^{sqC}$ is given by $G_{sqC}^{-1}(1-F_{L_T}(L_T))$. To derive the corresponding distribution function $G_{sqC}=F_{X_T^{sqC}}$, observe that the positive solution S_T^* of the quadratic equation $S_T^2-KS_T=x$, $x>0$, is given by $S_T^*=\frac{K}{2}+\sqrt{\frac{K^2}{4}+x}$, hence

$$G_{sqC}(x)=P\big(X_T^{sqC}\le x\big)=\begin{cases}0 & \text{, if } x<0,\\ F_{L_T}\left(\ln\left(\frac{\frac{K}{2}+\sqrt{\frac{K^2}{4}+x}}{S_0}\right)\right) & \text{, if } x\ge 0.\end{cases}$$

The inverse then can easily be shown to equal

$$G_{sqC}^{-1}(y)=S_0e^{F_{L_T}^{-1}(y)}\big(S_0e^{F_{L_T}^{-1}(y)}-K\big)_+,\quad y\in(0,1),$$

consequently the cost-efficient strategy for a long self-quanto call in the case $\bar\theta>0$ is

$$\underline{X}_T^{sqC}=G_{sqC}^{-1}(1-F_{L_T}(L_T))=S_0e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\big(S_0e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}-K\big)_+. \qquad (13)$$

A comparison of the payoff functions X_T^{sqC} and $\underline{X}_T^{sqC}$ of a standard resp. efficient self-quanto call on ThyssenKrupp with strike $K=16$ and maturity $T=22$ days can be found in Figure 1 below. The estimated NIG parameters for ThyssenKrupp used to calculate the efficient payoff profile can be found in Table 1 in Section 5.

Observe that in contrast to the standard payoff $X_T^{sqC}=h_{sqC}(S_T)=\tilde h_{sqC}(L_T)$, the payoff function $\underline{\tilde h}_{sqC}(L_T)$ of the efficient self-quanto call depends on the time to maturity because so do

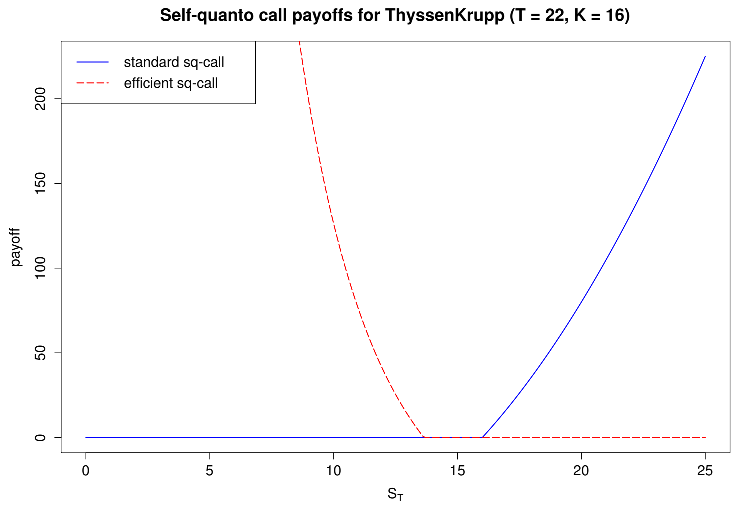


Figure 1: Payoff functions of a standard and efficient self-quanto call on ThyssenKrupp. The initial stock price is $S_0 = 15.25$, the closing price of ThyssenKrupp at July 1, 2013.

the distribution and quantile functions F_{L_T} resp. $F_{L_T}^{-1}$. However, if an investor buys an efficient self-quanto call, its payoff profile is fixed at the purchase date and will not be altered afterwards. Once bought or sold, the payoff distribution of a cost-efficient contract only equals that of its classical counterpart at the (initial) trading date, but no longer in the remaining time to maturity. To calculate the price $c(\underline{X}_{T,t}^{sqC})$ of an efficient self-quanto call with a payoff function fixed at time 0 at some later point in time $t > 0$, one has to resort to the fact that $S_T = S_0 e^{L_T} \stackrel{d}{=} S_0 e^{L_t + L_{T-t}} = S_t e^{L_{T-t}}$ and thus replace $L_T = \ln(S_T/S_0)$ in (13) by $\ln(S_t e^{L_{T-t}}/S_0)$, that is,

$$\underline{X}_{T,t}^{sqC} = S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(S_t e^{L_{T-t}}/S_0)))} \big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(S_t e^{L_{T-t}}/S_0)))} - K\big)_+.$$

The time-t-price of an efficient self-quanto call initiated at time 0 then can be calculated by

$$c(\underline{X}_{T,t}^{sqC}) = e^{-r(T-t)} S_0 \int_{-\infty}^{a} e^{F_{L_T}^{-1}(1-F_{L_T}(y+\ln(S_t/S_0)))} \big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(y+\ln(S_t/S_0)))} - K\big) d_{L_{T-t}}^{\bar\theta}(y)\, dy \tag{14}$$

where $a = F_{L_T}^{-1}(1 - F_{L_T}(\ln(K/S_0))) - \ln(S_t/S_0)$. If $t = 0$, one can alternatively use the general formula of Proposition 2.1, together with the representation of G_{sqC}^{-1} given above.

4. STANDARD AND EFFICIENT SELF-QUANTO PUTS

The payoff of a long self-quanto put with maturity T and strike price K is

$$X_T^{sqP} = S_T \cdot (K - S_T)_+ = S_0 e^{L_T} (K - S_0 e^{L_T})_+$$

and similar as in the call case, we find the time-0-price of a self-quanto put to equal

$$c(X_T^{sqP}) = K S_0\, E\big[Z_T^{\bar\theta+1} \mathbb{1}_{(-\infty,\ln(K/S_0))}(L_T)\big] - S_0^2 \frac{M_{L_T}(\bar\theta+2)}{M_{L_T}(\bar\theta+1)}\, E\big[Z_T^{\bar\theta+2} \mathbb{1}_{(-\infty,\ln(K/S_0))}(L_T)\big]$$

which can be specialized in the NIG and VG models to

$$c(X_T^{sqP}) = \begin{cases} KS_0F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta T,\mu T)}\big(\ln(K/S_0)\big) - S_0^2\frac{e^{\mu T+\delta T\sqrt{\alpha^2-(\beta+\bar{\theta}+1)^2}}}{e^{\delta T\sqrt{\alpha^2-(\beta+\bar{\theta}+2)^2}}}F_{NIG(\alpha,\beta+\bar{\theta}+2,\delta T,\mu T)}\big(\ln(K/S_0)\big) \\ KS_0F_{VG(\lambda T,\alpha,\beta+\bar{\theta}+1,\mu T)}\big(\ln(K/S_0)\big) - S_0^2 e^{\mu T}\left(\frac{\alpha^2-(\beta+\bar{\theta}+1)^2}{\alpha^2-(\beta+\bar{\theta}+2)^2}\right)^{\lambda T} F_{VG(\lambda T,\alpha,\beta+\bar{\theta}+2,\mu T)}\big(\ln(K/S_0)\big) \end{cases}$$

Again, the time-t-price of the self-quanto put for $0 \le t \le T$ is obtained from the above equations by replacing S_0 by S_t and T by $T-t$.

The payoff function $X_T^{sqP} = h_{sqP}(S_T)$ of a self-quanto put is a parabola which is open from below and has the roots 0 and K as well as a maximum at $S_T = \frac{K}{2}$. Hence, it is neither increasing nor decreasing in S_T and therefore not in $L_T = \ln(S_T/S_0)$ either, so Corollary 2.2 implies that a self-quanto put can never be cost-efficient unless $\bar{\theta} = 0$.

The lack of monotonicity also makes the determination of the distribution function G_{sqP} of the self-quanto put payoff and its inverse a little bit cumbersome. To derive them, first observe that the corresponding payoff function $\tilde{h}_{sqP}(x) = (S_0Ke^x - S_0^2e^{2x}) \cdot \mathbb{1}_{(-\infty,\ln(K/S_0))}$ is strictly increasing on $(-\infty, \ln(K/(2S_0)))$ and strictly decreasing on $(\ln(K/(2S_0)), \ln(K/S_0))$, and has a maximum at $x = \ln(K/(2S_0))$ with value $\tilde{h}_{sqP}\big(\ln(K/(2S_0))\big) = \frac{K^2}{4}$. For $y \in (0, \ln(K/S_0))$ we have

$$\tilde{h}_{sqP}(x) = y \iff x = \ln\left(\frac{K+\sqrt{K^2-4y}}{2S_0}\right) \;\vee\; x = \ln\left(\frac{K-\sqrt{K^2-4y}}{2S_0}\right)$$

from which we obtain

$$\begin{aligned} G_{sqP}(x) &= P\big(\tilde{h}_{sqP}(L_T) \le x\big) \\ &= \begin{cases} 1 & \text{for } x \ge \frac{K^2}{4}, \\ F_{L_T}\left(\ln\left(\frac{K-\sqrt{K^2-4x}}{2S_0}\right)\right) + 1 - F_{L_T}\left(\ln\left(\frac{K+\sqrt{K^2-4x}}{2S_0}\right)\right) & \text{for } \frac{K^2}{4} > x > 0, \\ 1 - F_{L_T}\big(\ln(K/S_0)\big) & \text{for } x = 0, \\ 0 & \text{for } x < 0. \end{cases} \end{aligned}$$

The shape of the payoff function here leads to two summands in the representation of the payoff distribution G_{sqP} on the interval $(0, \frac{K^2}{4})$, therefore its inverse G_{sqP}^{-1} needed to construct the cost-efficient self-quanto put payoff $\underline{X}_T^{sqP}$ according to Proposition 2.1 can only be evaluated numerically (using some suitable root-finding algorithms), but not given in closed form.

If $\bar{\theta} < 0$, then we have $\underline{X}_T^{sqP} = G_{sqP}^{-1}(F_{L_T}(L_T)) = G_{sqP}^{-1}(F_{L_T}(\ln(S_T/S_0))$, and from the above representation of G_{sqP} we conclude that $G_{sqP}^{-1}(F_{L_T}(\ln(S_T/S_0)) = 0$ if $S_T \le S_0e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(K/S_0)))}$ resp. $L_T \le F_{L_T}^{-1}(1-F_{L_T}(\ln(K/S_0)))$. Otherwise, the payoff is positive and tends to $\frac{K^2}{4}$ if S_T resp. L_T tend to infinity.

If $\bar{\theta} > 0$, then $\underline{X}_T^{sqP} = G_{sqP}^{-1}(1 - F_{L_T}(L_T)) = G_{sqP}^{-1}(1 - F_{L_T}(\ln(S_T/S_0))$ which is zero if $S_T \ge K$ resp. $L_T \ge \ln(K/S_0)$ and tends to $\frac{K^2}{4}$ if $S_T \to 0$ resp. $L_T \to -\infty$. Hence, for $\bar{\theta} > 0$ the efficient self-quanto put payoff shows just the opposite behaviour as for $\bar{\theta} < 0$. This is in line with Corollary 2.2 which states, in other words, that a cost-efficient payoff must alter its monotonicity properties if the sign of the risk-neutral Esscher parameter $\bar{\theta}$ changes. The two different payoff

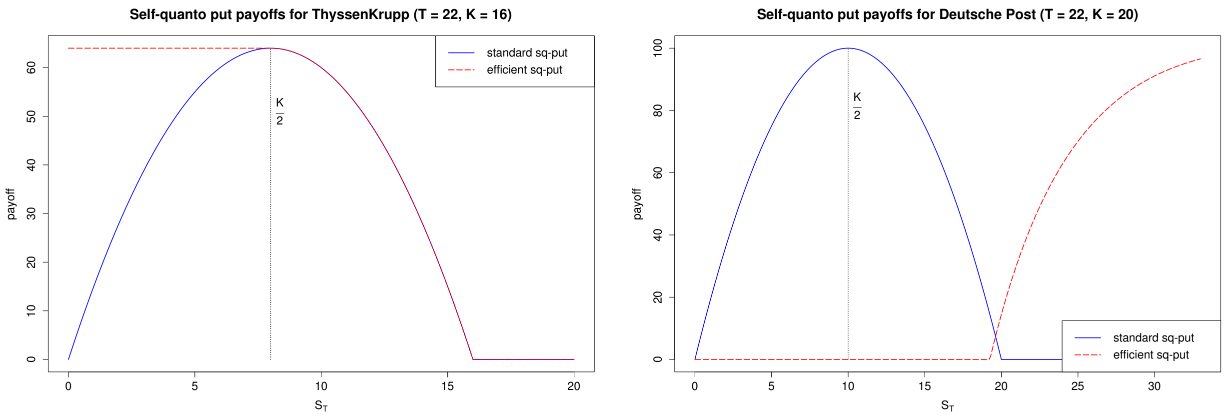


Figure 2: *Left:* Payoff functions of a standard and efficient self-quanto put on ThyssenKrupp ($\bar{\theta} > 0$). The initial stock price is $S_0 = 15.25$, the closing price of ThyssenKrupp at July 1, 2013. *Right:* Payoff functions of a standard and efficient self-quanto put on Deutsche Post ($\bar{\theta} < 0$). The initial stock price is $S_0 = 19.31$, the closing price of Deutsche Post at July 1, 2013.

profiles that can occur for an efficient self-quanto put are visualized in Figure 2 above. The estimated VG parameters for ThyssenKrupp and Deutsche Post that are used to calculate the efficient payoffs can be found in Table 1 in Section 5. As can be seen from the latter, the efficient payoff for ThyssenKrupp corresponds to the case $\bar{\theta} > 0$, whereas the efficient payoff for Deutsche Post has the typical shape for $\bar{\theta} < 0$.

For the time-t-price of an efficient self-quanto put that is issued at time 0, one obtains, with the same reasoning as in Section 3,

$$\begin{aligned} c(\underline{X}_{T,t}^{sqP}) &= e^{-r(T-t)} E\big[Z_{T-t}^{\bar{\theta}} \underline{X}_{T,t}^{sqP}\big] \\ &= \begin{cases} e^{-r(T-t)} \displaystyle\int_{a_-}^{\infty} G_{sqP}^{-1}\big(F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy =: c_t^-(S_t) & \text{if } \bar{\theta} < 0, \\ e^{-r(T-t)} \displaystyle\int_{-\infty}^{a_+} G_{sqP}^{-1}\big(1 - F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy =: c_t^+(S_t) & \text{if } \bar{\theta} > 0, \end{cases} \end{aligned} \tag{15}$$

where $a_- = F_{L_T}^{-1}(1 - F_{L_T}(\ln(K/S_0))) - \ln(S_t/S_0)$ and $a_+ = \ln(K/S_t)$. Due to the necessary numerical determination of $G_{sqP}^{-1}(x)$, the integrals in (15) have to be truncated in practical applications to obtain sensible and stable results from a numerical evaluation. The inequalities

$$\begin{aligned} & e^{-r(T-t)} \int_{a_-}^{z_-} G_{sqP}^{-1}\big(F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy \le c_t^-(S_t) \\ & \qquad \le \; e^{-r(T-t)} \int_{a_-}^{z_-} G_{sqP}^{-1}\big(F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy + e^{-r(T-t)} \frac{K^2}{4} \bar{F}_{L_{T-t}}^{\bar{\theta}}(z_-), \end{aligned}$$

$$\begin{aligned} & e^{-r(T-t)} \int_{z_+}^{a_+} G_{sqP}^{-1}\big(1 - F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy \le c_t^+(S_t) \\ & \qquad \le \; e^{-r(T-t)} \int_{z_+}^{a_+} G_{sqP}^{-1}\big(1 - F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy + e^{-r(T-t)} \frac{K^2}{4} F_{L_{T-t}}^{\bar{\theta}}(z_+), \end{aligned}$$

which hold for all $z_- > a_-$ resp. $z_+ < a_+$ allow to well control the error caused by the truncation.

5. APPLICATION TO REAL MARKET DATA

In this section we want to apply the theoretical results obtained so far to some real data and parameters to get an impression how large the potential efficiency losses of the standard options can be. For our calculations, we use NIG and VG parameters estimated from two German stocks, ThyssenKrupp and Deutsche Post. We used data from a two-year period starting at June 1, 2011, and ending on June 28, 2013, to estimate the parameters from the log-returns of both stocks. The stock prices within the estimation period are shown in Figure 3, and the obtained parameters are summarized in Table 1. The interest rate used to calculate $\bar{\theta}$ is $r = 4.3838 \cdot 10^{-6}$ which corresponds to the continuously compounded 1-Month-Euribor rate of July 1, 2013.

Observe that the risk-neutral Esscher parameters $\bar{\theta}_{NIG}$ and $\bar{\theta}_{VG}$ are negative for Deutsche Post and positive for ThyssenKrupp, therefore a self-quanto call can only be improved for ThyssenKrupp, for Deutsche Post it already is cost-efficient. For the former, we calculate the prices of standard and efficient self-quanto calls with strike $K = 16$ which are issued on July 1, 2013, and mature on July 31, 2013, so the time T to maturity is 22 trading days. The results are shown in Table 2. Apparently, the differences in prices and hence the efficiency losses are quite large, the standard self-quanto call costs almost twice as much as its efficient counterpart.

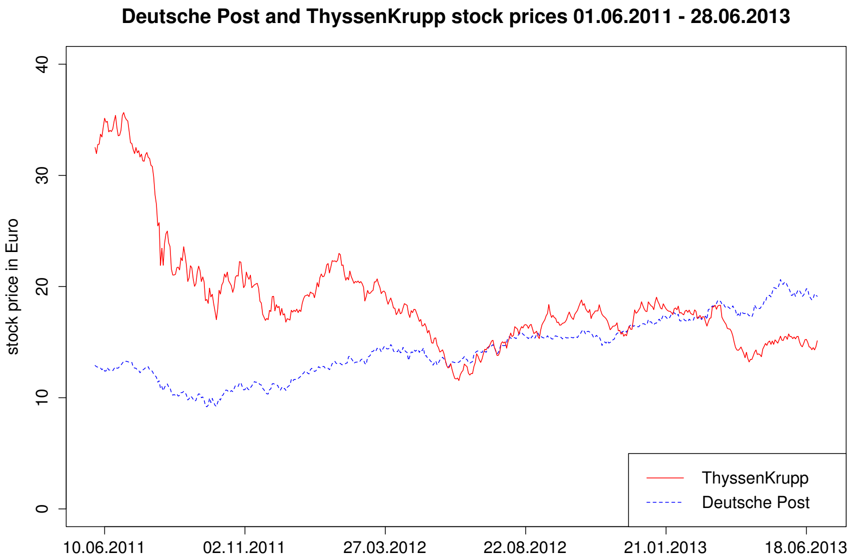


Figure 3: Daily closing prices of Deutsche Post and ThyssenKrupp used for parameter estimation.

Deutsche Post	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	75.059	1.758	0.019	0.000306	-3.4787
VG	1.942	126.266	3.719	0.0	-0.000165	-3.5220
ThyssenKrupp	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	53.065	-0.491	0.037	-0.001101	1.5823
VG	2.659	87.894	-0.613	0.0	-0.001025	1.6080

Table 1: Estimated parameters from daily log-returns of Deutsche Post and Volkswagen for the NIG- and the VG-model.

ThyssenKrupp	$c(X_T^{sqC})$	$c(\underline{X}_T^{sqC})$	**Efficiency loss in %**
NIG	8.3288	4.2251	49.27
VG	8.2629	4.1609	49.64

Table 2: Comparison of the prices of a self-quanto call on ThyssenKrupp with strike $K = 16$ and $T = 22$, and its cost-efficient counterpart in the NIG and VG models. The initial stock price is $S_0 = 15.25$, the closing price of ThyssenKrupp on July 1, 2013.

In contrast to the self-quanto call, the standard self-quanto put is—at least theoretically—inefficient for both stocks since the risk-neutral Esscher parameter $\bar{\theta}$ is different from zero in all cases. In our example we assume that the standard and efficient self-quanto puts on ThyssenKrupp and Deutsche Post have the same issuance day and maturity date as the self-quanto calls above, and the strikes are again $K = 16$ for ThyssenKrupp and $K = 20$ for Deutsche Post. The obtained results are listed in Table 3. Whereas the efficiency losses for Deutsche Post are of comparable magnitude as in the call example, one surprisingly does not save anything by investing in the efficient self-quanto put on ThyssenKrupp.

This becomes clearer if we take a look back on the corresponding payoff function $\underline{X}_T^{sqP}$. Recall that the risk-neutral Esscher parameters for ThyssenKrupp are always positive, therefore the left plot of Figure 2 applies here. If $\bar{\theta} > 0$, then obviously X_T^{sqP} and $\underline{X}_T^{sqP}$ are almost identical for $S_T \in (\frac{K}{2}, \infty)$ and only differ significantly if $S_T \in (0, \frac{K}{2})$. But if the risk-neutral probability $Q^{\bar{\theta}}(0 < S_T < \frac{K}{2})$ is very small, then it is intuitively evident that the prices $c(X_T^{sqP})$ and $c(\underline{X}_T^{sqP})$ should nearly coincide. This is the case here. The strike K is very close to the initial stock price S_0, and the risk-neutral measure $Q^{\bar{\theta}}$ is more right-skewed than the real-word one P (under the risk-neutral Esscher measure, only the skewness parameter β of the NIG and VG distributions changes to $\beta + \bar{\theta}$), so under $Q^{\bar{\theta}}$ it becomes even more unlikely that $S_T < \frac{K}{2}$.

The evolution of the prices $c(X_{T,t}^{sqC})$, $c(\underline{X}_{T,t}^{sqC})$ of the standard and efficient self-quanto call on ThyssenKrupp as well as that of the prices $c(X_{T,t}^{sqP})$, $c(\underline{X}_{T,t}^{sqP})$ of the self-quanto puts on Deutsche Post in the NIG models during the lifetime of the options is shown in Figure 4. The prices of the efficient options always roughly move in the opposite direction of that of the standard options which reflects the reversed resp. altered monotonicity properties of the underlying payoff profiles.

ThyssenKrupp	$c(X_T^{sqP})$	$c(\underline{X}_T^{sqP})$	**Efficiency loss in %**
NIG	16.1541	16.1541	0.0
VG	16.1226	16.1226	0.0
Deutsche Post	$c(X_T^{sqP})$	$c(\underline{X}_T^{sqP})$	**Efficiency loss in %**
NIG	17.6912	10.2613	42.00
VG	17.6593	10.2152	42.15

Table 3: Comparison of the prices of standard and efficient self-quanto puts on ThyssenKrupp and Deutsche Post with strikes $K = 16$ resp. $K = 20$, and $T = 22$, in the NIG and VG models. The initial stock prices are $S_0 = 15.25$ for ThyssenKrupp and $S_0 = 19.31$ for Deutsche Post, which are the closing prices on July 1, 2013.

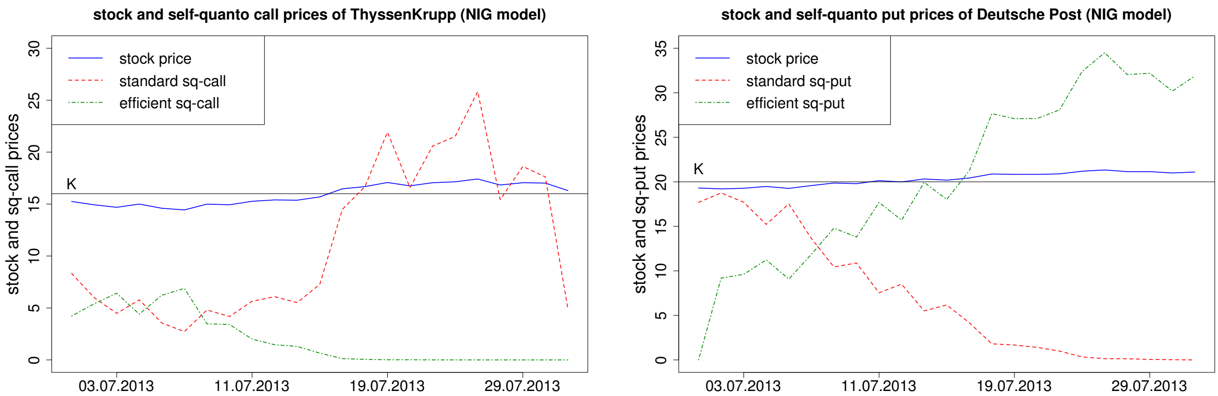


Figure 4: *Left:* Stock price of ThyssenKrupp from July 1, 2013, to July 31, 2013, and the prices $c(X^{sqC}_{T,t})$, $c(\underline{X}^{sqC}_{T,t})$ of the associated standard and efficient self-quanto calls. *Right:* Stock price of Deutsche Post from July 1, 2013, to July 31, 2013, and the prices $c(X^{sqP}_{T,t})$, $c(\underline{X}^{sqP}_{T,t})$ of the associated standard and efficient self-quanto puts.

6. SUMMARY AND CONCLUSION

We applied the concept of cost-efficiency to self-quanto puts and calls in exponential Lévy models where the risk-neutral measure is obtained by an Esscher transform. Whereas one can arrive—at least in principle—at closed-form solutions in the call case, things become more involved for the self-quanto put because of the lacking monotonicity properties of the corresponding payoff function. Nevertheless, the arising expressions and integrals remain numerically tractable and can be evaluated in an efficient and stable way which we demonstrated in a practical application using estimated parameters and real data from the German stock market. The observed efficiency losses are often quite large. However, the prices of the cost-efficient options are not always significantly lower than their classical counterparts. For efficient self-quanto puts that are issued at the money, the potential savings are negligible if the risk-neutral Esscher parameter is positive.

The evolution of the prices of standard and efficient options over time shows that they move in opposite directions: If the standard option expires worthless, its efficient counterpart typically ends up in the money, and vice versa. This should remind the reader that although cost-efficient options provide a cheaper way to participate in a certain payoff distribution, they are still speculative instruments which bear the risk of a total loss of one's investment.

References

D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, 2nd edition, 2009.

O.E. Barndorff-Nielsen. Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society London*, A 353:401–419, 1977.

O.E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2: 41–68, 1998.

O.E. Barndorff-Nielsen, T. Mikosch, and S. Resnick, editors. *Lévy Processes: Theory and Applications*. Birkhäuser, Boston, 2001.

C. Bernard and P.P. Boyle. Explicit representation of cost-efficient strategies. working paper, available at `http://ssrn.com/abstract=1695543`, 2010.

C. Bernard, P.P. Boyle, and S. Vanduffel. Explicit representation of cost-efficient strategies. *Finance*, 2014. forthcoming.

R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall / CRC Press, Boca Raton, 2004.

P. Dybvig. Distributional analysis of portfolio choice. *Journal of Business*, 61(3):369–393, 1988a.

P. Dybvig. Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock market. *Review of Financial Studies*, 1(1):67–88, 1988b.

F. Esscher. On the probability function in the collective theory of risk. *Scandinavian Actuarial Journal*, 15(3):175–195, 1932.

H.U. Gerber and E.S.W. Shiu. Option pricing by esscher transforms. *Transactions of the Society of Actuaries*, 46:99–191, 1994.

F. Hubalek and C. Sgarra. Esscher transforms and the minimal entropy martingale measure for exponential Lévy models. *Quantitative Finance*, 6(2):125–145, 2006.

D.B. Madan and E. Seneta. The variance gamma (V.G.) model for share market returns. *Journal of Business*, 63:511–524, 1990.

D.B. Madan, P.P. Carr, and E.C. Chang. The variance gamma process and option pricing. *European Finance Review*, 2:79–105, 1998.

S.T. Rachev, Y.S. Kim, M.L. Bianchi, and F.J. Fabozzi. *Financial Models with Lévy Processes and Volatility Clustering*. Wiley, Hoboken, 2011.

S. Raible. *Lévy processes in finance: theory, numerics, and empirical facts*. PhD thesis, University of Freiburg, 2000. Available at `http://www.freidok.uni-freiburg.de/volltexte/51/`.

K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.

W. Schoutens. *Lévy Processes in Finance: Pricing Financial Derivatives*. Wiley, Chichester, 2003.

E.A. v. Hammerstein. *Generalized hyperbolic distributions: Theory and applications to CDO pricing*. PhD thesis, University of Freiburg, 2011. Available at `http://www.freidok.uni-freiburg.de/volltexte/7974/`.

E.A. v. Hammerstein, E. Lütkebohmert, L. Rüschendorf, and V. Wolf. Optimality of payoffs in Lévy models. *International Journal of Theoretical and Applied Finance*, forthcoming, 2014.

POSTER SESSION

PRICING PARTICIPATING PRODUCTS UNDER REGIME-SWITCHING GENERALIZED GAMMA PROCESS

Farzad Alavi Fard

School of Economics, Finance and Marketing, RMIT, VIC 3001, Australia
Email: f.alavifard@gmail.com

We propose a model for valuing participating life insurance products under a regime-switching generalized gamma process. The Esscher transform is employed to determine an equivalent martingale measure in the incomplete market. The results are further manipulated through the utilization of the change of numeraire technique to reduce the dimensions of the pricing formulation. Due to the path dependency of the payoff of the insurance product and the non-existence of a closed-form solution for the PIDE, the finite difference method is utilized to numerically calculate the value of the product. To highlight some practical features of the product, we present a numerical example.

1. INTRODUCTION

Participating life insurance products are a popular class of equity linked insurance products around the world. In these policies the insured not only receives the guaranteed annual minimum benefit, but also receives proceeds from an investment portfolio. Grosen and Jorgensen (2000) provided a comprehensive discussion on different contractual features of participating policies.

In this article, we propose a model for valuing participating life insurance products under a regime-switching generalized gamma process, which is an extension of Fard and Siu (2013). Readers, may refer to this article for extended discussions for similar models and calculations.

2. MODELING FRAMEWORK

Consider a financial market, where an agent can either invest in a risk-free money market account or choose from a range of risky assets. All the parameters of the risk-free asset, as well as the risky assets, vary as the economy switches regimes, a process governed by a Markov-chain. We fix a complete probability space $(\Gamma, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the economy. Let $\{X_t\}_{t\in\mathcal{T}}$ on $(\Gamma, \mathcal{F}, \mathbb{P})$ be a continuous-time Markov chain with a finite state space $\mathcal{S} := (s_1, s_2, \cdots, s_N)$. The state space of the process $\{X_t\}_{t\in\mathcal{T}}$ is a finite set of unit vectors $\{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 1, \cdots, 0) \in \mathcal{R}^N$.

Let $\mathcal{Q}(t) = [q_{ij}(t)]_{i,j=1,2,\dots,N}, t \in \mathcal{T}$, denote a family of generators, or rate matrices, of the chain $\{X_t\}_{t\in\mathcal{T}}$ under $\mathbb{P}$. Then, as in Elliott et al. (1995):

$$X_t = X_0 + \int_0^t \mathcal{Q}X_s ds + M_t, \tag{1}$$

where M_t is a $\mathcal{R}^N$-valued martingale with respect to the filtration generated by $\{X_t\}_{t\in\mathcal{T}}$. In this article, any parameter Υ modulated by the Markov chain X_t is denoted by Υ_t, and defined as follows

$$\Upsilon_t := \Upsilon(t, X_t) := \langle \mathbf{\Upsilon}, X_t\rangle = \sum_{i=1}^{N} \Upsilon_i \langle X_t, e_j\rangle, \quad t \in \mathcal{T}, \tag{2}$$

where $\mathbf{\Upsilon} := (\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_N)$ with $\Upsilon_j > 0$ for each $j = 1, 2, \cdots, N$ and $\langle ., .\rangle$ denotes the inner product in the space $\mathcal{R}^N$.

Let $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ denote a measurable space, where $\mathcal{B}(\mathcal{T})$ is the Borel σ-field generated by the open subsets of $\mathcal{T}$. Let $\mathcal{X}$ denote $\mathcal{T} \times \mathcal{R}^+$, then $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is a measurable space. Let $N_{X_t}(., U)$ denote a Markov-switching Poisson random measure on the space $\mathcal{X}$. Write $N_{X_t}(dt, dz)$ for the differential form of measure $N_{X_t}(t, U)$. Let $\rho_{X_t}(dz|t)$ denote a Markov-switching Levy measure on the space $\mathcal{X}$ depending on t and the state X_t; η is a σ-finite (nonatomic) measure on $\mathcal{T}$. First, as in James (2005), consider the following Markov-modulated completely random measure

$$\mu_{X_t}(dt) := \int_{\mathcal{R}^+} h(z) N_{X_t}(dt, dz),$$

which is a kernel-biased. The generalized gamma (GG) process is a special case of the kernel-biased completely random measure and can be obtained by setting the kernel function h(z) = z. In this paper we are seeking a specific class of the GG processes that assist us in describing the impact of the states of an economy on the jump component. Hence, we use a MGG process, whose compensator switches over time, according to the states of the economy. Following, we present how to derive the intensity process for different classes of the MGG process.

Let $\alpha \geq 0$ and b_t denote the shape parameter and the scale parameter of the MGG process, defined according to (2). Then,

$$\rho_{X_t}(dz|t)\eta(dt) \quad = \quad \frac{1}{\Gamma(1-\alpha) z^{(1+\alpha)}} \sum_{i=1}^{N} e^{-b_i z} \langle \mathbf{b}, X_t\rangle dz \eta(dt). \tag{3}$$

When $\alpha = 0$, the MGG process reduces to a Markov modulated weighted gamma(MWG) process. When $\alpha = 0.5$ the MGG process becomes the Markov modulated inverse Gaussian (MIG) process.

Let $\{W_t\}_{t\in\mathcal{T}}$ denote a standard Brownian motion, and $\widetilde{N}_{X_t}(dt, dz)$ denote the compensated Poisson random measure. Further, let r_t be the instantaneous market interest rate, and μ_t and σ_t

denote the drift and volatility of the market value of the reference asset, respectively, all defined as per (2). Then, consider a generalized jump-diffusion process $A := \{A(t)|t \in \mathcal{T}\}$, such that

$$dA_t \;=\; A_{t^-}\left[\mu_t dt + \sigma_t dW_t + \int_{\mathcal{R}^+} z\widetilde{N}_{X_t}(dt, dz)\right], \tag{4}$$

where $A_0 = 0$. We assume under $\mathbb{P}$ the price process $\{S_t\}_{t\in\mathcal{T}}$ is defined as $S_t := \exp(A_t)$.

For each time $t \in \mathcal{T}$, let R_t and D_t denote the book value of the policy reserve and the bonus buffer, respectively. R_t is considered as the policyholder's account balance. Let S_t denote the market value of the asset backing the policy, so that $S_t = R_t + D_t,\ t \in \mathcal{T}$. The funds are distributed between two components of liability over time according to the bonus policy described by the continuously compounded interest rate credited to the policy reserve c_R,

$$dR_t \;=\; c_R(S, R)R_t dt.$$

Definition 2.1 *Let $g(S, R, X, T)$ denote the terminal payoff of the policy. Then the fair value*

$$V(S_T, R_T, X_T) = R_T + \gamma P_{1T} - P_{2T}, \tag{5}$$

where γ is the terminal bonus distribution rate, $P_{1T} := \max(\alpha_p S_T - R_T, 0)$ is the terminal bonus option, $P_{2T} := \max(R_T - S_T, 0)$ is the terminal default option, and R_T is the guaranteed benefit.

FAIR VALUATION OF THE PARTICIPATING POLICY

Define $\mathcal{G}_t$ for the σ-algebra $\mathcal{F}^X_t \vee \mathcal{F}^A_t$ for each $t \in \mathcal{T}$. For each θ_t defined as in (2) write $(\theta.A)_t := \int_0^t \theta(u)dA(u)$ such that θ is integrable with respect to the return process. Let $\mathcal{M}(\theta)_t := E[e^{(\theta.A)_t}|\mathcal{F}^X_t]$ be a Laplace cumulant process. Denote also in the following $\eta'(t)dt := \eta(dt)$. Then, let $\{\Lambda_t\}_{t\in\mathcal{T}}$ denote a $\mathcal{G}$-adapted stochastic process

$$\begin{aligned}\Lambda_t \;&:=\; e^{(\theta.A)_t}.\mathcal{M}(\theta)^{-1}_t, \\ &=\; e^{\int_0^t \theta_s\sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2\sigma_s^2 ds + \int_0^t\int_{\mathcal{R}^+}\theta_s z\widetilde{N}_{X_s}(dz,ds) - \int_0^t\int_{\mathcal{R}^+}(e^{\theta_s z}-1+\theta_s z)\rho_{X_s}(dz|s)\eta(ds)}.\end{aligned} \tag{6}$$

Proposition 2.1 *Λ_t is $\mathbb{P}$ martingale w.r.t. $\mathcal{G}_t$.*

Definition 2.2 *For each $\theta \in L(A)$ define $\mathbb{P}^\theta \sim \mathbb{P}$ on $\mathcal{G}(T)$ by the Radon-Nikodym derivative $\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\big|_{\mathcal{G}(T)} := \Lambda_T$.*

Proposition 2.2 *For each $t \in \mathcal{T}$, let the discounted price of the risky asset at time t be $\widetilde{S}(t) := e^{-rt}S(t)$. Then, $\widetilde{S} := \{\widetilde{S}(t)|t \in \mathcal{T}\}$ is a $\mathbb{P}^\theta$-local-martingale if and only if θ_t satisfies :*

$$\theta_t\sigma_t^2 \;+ \int_{\mathcal{R}^+}\left\{e^{\theta_t z}(e^z - 1) - z\right\}\rho_{X_t}(dz|t)\eta'(t) = r_t - \mu_t. \tag{7}$$

Proposition 2.3 *Suppose $\widetilde{W}_t = W_t - \int_0^t \sigma_s \theta_s ds$ is a $\mathbb{P}^\theta$-Browning motion, $\rho^\theta_{X_t}(dz|t) := e^{\theta_s z}\rho_{X_t}(dz|t)$ is the $\mathbb{P}^\theta$ compensator of $N^\theta_{X_t}(dz, dt)$ then*

$$dA_t = \big(\mu_t + 2\theta_t\sigma_t^2 - \frac{1}{2}\sigma_t^2\big)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+} z\big(1 - e^{-\theta_t z}\big)\rho^\theta_{X_t}(dz|t)\eta(dt) + \int_{\mathcal{R}^+} z\widetilde{N}^\theta_{X_t}(dz, dt).$$

As in Hansen and Jorgensen (2000), we choose an alternative numeraire to reduce the number of state variables. Define a new state variable $Z := \ln(\frac{S}{R})$, so that $C_Z(Z) = C_R(S, R, X, t)$. Then, under $\mathbb{P}^\theta$ we define

$$\mathcal{E}(t) := \exp\Bigg(- \int_0^t (r_s - \frac{1}{2}\sigma_s^2)ds\Bigg)\frac{S_t}{S_0} \tag{8}$$

We notice that $\mathcal{E}(t)$ is martingale w.r.t. $\mathcal{G}_t$ (Oksendal and Sulem (2005), chapter 1).

Definition 2.3 *Define $\mathbb{Q} \sim \mathbb{P}^\theta$ on $\mathcal{G}(T)$ by the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}^\theta}\big|_{\mathcal{G}(T)} := \mathcal{E}(T)$.*

Also, define $W^{\mathbb{Q}}_t := \widetilde{W}_t - \int_0^t \sigma_s ds$, and $N^{\mathbb{Q}}_{X_t}(dz, dt) := N^\theta_{X_t}(dz, dt) - f(z, \theta)\rho^\theta_{X_t}(dz|t)\eta(dt)$, where $f(z, \theta_t) := e^{\theta_t z}(e^z - 1) - z$ for convenience in presentation. Then by a version of the Girsanov theorem

$$dS_t = (r_t + \tfrac{1}{2}\sigma_t^2)dt + \sigma_t dW^{\mathbb{Q}}_t + \textstyle\int_{\mathcal{R}^+}\Big(e^{\theta_t z}(e^z - 1) - z\Big)\widetilde{N}^{\mathbb{Q}}_{X_t}(dz, dt).$$

Consequently, by Ito's lemma, the dynamic of Z under $\mathbb{Q}$ is

$$\begin{aligned} dZ_t &= \Big(r_t - C_Z(Z_t)\Big)dt + \sigma_t dW^{\mathbb{Q}}_t + \int_{\mathcal{R}^+} \ln(1 + f(z, \theta_t))\widetilde{N}^{\mathbb{Q}}_{X_t}(dt, dz), \\ &+ \int_{\mathcal{R}^+}\Big(\ln(1 + f(z, \theta_t)) - f(z, \theta_t)\Big)\rho^{\mathbb{Q}}_{X_t}(dz|t)\eta(dt). \end{aligned} \tag{9}$$

Where, $\rho^{\mathbb{Q}}_{X_t}(dz|t)\eta(dt)$ is defined under $\mathbb{Q}$ for $\widetilde{N}^{\mathbb{Q}}(dt, dz)$.

Proposition 2.4 *The valuation of V_t using the process Z under $\mathbb{Q}$, is equivalent to that from process S under $\mathbb{P}^\theta$.*

Proof. Let $E^{\mathbb{Q}}$ and E^θ be the expectation operator under $\mathbb{Q}$ and $\mathbb{P}^\theta$, respectively. Then, by Bayes' rule

$$\begin{aligned} V_t &= E^\theta\Big[\exp\Big(- \int_t^T r_s ds\Big)V(S, R, X, T)\Big|\mathcal{G}_t\Big] \\ &= S_t E^{\mathbb{Q}}\Big[e^{-Z_T}V_Z(Z, X, t)\Big|(Z_t, X_t) = (Z, X)\Big]. \end{aligned}$$

We call $\bar{V}_Z(Z, X, t)$, S-denominated value of the contract; that is

$$\bar{V}_Z(Z, X, t) = E^{\mathbb{Q}}\Big[e^{-Z_T}V_Z(Z, X, T)\Big|(Z_t, X_t) = (Z, X)\Big].$$

■

Corollary 2.5 *(Z_t, X_t) is a two-dimensional Markov process with respect to the filtration $\mathcal{G}_t$.*

Corollary 2.6 *The S-denominated value of the participating product $\bar{V}_Z(Z, X, t)$ is $\mathbb{Q}$ martingale.*

Further, write $\bar{V}_i$ for $\bar{V}(Z, e_i, t)$, where $i = 1, 2, ..., N$ and $\bar{\mathbf{V}} := \{\bar{V}_1, \bar{V}_2, ..., \bar{V}_N\}$. Then, as in Buffington and Elliot (2002), $\bar{\mathbf{V}}$ satisfies the following N PIDEs:

$$\mathcal{L}_{Z,e_i}(\bar{V}_i) + \langle \bar{\mathbf{V}}, \mathcal{Q}e_i \rangle = 0, \qquad i = 1, 2, ..., N, \tag{10}$$

where $\mathcal{L}_{Z,e_i}(\bar{V}_i)$ is the differential operator. Further, as a result of the Corollary2.6, we have

$$\begin{aligned} d\bar{V}_Z(Z, X, t) &= \int_{\mathcal{R}^+} \Big\{ \bar{V}_Z\Big(Z_{t^-} + \ln(1 + f(z, \theta_t)) \Big) - \bar{V}_Z(Z_{t^-}) \Big\} \widetilde{N}^{\mathbb{Q}}_{X_t}(dt, dz) \\ &+ \frac{\partial \bar{V}_Z}{\partial Z}(Z, X, t)\sigma dW^{\mathbb{Q}}_t + \langle \bar{\mathbf{V}}, \mathcal{Q}X_t \rangle dt. \end{aligned}$$

With the auxiliary condition $\bar{V}_Z(Z, X, T) = e^{-Z_T} + \gamma \max(\alpha - e^{-Z_T}, 0) + \max(e^{-Z_T} - 1, 0)$.

3. NUMERICAL ANALYSIS

There is no known analytical solution to (10). We employ the explicit finite difference method to approximate the solution numerically. Let $[0, Z^{\max}] \times [0, T]$ denote the finite computational domain, where the width of the spatial interval is chosen to be sufficiently large. The derivatives of the value function $V(Z, X, t)$ in equations (10) can be replaced by the finite differences and the integral terms are approximated by using the trapezoidal rule at first. The computational domain is discretized into a finite difference mesh, where ΔZ and Δt are the step-width and time step, respectively. In order to approximate the integral term, we adopt the trapezoidal rule with the same spatial grids. By the explicit finite difference scheme, we start from the terminal values, and move backwards in time so that we can calculate the value function.

Assume an economy with two states, namely, where $X_t = 1$ and $X_t = 2$ represent 'Good' and 'Bad' economies, respectively. Let $\mathcal{P}(t) = [p_{ij}]$ be the transition probability matrix for time t. Note that in Section 2, we characterize the Markov chain using the matrix of transition rates, $\mathcal{Q}$, so $\mathcal{P}(t)$ must be calculated by solving (1), first. To illustrate our model, we assume $p_{11} = p_{22} = 0.40$, $r_1 = 0.035$, $r_2 = 0.015$, $b_1 = 200.00$, $b_2 = 500.00$, $\beta = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\mu_1 = 0.10$, $\mu_2 = 0.05$, $S_0 = 100$ and $\gamma = 0.7$. The term to maturity of the contract is $T = 20$ years, Δt is assumed to be one trading day ($\Delta t = 1/252$). For the Merton jump diffusion model, we consider the drift and the dispersion of the reference portfolio as well as the risk-free rate to be equal to the corresponding parameters in the no-regime-switching version of the model. In addition, we assume the intensity parameter of the model to be 60%, and the jump size of the compound Poisson process follows a normal distribution of $N(-0.05, 0.49)$.

Figure 1 presents the impact of α on the fair values of the participating policy, calculated with the above model specifications. The graph shows a meaningful difference between the fair values of the policy, with and without switching regimes. For example, when $\alpha = 0.2$, the fair

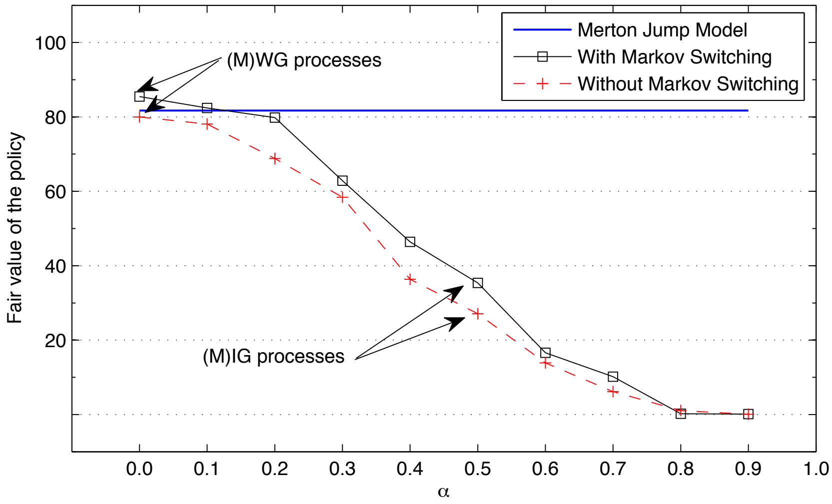


Figure 1: The fair value of the participating policy, with general (M)GG processes.

value calculated without regime-switching is 16.01% lower than the fair value of the contract with regime-switching. This difference is as high as 73.33% for the fair values under the two scenarios with $\alpha = 0.9$. We also document the significant effect of α on the values of the contracts for both cases.

References

J. Buffington and R.J. Elliot. American options with regime switching. *International Journal of Theoretical and Applied Finance*, 5:497–514, 2002.

R. J. Elliott, L. Aggoun, and J. B. Moore. *Hidden Markov Models: Estimation and Control, vol. 29 of Applications of Mathematics*. Springer, 1995.

F.A. Fard and T.K. Siu. Pricing participating products with markov-modulated jumpdiffusion process: An efficient numerical pide approach. *Insurance: Mathematics and Economics*, 53(3): 712 – 721, 2013. ISSN 0167-6687. doi: http://dx.doi.org/10.1016/j.insmatheco.2013.09.011.

A. Grosen and P.L. Jorgensen. Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender options, and bonus policies. *Insurance, Mathmatics and Economics*, 26: 37–57, 2000.

M. Hansen and P.L. Jorgensen. Analytical valuation of American-style Asian options. *Management Science*, 46:1116–1136, 2000.

L.F. James. Bayesian Poisson process partition calculus with an application to Bayesian Levy moving averages. *The Annals of Statistics*, 33:1771–1799, 2005.

B. Oksendal and A. Sulem. *Applied Stochastic Control of Jump Difussion*. Springer, 2005.

RISK CLASSIFICATION FOR CLAIM COUNTS USING FINITE MIXTURE MODELS

Lluís Bermúdez[†] **and Dimitris Karlis**[§]

[†]*Riskcenter, University of Barcelona*
[§]*Athens University of Economics and Business*
Email: `lbermudez@ub.edu, karlis@aueb.gr`

When modelling insurance claim counts data, the actuary often observes overdispersion and excess of zeros that may be caused by the unobserved heterogeneity. A common approach for accounting for overdispersion is to consider models with some overdispersed distribution as opposed to Poisson models. Zero-inflated, hurdle and compound frequency models are usually applied to insurance data to account for such features of the data. A natural way to allow for overdispersion is to consider mixtures of a simpler model. In this paper, we consider a K-finite mixture of Poisson and Negative Binomial regressions. This approach has interesting features: first, the zero-inflated model represents a special case; and second, it allows for an elegant interpretation based on the typical clustering application of finite mixture models. These models are applied to an automobile insurance claims data set in order to analyse the consequences for risk classification.

1. INTRODUCTION AND MOTIVATION

Risk classification based on generalized linear models is usually accepted. A regression component is included in the claim count distribution to take the individual characteristics into account.

In insurance data sets, for claim count modelling purposes, the Poisson regression model is usually rejected because of the presence of overdispersion. This rejection may be interpreted as the sign that the portfolio is heterogeneous.

In automobile insurance, the problem of unobserved heterogeneity is caused by the differences in driving behaviour among policyholders that cannot be observed or measured by the actuary (i.e. driving ability, driving aggressiveness, obedience of traffic regulations). The omission of these important classification variables may be the reason for the overdispersion detected. Meanwhile, the presence of excess of zeros can be also seen as a consequence of unobserved heterogeneity.

Many attempts have been made in the actuarial literature to account for such features of the data. Compound frequency models, zero-inflated models and hurdle models are usually applied to insurance claim count data. Boucher et al. (2007) present and compare different risk classification models for the annual number of claims reported to the insurer: Poisson-Gamma (or Negative Binomial) model, Poisson-Inverse Gaussian model, Poisson-Log Normal model, Zero-inflated models and Hurdle models.

In the bivariate setting, in a recent paper, Bermúdez (2009) used bivariate Poisson regression models for ratemaking in car insurance to account for the dependence between two different types of claims, including zero-inflated models to account for the excess of zeros and the overdispersion in the data set. In Bermúdez and Karlis (2012), these models were revisited in order to consider alternatives, proposing a 2-finite mixture of bivariate Poisson regression models to demonstrate that the overdispersion in the data requires more structure if it is to be taken into account and that a simple zero-inflated bivariate Poisson model does not suffice.

In this paper, to account for overdispersion and excess of zeros in the univariate setting, we consider a K-finite mixture of Poisson and Negative Binomial (NB) regressions. The finite mixtures of Poisson or NB regression models are especially useful where count data were drawn from heterogeneous populations as Park and Lord (2009) show for vehicle crash data analysis.

2. K-FINITE MIXTURE OF REGRESSION MODELS

Let Y be the number of claims for automobile insurance, we consider the K-finite mixture of Poisson or NB regressions as

$$P(Y_i = y_i) = \sum_{j=1}^{K} p_j P(y_i; \mu_{ij}) \quad \text{or} \quad P(Y_i = y_i) = \sum_{j=1}^{K} p_j \mathit{NB}(y_i; \mu_{ij}, \theta_j)$$

respectively, where $p_j > 0$ ($j = 1, \ldots, K$) are the mixing proportions with $\sum p_j = 1$.

For the NB regression model, we assume

$$\mathit{NB}(y; \mu, \theta) = \frac{\Gamma(\theta + y)}{\Gamma(\theta) y!} \left(\frac{\mu}{\theta + \mu}\right)^{y} \left(\frac{\theta}{\theta + \mu}\right)^{\theta}, \quad \mu, \theta > 0, \ \ y = 0, 1, \ldots$$

i.e. the probability function of a NB with mean μ and variance $\mu + \frac{\mu^2}{\theta}$.

Furthermore, in both cases, we assume for the mean of the j-th components that it relates to some covariate vector, namely we assume that

$$\log(\mu_{ij}) = \mathbf{x}_i \beta_j$$

where $\mathbf{x}_i$ is the vector of covariates related to the i-th individual and β_j is the vector of regression coefficients for the j-th component.

This modelling has some interesting features: first of all, the zero inflated model is a special case; secondly, it allows for overdispersion; and thirdly, it allows for a neat interpretation based on the typical clustering usage of finite mixture models.

We fitted the model by a using standard EM algorithm. For the NB mixture, at the E-step we estimated the weights w_{ij} as

$$w_{ij} = p_j \mathit{NB}(y_i; \mu_{ij}, \theta_j) \, / \, \sum_{j=1}^{K} p_j \mathit{NB}(y_i; \mu_{ij}, \theta_j)$$

and during the M-step we update β_j by fitting a standard NB regression model with response $\mathbf{y}$, covariates $\mathbf{x}$ and weights $\mathbf{w}_j = (w_{1j}, \dots, w_{nj})$.

Initial values for $K = 2$ were selected by perturbing a simple NB regression model. Namely we fitted a single NB regression and keeping the fitted values, we split them in two components with mixing probabilities 0.5 each one and means equal to 1.2 and 0.8 of the fitted values. Then in order to fit a model with $K + 1$ components we used the solution with K components and a new component at the center (that of a single NB regression), with mixing probability 0.05. The rest mixing probabilities were rescaled to sum to 1. Extensive simulation have shown that this approach works well to locate the maximum.

3. APPLICATION

The data contains information for 80,994 policyholders from a major insurance company operating in Spain. Twelve exogenous variables were considered plus the annual number of accidents recorded. The description of the explanatory variables is presented in table 1.

We have fitted models of added complexity to this data set, starting from a simple Poisson regression model. We have used AIC to select the best among a series of candidate models. All models were run in `R`. In table 2 we have compared the fitted models, resulting that the best fit was obtained with the 2-Finite NB mixture model. Finite mixture models with $K > 2$ were also fitted, but no improvement in terms of AIC were achieved. In table 3 the results for the 2-Finite NB mixture model are summarized.

Variable	Definition
V1	equals 1 for women and 0 for men
V2	equals 1 when driving in urban area, 0 otherwise
V3	equals 1 when zone is medium risk (Madrid and Catalonia)
V4	equals 1 when zone is high risk (Northern Spain)
V5	equals 1 if the driving license is between 4 and 14 years old
V6	equals 1 if the driving license is 15 or more years old
V8	equals 1 if the client is in the company for more than 5 years
V9	equals 1 of the insured is 30 years old or younger
V10	equals 1 if includes comprehensive coverage (except fire)
V11	equals 1 if includes comprehensive and collision coverage
V12	equals 1 if horsepower is greater than or equal to 5500cc

Table 1: Explanatory variables used in the models

Model	Log-lik	Parameters	AIC
Poisson	-24172.5	12	48369.00
Negative Binomial	-22442.8	13	44911.60
Poisson-IG	-22464.0	13	44954.00
Poisson-LN	-22509.7	13	45045.46
ZIP	-22515.4	13	45056.86
ZIPIG	-22464.0	14	44956.00
ZINB	-22442.8	14	44913.60
Hurdle Poisson	-22554.2	13	45134.38
Hurdle NB	-22489.8	14	45007.60
2-Finite Poisson mixture	-22493.2	25	45036.46
2-Finite NB mixture	-22419.0	27	44892.06

Table 2: Information criteria for selecting the best model for the data

variable	1st component			2nd component		
	coef.	s.e.	p-value	coef.	s.e.	p-value
const.	-3.0017	0.7981	0.0002	-1.8786	0.6391	0.0033
V1	-0.2562	0.1119	0.0220	0.1363	0.0566	0.0160
V2	-0.0002	0.0744	0.9984	-0.0698	0.0385	0.0700
V3	0.1091	0.0847	0.1978	-0.0267	0.0463	0.5641
V4	0.1230	0.0928	0.1851	0.1972	0.0468	< 0.0001
V5	1.2946	0.5298	0.0145	-2.0342	0.9584	0.0338
V6	-5.5294	11.7203	0.6371	-0.2416	0.2635	0.3593
V8	0.0076	0.0845	0.9279	0.2290	0.0553	< 0.0001
V9	0.1558	0.0839	0.0634	-0.0523	0.2585	0.8398
V10	-0.0690	0.1128	0.5407	0.0997	0.0537	0.0636
V11	0.0918	0.0735	0.2121	0.0504	0.0416	0.2262
V12	-0.2065	0.0944	0.0287	0.1533	0.0492	0.0018
θ	0.3040	0.2130		0.3270	0.2460	
p	0.4788			0.5212		

Table 3: 2-Finite mixture of NB regression model

4. CONCLUSIONS

In this paper, we have proposed the use of a 2-finite mixture of Poisson and Negative Binomial regressions to allow for the overdispersion and the excess of zeros usually detected in automobile insurance dataset. Assuming the existence of two type of clients described separately by each component of the mixture improves the modelling of the dataset. The idea is that the data consist of subpopulations for which the regression structure is different. The model corrects for zero inflation and overdispersion.

The existence of “true” zeros assumed by zero-inflated or Hurdle models may be a too strong assumption in some cases. However, the 2-finite mixture of Poisson or Negative Binomial regression does not make this somewhat strict assumption and allows mixing with respect to both zeros and positives. This idea is more flexible than zero-inflated and Hurdle models and it holds better in our case. As it can be seen in figure 1, the group separation is characterized by low mean for the first component (“good” drivers) and high mean with higher variance for the second one (“bad” drivers).

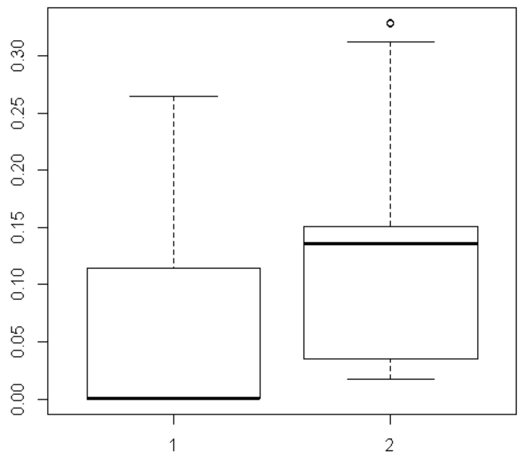


Figure 1: Boxplots of the fitted means for each of the two components

Finally, as it seems that the data set may have been generated from two distinct subpopulations, the model allows for a net interpretation of each cluster separately. Note that different regression coefficients can be used to account for the “observed” heterogeneity within each population.

References

L. Bermúdez. A priori ratemaking using bivariate Poisson regression models. *Insurance: Mathematics and Economics*, 44(1):135–141, 2009.

L. Bermúdez and D. Karlis. A finite mixture of bivariate poisson regression models with an application to insurance ratemaking. *Computational Statistics & Data Analysis*, 56:3988–3999, 2012.

J.P. Boucher, M. Denuit, and M. Guillén. Risk classification for claim counts: a comparative analysis of various zero-inflated mixed poisson and hurdle models. *North American Actuarial Journal*, 11(4):110–131, 2007.

B.J. Park and D. Lord. Application of finite mixture models for vehicle crash data analysis. *Accident Analysis and Prevention*, 41(4):683–691, 2009.

TWO EFFICIENT VALUATION METHODS OF THE EXPOSURE OF BERMUDAN OPTIONS UNDER THE HESTON MODEL

Q. Feng[1], C.S.L. de Graaf [2], D. Kandhai[2,3] and C.W. Oosterlee[1,4]

[1]*Center for Mathematics and Computer Science (CWI), Science Park 123, 1098 XG Amsterdam, The Netherlands*
[2]*University of Amsterdam, Kruislaan 403, 1098 SJ Amsterdam, The Netherlands*
[3]*ING BANK, Amsterdamse Poort Bijlmerplein 888, 1102 MG Amsterdam, The Netherlands*
[4]*Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands*
Email: qian@cwi.nl

1. INTRODUCTION

Expected Exposure (EE) is one of the key elements in Credit Value Adjustment (CVA) (Gregory 2012). In credit risk management, the Potential Future Exposure (PFE) is another important indicator. It indicates the maximum expected exposure at some level of confidence. Valuation of either EE or PFE of path-dependent derivatives has the additional difficulty that some event may happen at any time over the entire path, either earlier or later than the valuation time.

A Bermudan option is an option where the buyer has the right to exercise at a set of times, which we may call watch times. If the option is exercised, the exposure disappears as the holder has realized the value of the option; otherwise, the holder will lose the current value of the option if the counterparty defaults. With simulated scenarios, the option value at each exercise time can be calculated; when a large number of scenarios is available, we can get an empirical distribution of the option value at each time point. It is easy to get the EE and PFE when the distribution of the exposure is known.

In de Graaf et al. (2014), three computational techniques for approximation of counterparty exposure for financial derivatives are presented. This abstract focuses on the introduction on one of them: the Stochastic Grid Bundling method (SGBM). We will show that the EE and PFE are natural by-products when pricing a Bermudan option applying SGBM. The Greeks of the exposure can be calculated at the same time without additional computational costs.

The COS method in Fang (2010) also offers an efficient way for pricing a Bermudan option under the Heston model. We extend it for pricing the exposure values of the Bermudan options. We use the results of the COS method as the reference value.

COS and SGBM are fundamentally different: one is aimed to recover the conditional density from the characteristic function based on Fourier-cosine expansion while the other employs regression to approximate the conditional distribution based on bundling and simulation. Monte Carlo simulation is needed for generating the scenarios/the stochastic grid. We apply the QE scheme (Andersen 2007) with a martingale correction for generating the paths under the Heston model.

2. BERMUDAN OPTIONS AND THE HESTON MODEL

A Bermudan option is defined as an option where the buyer has the right to exercise at a set number of (discretely spaced) times. The exercise time set is denoted by

$$\mathcal{T} = \{t_1, t_2, \ldots, t_M\}, \tag{1}$$

where M denotes the number of exercise times, and the time difference is Δt. At initial time t_0 exercising is not allowed.

The dynamics of the Heston stochastic volatility model (Heston 1993) are given by

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1, \\ dv_t = \kappa(\eta - v_t)dt + \sigma\sqrt{v_t} dW_t^2, \\ dW_t^1 dW_t^2 = \rho dt, \end{cases} \tag{2}$$

where

- W_t^1 and W_t^2 are two Wiener processes correlated by ρ;
- κ is the mean-reversion speed;
- η the mean reversion level;
- σ the so-called volatility of volatility;
- r the risk-free interest rate.

Knowing the market condition (S_m, v_m) at time t_m, the immediate exercise and the continuation value for the Bermudan option at time t_m are, respectively, defined as:

$$g(S_m) = \max(\gamma(S_m - K), 0) \quad \text{with} \quad \gamma = \begin{cases} 1, & \text{for a call} \\ -1, & \text{for a put} \end{cases}, \tag{3}$$

$$c(S_m, v_m, t_m) = e^{-r\Delta t}\mathbb{E}\left[U(S_{m+1}, v_{m+1}, t_{m+1}) \middle| (S_m, v_m)\right], \tag{4}$$

where $U(S_{m+1}, v_{m+1}, t_{m+1})$ is the option value at time t_{m+1}.

The holder of the option will exercise when the exercise value is higher than the continuation value, and then the contract terminates. At maturity t_M, the option value is equal to the exercise value. The following recursive scheme based on dynamic programming can be set up to price a Bermudan option:

$$U(S_m, v_m, t_m) = \begin{cases} g(S_M), & \text{for } m = M; \\ \max\left[c(S_m, v_m, t_m), g(S_m)\right], & \text{for } m = 1, 2, \cdots, M-1; \\ c(S_0, v_0, t_0), & \text{for } m = 0. \end{cases} \tag{5}$$

By definition of exposure, the Bermudan option exposure at time t_m can thus be formulated as:

$$\mathrm{E}(t_m) = \begin{cases} 0, & \text{if exercised,} \\ c(S_m, v_m, t_m), & \text{if not exercised,} \end{cases} \quad m = 1, 2, \cdots, M-1. \tag{6}$$

In addition, $\mathrm{E}(t_0) = c(S_0, v_0, t_0)$ and $\mathrm{E}(t_M) = 0$.

CVA can be seen as the price of counterparty credit risk, while PFE is a measure for the potential loss (Gregory 2012). In other words, CVA depends on the EE, while PFE is the loss given a fixed confidence interval. Both measures depend on the future distribution of exposure. The key problem becomes how to determine the exposure distribution along the time horizon. in particular, it is a difficult task to calculate the continuation value at each exercise time.

3. STOCHASTIC GRID BUNDLING METHOD

The Stochastic Grid Bundling method (SGBM) is based on *simulation*, *bundling* and *regression* for pricing Bermudan options. It consists of the following steps:

Step 1: (forward) simulation Generating forward scenarios/paths by Monte Carlo simulation. It is the stochastic grid on which we will make the calculation. The QE method (Andersen 2007) is applied to generate paths.

Step 2: (backward) bundling At time t_m, $m = M-1, \ldots, 1$, all paths are clustered into β bundles; the bundle set at time t_m is denoted by $\{\mathcal{B}^{p,m}\}_{p=1}^{\beta}$. There are several schemes available to make bundles, and we choose the recursive bifurcation method (Jain and Oosterlee 2013). Figure 1 shows how the bundles are made at time t_{M-1}.

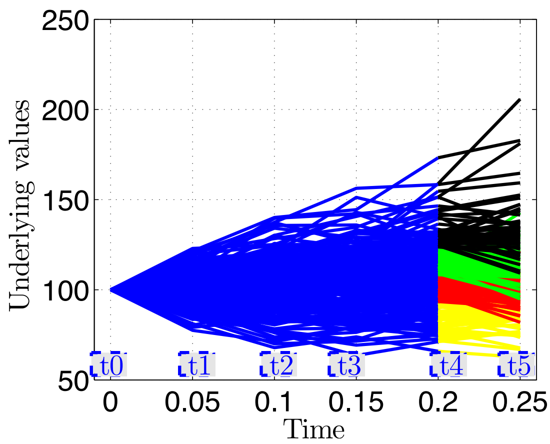


Figure 1: Bundling at time t_{M-1}.

Step 3: (backward) regression Regression is used to calculate the continuation value at time t_m, $m = 0, \ldots, M-1$. The essential idea of SGBM is that the option value can be written as a linear combination of a set of basis functions. We choose the set of the basis functions $\{g_k(S,v)\}_{k=0}^{B}$ in such a way that analytic formulas of their conditional expectations are available. The basic idea is the same as the Longstaff-Schwarz method (Longstaff and Schwartz 2001).

However, there is an important difference between SGBM and the Longstaff-Schwartz method. In SGBM, the coefficients are different among each bundle. We assume that, for paths in the p-th bundle $\mathcal{B}^{p,m}$ at time t_m, a set of coefficients $\{\alpha_k^{p,m}\}_{k=0}^{B}$ exists, so that for the option values of these paths at time t_{m+1}, the following relationship holds

$$U\left(S_{m+1}, v_{m+1}, t_{m+1}\right) \approx \sum_{k=0}^{B} \alpha_k^{p,m} g_k(S_{m+1}, v_{m+1}), \tag{7}$$

where the coefficient set $\{\hat{\alpha}_k^{p,m}\}_{k=0}^{B}$ can be approximated by regression when the option values $U\left(S_{m+1}, v_{m+1}, t_{m+1}\right)$ at the stochastic paths have been determined.

Equation (7) can be substituted into (4), which gives us:

$$\begin{aligned} c(S_m, v_m, t_m) &= e^{-r\Delta t}\mathbb{E}\left[U(S_{m+1}, v_{m+1}, t_{m+1})\middle|(S_m, v_m)\right] \\ &\approx e^{-r\Delta t}\mathbb{E}\left[\sum_{k=0}^{B} \hat{\alpha}_k^{p,m} g_k(S_{m+1}, v_{m+1})\middle|(S_m, v_m)\right] \\ &= e^{-r\Delta t}\sum_{k=0}^{B} \hat{\alpha}_k^{p,m} f_k(S_m, v_m), \end{aligned} \tag{8}$$

where $f_k(S_m, v_m)$ represents the conditional expectations of the basis functions $g_k(S_{m+1}, v_{m+1})$. Consequently, we can approximate the option and exposure value of each path at time t_m.

We repeat Steps 2 and 3 backwards in time, until the initial time t_0.
Moreover, the sensitivity of EE w.r.t. S_0 can be obtained directly from (8), as:

$$\Delta(S_m, v_m, t_m) = \begin{cases} 0, & \text{if exercised}, \\ e^{-r\Delta t}\sum_{k=0}^{B} \hat{\alpha}_k^{p,m} \frac{\partial f_k(S_m, v_m)}{\partial S_m} \cdot \frac{\partial S_m}{\partial S_0}, & \text{if not exercised}, \end{cases} \tag{9}$$

where $m = 1, 2, \cdots, M-1$, and for calculation of the $\frac{\partial S_m}{\partial S_0}$ term, we note that at time τ

$$S_\tau = S_0 e^{(r-\frac{1}{2}v_\tau)\tau + \sqrt{v_\tau}W_\tau^1}, \tag{10}$$

as the variance follows CIR dynamics, we can write the derivative of S_τ w.r.t S_0 as

$$\frac{\partial S_\tau}{\partial S_0} = e^{(r-\frac{1}{2}v_\tau)\tau + \sqrt{v_\tau}W_\tau^1} = \frac{S_\tau}{S_0}. \tag{11}$$

At time t_0, the sensitivity of EE w.r.t. S_0 is then given by

$$\Delta(S_0, v_0, t_0) = e^{-r\Delta t}\sum_{k=0}^{B} \hat{\alpha}_k^{0} \frac{\partial f_k(S_0, v_0)}{\partial S_0}. \tag{12}$$

Notice that there is no need for bundling at time t_0.

4. NUMERICAL RESULTS

In this paper, we choose the basis functions as

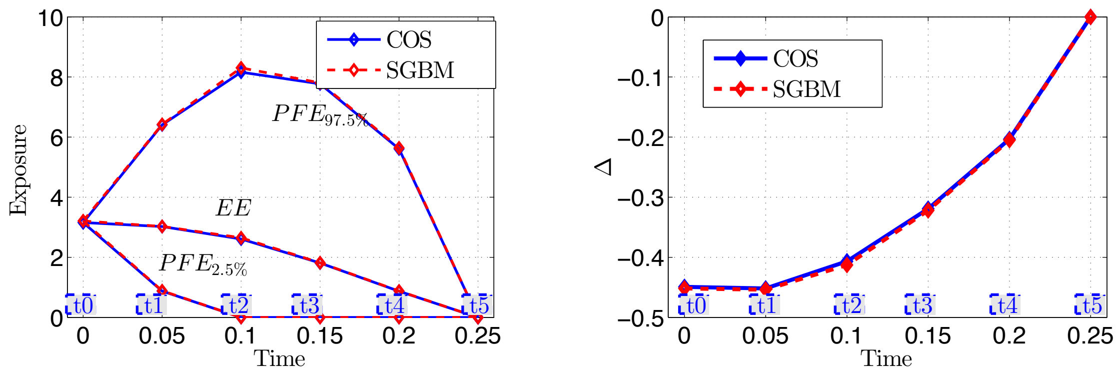
$$\text{constant}(1), \log(S), (\log(S))^2, (\log(S))^3, (\log(S))^4. \tag{13}$$

We choose the set of parameters presented in Table 1.

Parameter	Value
Spot (S_0)	100.0
Strike (K)	100
Interest (r)	0.04
Variance (v_0)	0.0348
Tenor (T)	0.25
Mean Reversion (κ)	1.15
Mean Variance (η)	0.0348
Vol of Var (σ)	0.459
Correlation (ρ)	0

Table 1: Parameter set for test.

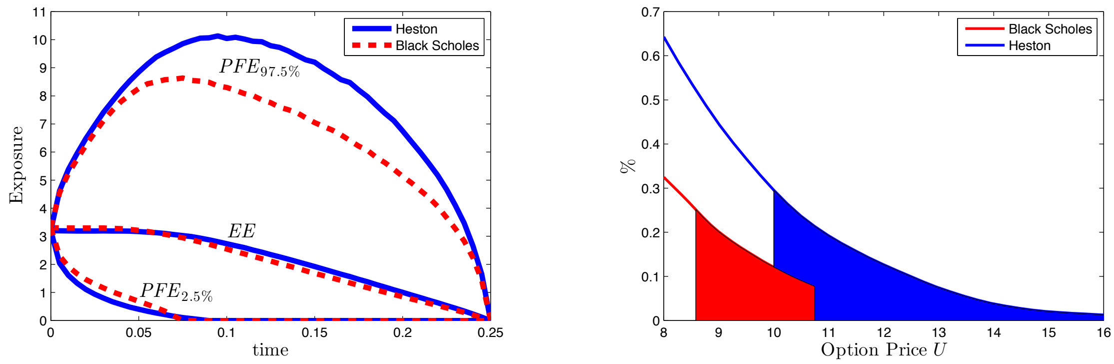
The COS method is used to get the reference value. The results of exposure and sensitivity of a Bermudan put option based on the parameter set with 5 exercise times is presented in Figure 2.



(a) EE and PFE. (b) sensitivity Δ of exposure.

Figure 2: Comparison between SGBM and COS.

We compare the results under the Black-Scholes model and the Heston model separately and see the impact of the stochastic volatility. The constant variance level in the Black-Scholes model is made equal to the mean reversion level in Heston model. Figure 3 shows that the PFE is more effected by a stochastic volatility compared to EE. It is because that the exposure distribution under stochastic volatility give rise to a fatter right tail.



(a) EE and PFE.

(b) The right tail of the distribution.

Figure 3: Impact of the stochastic volatility: comparison between results under the Black-Scholes model and the Heston model.

5. CONCLUSION

SGBM is a quite efficient method with high accuracy. There are some similarities with the well-known Longstaff-Schwartz method, but fundamentally different from it in the following ways:

- All the paths are used for regression instead of only 'in the money' paths;
- The optimal stopping strategy and cash flow is merely a by-product of SGBM;
- By applying bundling, the approximation of the regression coefficients is optimized locally;
- Information from the dynamics is included by using the analytic formulas for the expectation of the basis functions.

Comparison between the results under the Black-Scholes model and the Heston model indicates that the stochastic volatility has a strong impact on PFE.

References

L.B.G. Andersen. Efficient simulation of the Heston stochastic volatility model, 2007. URL `http://ssrn.com/abstract=946405orhttp://dx.doi.org/10.2139/ssrn.946405`.

C.S.L. de Graaf, Q. Feng, D. Kandhai, and C.W. Oosterlee. Efficient computation of exposure profiles for counterparty credit risk, 2014. URL `http://dx.doi.org/10.2139/ssrn.2395183`.

F. Fang. *The COS Method: An Efficient Fourier Method for Pricing Financial Derivatives*. PhD thesis, Technische Universiteit Delft, 2010.

J. Gregory. *Counterparty credit risk and credit value adjustment: A continuing challenge for global financial markets - Second Edition*. John Wiley & Sons, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8Q, United Kingdom, 2012.

S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

S. Jain and C.W. Oosterlee. The stochastic grid bundling method: Efficient pricing of bermudan options and their greeks, 2013. URL `http://dx.doi.org/10.2139/ssrn.2293942`.

F.A. Longstaff and E.S. Schwartz. Valuing american options by simulation: a simple least-squares approach. *Review of Financial Studies*, 14(1):113–147, 2001.

EVOLUTION OF COPULAS AND ITS APPLICATIONS

Naoyuki Ishimura

Graduate School of Economics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan
Email: `ishimura@econ.hit-u.ac.jp`

Copulas are known to provide a flexible tool for describing possible nonlinear relations among risk factors. Except for several examples, however, copulas are mainly concerned with the static problems, not with the time-dependent processes. Here, we review our recent studies on the evolution of copulas, which assumes that a copula itself evolves according to the time variable, and consider its slight generalization in discrete processes. Possible applications of our evolution of copulas are also discussed.

1. INTRODUCTION

A copula is a well-employed tool for investigating the dependence structure among risk factors. Copulas make a link between multivariate joint distributions and univariate marginal distributions. Because of their flexibility, copulas have been extensively studied and applied in a wide range of areas concerning the problem of dependence relations, which include, to name a few, actuarial and insurance mathematics, financial mathematics, hydrology, seismology, and so on. For a general reference on the theory of copulas, we refer to the book by Nelsen (2006).

Copulas, however, are concerned mainly with the stationary situation and not with the time-dependent circumstance. There exist of course a few attempts which deal with the time variable in the copula theory. We recall the study on copulas with Markov processes by Darsow et al. (1992), and also on dynamic copulas by Patton (2006).

In our research, we introduce the concept of the evolution of copulas both in continuous and discrete time, which proclaims that a copula itself evolves according to the time variable. To start with the continuous processes, let $\{C(u, v, t)\}_{t \geq 0}$ be a time parameterized family of bivariate copulas, which satisfy the heat equation:

$$\frac{\partial C}{\partial t}(u, v, t) = \Big(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\Big) C(u, v, t). \tag{1}$$

By the definition of a copula, $C(\cdot, \cdot, t)$ verifies the following conditions:

(i) $C(u, 0, t) = C(0, v, t) = 0$, $C(u, 1, t) = u$ and $C(1, v, t) = v$;

(ii) For every $(u_i, v_i, t) \in I^2 \times (0, \infty)$ $(i = 1, 2)$ with $u_1 \leq u_2$ and $v_1 \leq v_2$, it follows that $C(u_1, v_1, t) - C(u_1, v_2, t) - C(u_2, v_1, t) + C(u_2, v_2, t) \geq 0$. Here, we write $I := [0, 1]$.

It is proved in Ishimura and Yoshizawa (2011) that for any given bivariate copula C_0 as the initial condition, there exists a family of copulas $\{C(u, v, t)\}_{t \geq 0}$, which satisfy (1), with $C(u, v, 0) = C_0(u, v)$. Moreover, we have

$$C(u, v, t) \longrightarrow \Pi(u, v) := uv \quad \text{as} \quad t \to \infty.$$

Remark that the product copula Π is also a steady solution of (1) and thus it is customarily called a harmonic copula.

From a practical point of view, however, the discrete version of the above evolution may be much more useful. In this short note, we deal with the evolution of multivariate copulas in discrete processes and discuss the possibility of applications. First, we clarify the notion of evolution of copulas in discrete time.

Definition 1.1 *A discretely parameterized family of functions $\{C^n\}_{n \in \mathbb{N}}$ defined on $I^d := [0, 1]^d$ and valued in I is called the evolution of copulas in a discrete process if the following three conditions are satisfied:*

(i) For every $n \in \mathbb{N}$, $C^n(u_1, u_2, \ldots, u_d)$ is increasing in each component u_i.

(ii) For every $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, d\}$ with $u_i \in I$,

$$C^n(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_d) = 0 \quad \textit{and} \quad C^n(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i.$$

(iii) (d-increasing condition) For every $n \in \mathbb{N}$ and $(u_{1i}, u_{2i}, \ldots, u_{di}) \in I^d$ with $u_{1i} \leq u_{2i}$ $(i = 1, 2)$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+i_2+\cdots+i_d} C^n(u_{1i_1}, u_{2i_2}, \ldots, u_{di_d}) \geq 0.$$

In the next section, we exhibit the construction of such a family of copulas.

2. EVOLUTION OF COPULAS IN DISCRETE PROCESSES

We now turn our attention to the investigation of the evolution of copulas in discrete processes, with the intention to undertake a slight generalization of Ishimura and Yoshizawa (2012).

Let $d \geq 2$ and assume that $N \gg 1$ and $0 < h \ll 1$. We define

$$\Delta u_1 = \Delta u_2 = \cdots = \Delta u_d := \frac{1}{N} \qquad \text{and} \qquad \Delta t := h,$$

such that

$$\lambda := \frac{\Delta t}{(\Delta u_i)^2} = hN^2 \quad (i = 1, 2, \ldots, d),$$

and for $i = 1, 2, \ldots, d$, we have

$$u_{i,k_i} := k_i \Delta u_i = \frac{k_i}{N}, \qquad (k_i = 0, 1, \ldots, N).$$

A family of copulas $\{C^n(u_1, u_2, \ldots, u_d)\}_{n \in \mathbb{N}}$ is now defined as follows.
The initial condition is defined by

$$C^0(u_1, u_2, \ldots, u_d) := C_0(u_1, u_2, \ldots, u_d),$$

where C_0 denotes a given initial copula.

Let $C^n_{k_1,k_2,\ldots,k_d} := C^n(u_{1,k_1}, u_{2,k_2}, \ldots, u_{d,k_d})$, where $\{(u_{1,k_1}, u_{2,k_2}, \ldots, u_{d,k_d})\}_{k_1,k_2,\ldots,k_d=0,1,\ldots,N}$ denote the lattice points. Its values for $k_1, k_2, \ldots, k_d = 1, 2, \ldots, N-1$ are governed by the system of difference equations

$$C^{n+1}_{k_1,k_2,\ldots,k_d} = \alpha_n C^n_{k_1,k_2,\ldots,k_d} + \beta_{ni} \sum_{i=1}^{d} (C^n_{k_1,\ldots,k_{i-1},k_i+1,k_{i+1},\ldots,k_d} + C^n_{k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_d}). \tag{2}$$

Here, we postulate that

$$\alpha_n > 0, \quad \beta_{ni} > 0 \quad (i = 1, 2, \ldots, d) \quad \text{and} \quad \alpha_n + 2\sum_{i=1}^{d} \beta_{ni} = 1.$$

If for some j, $k_j = N$, then the j-th term in the sum of the right hand side of (2) should be replaced by

$$C^n_{k_1,\ldots,k_{j-1},N+1,k_{j+1},\ldots,k_d} + C^n_{k_1,\ldots,k_{j-1},N-1,k_{j+1},\ldots,k_d} \longrightarrow 2C^n_{k_1,\ldots,k_{j-1},N,k_{j+1},\ldots,k_d}.$$

If for some j, $k_j = 0$, then

$$C^n_{k_1,\ldots,k_{j-1},1,k_{j+1},\ldots,k_d} + C^n_{k_1,\ldots,k_{j-1},-1,k_{j+1},\ldots,k_d} \longrightarrow 2C^n_{k_1,\ldots,k_{j-1},0,k_{j+1},\ldots,k_d} = 0.$$

Given these adjustments, we see that the boundary condition

$$\begin{cases} C^n_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_d} = 0 \\ C^n_{N,\ldots,N,k_i,N,\ldots,N} = u_{i,k_i}, \end{cases} \qquad \text{for } k_i = 0, 1, \ldots, N \quad (i = 1, 2, \ldots, N)$$

is properly imposed.

For a point $(u_1, u_2, \ldots, u_d) \in I^d$ other than $\{(u_{1,k_1}, u_{2,k_2}, \ldots, u_{d,k_d})\}_{k_1,k_2,\ldots,k_d=0,1,\ldots,N}$, the value $C^n(u_1, u_2, \ldots, u_d)$ is provided by interpolation; that is, if

$$u_{1,k_1} \leq u_1 \leq u_{1,k_1+1}, \; u_{2,k_2} \leq u_2 \leq u_{2,k_2+1}, \ldots, u_{d,k_d} \leq u_d \leq u_{d,k_d+1},$$

then we define

$$C^n(u_1, u_2, \ldots, u_d) = \sum_{j_1=0}^{1} \cdots \sum_{j_d=0}^{1} S(k_1+j_1, k_2+j_2, \ldots, k_d+j_d)(u_1 - u_{1,k_1})^{j_1}(u_2 - u_{2,k_2})^{j_2} \cdots (u_d - u_{d,k_d})^{j_d}, \tag{3}$$

where we have

$$S(k_1+j_1,k_2+j_2,\dots,k_d+j_d)=\sum_{l_1=0}^{j_1}\cdots\sum_{l_d=0}^{j_d}(-1)^{j_1+\cdots+j_d+l_1+\cdots+l_d}C^n_{k_1+l_1,k_2+l_2,\dots,k_d+l_d}.$$

We remark that the values $S(k_1+j_1,k_2+j_2,\dots,k_d+j_d)$ are all non-negative by virtue of the d-increasing condition above, which makes $C^n(u_1,u_2,\dots,u_d)$ indeed a copula. We also note that if $C^n_{k_1+l_1,k_2+l_2,\dots,k_d+l_d}=\prod_{i=1}^d u_{i,k_i+l_i}$ for all values, which means that the product copula is concerned, then $C^n(u_1,u_2,\dots,u_d)$ reduces to $\prod_{i=1}^d u_i$ for $u_{1,k_1}\le u_1\le u_{1,k_1+1}$, $u_{2,k_2}\le u_2\le u_{2,k_2+1}$, ..., $u_{d,k_d}\le u_d\le u_{d,k_d+1}$.

It is easy to check that a sequence of copulas $\{C^n(u_1,u_2,\dots,u_d)\}_{n\in\mathbb{N}}$ is well defined and gives the evolution of copulas in discrete time. Furthermore, the difference scheme is stable and we infer that

$$\max_{(u_1,u_2,\cdots,u_d)\in I^d}\left|C^n(u_1,u_2,\dots,u_d)-\prod_{i=1}^d u_i\right|\le K\theta^n,$$

for some constants K and θ with $0<\theta<1$. In summary, we can state the next theorem.

Theorem 2.1 *Let $d\ge 2$. For any initial copula C_0 of dimension d, there exists an evolution of copulas $\{C^n(u_1,u_2,\dots,u_d)\}_{n\in\mathbb{N}}$ in discrete time, which satisfies the system of difference equations (2) at every $\{(u_{1,k_1},u_{2,k_2},\dots,u_{d,k_d})\}_{k_1,k_2,\dots,k_d=0,1,\dots,N}$ and are bridged by the interpolation (3). Additionally, we have*

$$C^n(u_1,u_2,\dots,u_d)\to\prod_{i=1}^d u_i\quad\textit{uniformly on } I^d \textit{ as } n\to\infty.$$

We may omit the details of the proof and other properties.

Remark 2.1 *The difference scheme (2) originally comes from the discretization of the heat equation; e.g. Ishimura and Yoshizawa (2012) deals with*

$$\begin{aligned}&\frac{C^{n+1}_{k_1,k_2,\dots,k_d}-C^n_{k_1,k_2,\dots,k_d}}{\Delta t}\\&\qquad=\sum_{i=1}^d\frac{C^n_{k_1,\dots,k_{i-1},k_i+1,k_{i+1},\dots,k_d}-2C^n_{k_1,\dots,k_{i-1},k_i,k_{i+1},\dots,k_d}+C^n_{k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_d}}{(\Delta u_i)^2}.\end{aligned}$$

Our current scheme (2) is thus regarded as a slightly generalized version.

3. DISCUSSIONS

We have developed the concept of time-dependent copulas, and in particular, the evolution of copulas in discrete processes. Compared to other time-related copulas, in our notion, a copula evolves according to the time variable. It may be employed in certain inference on the time-varying deformation of structures among random events. Since the dependence between random factors usually changes in time, one can expect that a mathematical modeling with some evolution system has a place to be involved. We hope that such a description is realized with the use of our theory.

A major drawback of our evolution, however, is that every copula converges to the simplest copula, the product copula. The relation between random variables, which is described by our system, is gradually becoming simple in a sense. It may be challenging to bypass this difficulty. A possible way would be to presume that it is rather suitable to invoke the backward type of the scheme (2); we assign the maturity state and consider the backward evolution which starts from this maturity. We believe that other good applications of the evolution of copulas are still unexplored and we continue to make an effort on finding a relevant example.

References

W.F. Darsow, B. Nguyen, and E.T. Olsen. Copulas and Markov Processes. *Illinois J. Math.*, 36: 600–642, 1992.

N. Ishimura and Y. Yoshizawa. On time-dependent bivariate copulas. *Theor. Appl. Mech. Japan*, 59:303–307, 2011.

N. Ishimura and Y. Yoshizawa. Evolution of multivariate copulas in discrete processes. *Procedia Economics and Finance*, 1:186–192, 2012.

R.B. Nelsen. *An Introduction to Copulas*. Springer, New York, 2006.

A.J. Patton. Modelling asymmetric exchange rate dependence. *International Econ. Review*, 47: 527–556, 2006.

VALUATION OF EMPLOYEE STOCK OPTIONS IN THE HESTON MODEL

Tilman Sayer

Department of Financial Mathematics, Fraunhofer Institute for Industrial Mathematics, Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany
Email: tilman.sayer@itwm.fraunhofer.de

The paper deals with the valuation of employee stock options within the stochastic volatility model of Heston (1993). We introduce personal market perspectives into the model and apply a two-dimensional tree algorithm to value the options for both subjective and objective beliefs. Pseudo code that illustrates the valuation principle is presented.

1. INTRODUCTION AND MOTIVATION

Employee stock options (ESOs) are a popular part of employee remuneration. Typically these options are non-tradable calls on the company's stock and are given to an employee as part of her compensation. Often they last as long as ten or fifteen years, are issued at-the-money and are inaccessible during the vesting period, an interval of variable length after grant. After vesting, the options can be exercised at any time, however, when the employee quits the job during the vesting period, the ESOs forfeit worthless.

In the literature, three values arise when pricing employee stock options. The first one is the subjective value, which is the price the employee assigns to the option. Due to her trading restrictions, she tends to exercise the ESO sub-optimally from market perspectives. Mathematically this is incorporated into the model through subjective market beliefs and risk aversion. The second value is the objective price, that equals the costs for setting up a hedging portfolio of the stock and the riskless asset that exactly mimics the subjective exercise policy. Since the company can act as an unconstrained market player, the portfolio follows the risk-neutral processes. The third option price equals the value an unrestricted market participant is willing to pay, i.e. it is determined under risk neutrality and optimal exercise.

The structure of the paper is as follows. Section 2 introduces the model framework in the context of the stochastic volatility model of Heston (1993) and sets up the refined tree model. Pseudo code as well as implementation details are given in Section 3. Section 4 shows numerical results, Section 5 concludes and gives an outlook.

Parts of this work are included in the PhD thesis of Tilman Sayer, see Sayer (2012).

2. CONTINUOUS AND TREE MODEL

Here, we briefly describe the stochastic volatility model of Heston (1993) and the tree algorithm proposed in Ruckdeschel et al. (2013). The company's stock is assumed to follow stochastic volatility and further, option specific characteristics like e.g. early exercise, can easily be incorporated when employing the tree to value the ESO.

Throughout the paper, we consider the probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $\mathbb{P}$ denotes the risk-neutral probability measure, obtained through calibration of the model to market data. The dynamics of the stock price and variance process respectively are given by

$$\mathrm{d}S_t = (r + \mu - d)\, S_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^S\,, \qquad S_0 = s_0 > 0\,, \tag{1}$$

$$\mathrm{d}V_t = \kappa\,(\theta - V_t)\,\mathrm{d}t + \eta\sqrt{V_t}\mathrm{d}W_t^V\,, \qquad V_0 = v_0 > 0\,, \tag{2}$$

with constant interest rate $r \geq 0$, dividend yield $d \geq 0$ and subjective excess return μ. Due to the trading restrictions and the resulting conceivably sub-optimal exercise, subjective expectations of the stock return are essential when determining the personal value of the option, compare for instance Kulatilaka and Marcus (1994). Further, $\kappa > 0$ denotes the speed of mean reversion to the long-term variance level $\theta > 0$ and $\eta > 0$ is the volatility of the variance. The Brownian motions W_t^S and W_t^V are correlated with constant correlation $\rho \in [-1, 1]$.

Our first step in the generation of the approximating tree model is to build a recombining binomial tree $\hat{V}$ that approximates the variance process. Based on its nodes, we approximate the stock price process by a recombining trinomial tree. Then, both separate trees are joint by defining transition probabilities that properly incorporate the correlation structure between the Brownian motions, i.e. without obtaining negative weights. To ensure weak convergence of the tree model to the continuous one, we employ the concept of moment matching. In particular, we employ Lemma 1 of Ruckdeschel et al. (2013) for the continuous values. In the following, let T, N and $\Delta = T/N$ represent the investment horizon, the number of discretisation steps and the step size, respectively.

Due to the heteroscedasticity of the variance process, a naively constructed tree would not recombine and the number of generated nodes grows exponentially. Computationally, this results in an inefficient approximation. To construct an efficient tree, we employ Itô's Lemma on the transformation $Z_t = 2\sqrt{V_t}/\eta$ to obtain

$$\mathrm{d}Z_t = \left[\left(\frac{2\kappa\theta}{\eta^2} - \frac{1}{2}\right)\frac{1}{Z_t} - \frac{\kappa}{2}Z_t\right]\mathrm{d}t + \mathrm{d}W_t^V\,, \quad Z_0 = \frac{2\sqrt{V_0}}{\eta}\,.$$

Since Z_t features constant variance, its binomial tree approximation $\hat{Z}$ recombines and we obtain a recombining tree model $\hat{V}$ by employing the inverse transformation on the nodes of $\hat{Z}$. In order to obtain proper transition probabilities, for each variance node, we choose the subsequent successor nodes as nodes surrounding the drift. Due to the mean reverting property of (2) this might result in multiple jumps, as detailed in Figure 1. Let v_1 and v_2 be the nodes that respectively result from a down and up jump. The probabilities $\mathbb{P}_V(v_1)$ and $\mathbb{P}_V(v_2) = 1 - \mathbb{P}_V(v_1)$ are determined by matching the tree moments against the continuous ones.

The trinomial stock price tree is built for the growth adjusted logarithmic state space transfor-

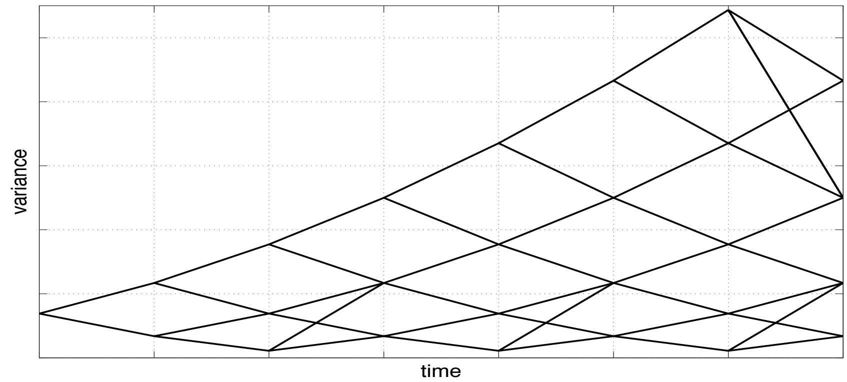


Figure 1: Binomial tree $\hat{V}$. Due to mean reversion and heteroscedasticity of $\mathrm{d}V_t$, the approximating tree is self-truncating with unevenly spaced nodes.

mation of (1), i.e. we consider the process

$$\mathrm{d}X_t = \mathrm{d}\log\left(S_t e^{-(r+\mu-d)t}\right) = -\frac{V_t}{2}\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^S\,, \quad X_0 = \log\left(S_0\right)\,,$$

and its tree approximation $\hat{X}$. Yet, the diffusion $\sqrt{V_t}$ of $\mathrm{d}X_t$ depends on V_t and consequently on the nodes of $\hat{V}$. As a result, the jump heights of $\hat{X}$ depend on the variance nodes and the tree would in general not recombine. We circumvent this by defining the distance between two nodes of $\hat{X}$ as $\sqrt{\alpha\Delta}$ for a given value α. By putting the variance nodes of $\hat{V}$ into relation with α, we respectively determine multipliers of this spacing that are needed in order to produce jumps wide enough to account for the instantaneous variance nodes. For a particular tree node, let x_1, x_2 and x_3 denote its successor nodes, where x_1 and x_3 respectively label the nodes that result from a down and up jump. The node x_2 lies between x_1 and x_3 and results from a pure horizontal jump. Again, we obtain the transition probabilities $\mathbb{P}_X(x_1)$, $\mathbb{P}_X(x_2)$ and $\mathbb{P}_X(x_3) = 1 - \mathbb{P}_X(x_1) - \mathbb{P}_X(x_2)$ by matching the tree moments against the ones of the limit distribution.

For $\rho = 0$, the joint tree model is obtained by defining the product probabilities

$$\mathbb{P}\left(x_i, v_j\right) = \mathbb{P}_X\left(x_i\right)\mathbb{P}_V\left(v_j\right)\,, \quad i = 1,2,3\,, \quad j = 1,2\,.$$

For correlated Brownian motions, we modify these probabilities in a way that both the already matched marginal moments are maintained and the match between tree and model correlation is optimised.

With this tree model, we can describe the movement of the company's stock and employ it to value the option. In general, a risk-averse employee compensated with the ESO chooses the exercising time τ such that her expected utility

$$\mathbb{E}\left[U\left(e^{-r\tau}\left(S_\tau - K\right)^+\right)\right]$$

is maximised, where K is the strike price, $(S_\tau - K)^+$ is the payout of the ESO at τ and $U(.)$ is a utility function. Intuitively, as soon as the option is vested, the employee either exercises the ESO and invests the profit in the riskless asset or continues to hold the option for the subsequent period. Naturally, she decides for the strategy with the greater utility. Let U_0 be the expected utility the employee assigns to the ESO at grant date. The monetary amount $I = e^{-rT}U^{-1}(U_0)$ corresponds to the subjective price of the ESO and is typically interpreted as an indifferent payment, i.e. as the particular value for which the employee is neutral between I and the ESO. Given the exercise policy of the employee, the value of the hedging portfolio is calculated respectively, even if exercising the option is sub-optimal from market perspectives. Its value at grant date defines the objective value of the ESO.

3. IMPLEMENTATION DETAILS

The tree model is implemented using Matlab programming language. Pseudo code that illustrates the option pricing by backward induction is given in Figure 2, where we consider the subjective and objective value. Let `n` denote a node in the joint tree. In the code, `n.Ssub` and `n.Sobj`

```
Input: parameters % relevant market, option & tree parameters
Output: sub, obj % subjective & objective value
for t = T, T - Δ, ..., 0 do
 N = getRelevantNodes(t); % obtain active tree nodes
 sub_e = 0;
 obj_e = 0;
 for each n in N do
  if (isESOVested) % exercise values from subjective & market perspectives
   sub_e = U(exp(r * (T - t)) * max(0, n.Ssub - K));
   obj_e = max(0, n.Sobj - K);
  end
  if (t == T)
   n.sub = sub_e;
   n.obj = obj_e;
  else
   (n1, ..., n6, P1, ..., P6) = getNodesAndWeights(n);
   sub_h = P1 * n1.sub + ... + P6 * n6.sub;
   obj_h = P1 * n1.obj + ... + P6 * n6.obj;
   n.sub = max(sub_h, sub_e);
   if (n.sub == sub_h) n.obj = exp(-r * Δ) * obj_h;
   else n.obj = obj_e;
   end
  end
 end
end
sub = exp(-r * T) * InvU(n.sub); % subjective value
obj = n.obj; % objective value
return sub, obj;
```

Figure 2: Pseudo code illustrating the backward induction and the calculation of the subjective and objective value. The functions `U(.)`, `InvU(.)` and `getNodesAndWeights(.)` respectively return utility, inverse utility and the successor nodes as well as the corresponding transition probabilities. At each time step, the active nodes are obtained through `getRelevantNodes(.)`.

label the stock price from subjective and market perspectives. Furthermore, `n.sub` and `n.obj` denote the subjective and objective option value at node n. Since the marginal trees $\hat{V}$ and $\hat{X}$ as well as the correlated transition probabilities $\mathbb{P}(x_i, v_j)$ for $i = 1, 2, 3$ and $j = 1, 2$ do not depend on the current time step, these values can be determined before running the backward induction, i.e. before valuing the tree from its leafs to the root. Lines (7) and (8) determine the exercise values for subjective and market beliefs. The values for holding the ESO are obtained between lines (14) and (16). Depending on the subjective policy of the employee, i.e. the maximum value of `sub_h` and `sub_e`, the hedging portfolio mimics her decision in the lines (18) to (20), where the objective value of node n is set. Finally, line (24) determines the indifferent payment.

4. NUMERICAL RESULTS

The relation between I and the price of the hedging portfolio is illustrated in Figure 3 for different values of μ and for a spot price range from 20 to 300. The remaining parameters are $r = 5\%$,

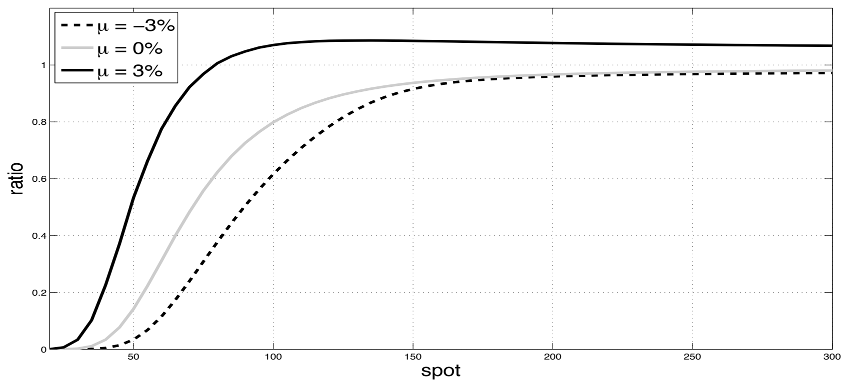


Figure 3: Relation between the subjective and objective option value for varying subjective excess return μ and spot value.

$d = 0\%$, $v_0 = \theta = 4\%$, $\kappa = 2$, $\eta = 20\%$ and $\rho = -0.85$. We further assume power utility $U(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma = 0.25$. The option lives for two years and vests after six months. The strike is set to 100, $N = 200$ and $\alpha = 2\%$. For $\mu = 0\%$, the subjective and market beliefs coincide. Since the employee acts risk-averse, she exercises sub-optimally, hence waives option value. As the spot increases, this reduction becomes irrelevant, i.e. the particular ratio tends to one. For $\mu = -3\%$, the effect of waiving option value is further stressed due to the pessimistic subjective beliefs. In the case of $\mu = 3\%$, the optimistic personal perceptions cause the employee to delay her exercise decision. Further, the indifferent payment I even exceeds the value of the portfolio, despite risk aversion.

5. CONCLUSION AND OUTLOOK

In this paper, we have introduced subjective excess returns in the stochastic volatility model of Heston (1993) and modified a recombining tree model to value employee stock options accordingly. Due to the structure of the tree, many specific characteristics of ESOs can easily be incorporated. We determined price ratios between subjective and objective option values, where we focused on the excess returns. Future work could for instance cover further ESO specifications or employ the tree to value similar sophisticated derivative types.

References

S. L. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*, 6(2):327–343, 1993.

N. Kulatilaka and A. J. Marcus. Valuing Employee Stock Options. *Financial Analysts Journal*, 50 (6):46–56, 1994.

P. Ruckdeschel, T. Sayer, and A. Szimayer. Pricing American Options in the Heston Model: A Close Look at Incorporating Correlation. *Journal of Derivatives*, 20(3):9–29, 2013.

T. Sayer. *Valuation of American-style derivatives within the stochastic volatility model of Heston*. PhD Dissertation, Dr. Hut, München, Germany, 2012.

MARKOV SWITCHING AFFINE PROCESSES AND APPLICATIONS TO PRICING

Misha van Beek[†], Michel Mandjes[†,‡,§], Peter Spreij[†] and Erik Winands[†]

[†]*Korteweg-de Vries Institute for Mathematics, University of Amsterdam,*
1090 GE Amsterdam, The Netherlands
[‡]*EURANDOM, 5600 MB Eindhoven, The Netherlands*
[§]*CWI, 1090 GB Amsterdam, The Netherlands*
Email: mishavanbeek@uva.nl, M.R.H.Mandjes@uva.nl, spreij@uva.nl, E.M.M.Winands@uva.nl

1. INTRODUCTION

This paper extends derivative pricing based on multivariate affine processes to affine models with Markov switching drift and diffusion coefficients. In the economic and finance literature, models with Markov switching parameters are often said to be regime switching.

In many economic and finance applications, processes fall prey to changes in regime. Regimes are time periods between which the dynamics of these processes are substantially different (Hamilton 1989). E.g. the mean returns, correlations and volatilities of stock prices are different in bull and bear markets, and the mean reversion level of interest rates may be lower in crisis scenarios. Based on this observation, the pricing of derivatives should account for the existence of different regimes.

Furthermore, many financial products benefit from multidimensional analysis. The price of a European call option is better modeled by allowing for stochastic interest rates and stochastic volatility. Also other products require multidimensional analysis directly through their structure. The price of a credit default swap (CDS) is derived from the dynamics of the interest rate and the hazard rate of default of the underlying. When we want to adjust to a price of a derivative for the creditworthiness of its seller, an additional process for the hazard rate of the seller enters into the game. This is known as a credit valuation adjustment (CVA), and together with a similar adjustment for the buyer's creditworthiness, the debit valuation adjustment (DVA), these are common and increasingly important drivers of multivariate analysis (Hull and White 2013).

In this paper, we consider the popular and broad class of multivariate affine processes that is often used to jointly model time series such as interest rates, stochastic volatility, hazard rates and log-asset prices (Duffie et al. 2003). Affine processes include the Vasicek and Cox-Ingersoll-Ross short rate models as special univariate cases. The primary advantage of affine processes over general multivariate processes in general is that the price of many derivatives has a closed form or is implicit in a system of ordinary differential equations (ODEs). ODE solutions are markedly more tractable than the partial differential equations (PDEs) that multivariate processes produce. We generalize multivariate affine processes to include Markov switching drift and diffusion coefficients. Our resulting Markov switching- (MS-)affine process maintains the property of ODE pricing solutions.

There is a rather restraint body of literature on this problem. Elliott and Mamon (2002) consider pricing a bond based on a short rate that follows a univariate Vasicek model with Markov switching mean reversion level. Elliott and Siu (2009) extend this result to bond prices based on a short rate that follows a univariate affine process with Markov switching mean reversion level and (in the Vasicek case) diffusion.

We take a more formal approach and follow the line of argumentation of Filipović (2009, Chapter 10). We derive the characteristic function of the MS-affine process and show that it can be expressed using the solutions of two systems of ODEs. We also prove that these solutions exist and are unique, provided that the parameters of the process are admissible in some sense. The characteristic function is the basis to price a wide variety of payoffs.

Effectively, our main theorem extends all pricing ODEs for affine processes to MS-affine processes. These include CVA and DVA adjustments, CDSs, exchange options, and many more. Moreover, for all these derivatives we may have regime dependent payoffs. The regime dependent payouts are used, for example, when the payoff of a derivative relies on the rating of a counterparty, and for this counterparty we have a rating migration matrix. Each rating (e.g. AAA, AA, etc.) can be seen as a regime in which the dynamics of the processes are different. Another example is when the dynamics of the affine process are different after some policy is introduced, but we are unsure when this policy takes effect. The different regimes would be the different states that the development and implementation of this policy can be in.

This remainder of this paper is outlined as follows. First we define the MS-affine process and the admissibility of its parameters. Then we provide two theorems that can be used for derivative pricing. We conclude with a simple example on how to apply these theorems to a bond price.

2. MODEL AND ANALYSIS

Let W_t be a d-dimensional Brownian motion with filtration $\{\mathcal{G}_t\}$. Let S_t be a continuous time Markov chain with state space $\mathcal{S} = \{1, \ldots, h\}$, filtration $\{\mathcal{H}_t\}$ and generator Q that switches between the regimes in $\mathcal{S}$. W_t and S_t are independent and defined on a filtered probability space $\langle \Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P} \rangle$, where $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$.

Definition 2.1 *We call the process X on the canonical state space $\mathcal{X} = \mathbb{R}_+^m \times \mathbb{R}^n$, $m \geq 0$, $n \geq 0$, $m + n = d \geq 1$, MS-affine if*

$$\mathrm{d}X_t = \mu_{S_t}(X_t)\mathrm{d}t + \sigma_{S_t}(X_t)\mathrm{d}W_t, \tag{1}$$

where

$$\sigma_s(x)\sigma_s^\top(x) = a_s + \sum_{i=1}^d x_i\alpha_i, \qquad \mu_s(x) = b_s + \sum_{i=1}^d x_i\beta_i = b_s + \mathcal{B}x \tag{2}$$

for some $d \times d$*-matrices* a_s *and* α_i*, and* d*-vectors* b_s *and* β_i*, with* $\mathcal{B} = \begin{bmatrix}\beta_1 & \cdots & \beta_d\end{bmatrix}$.

Hence only a_s and b_s are regime dependent, not α_i and β_i.[1]

X_t may stack all sorts of financial variables. For example, if r_t is the short rate, A_t some asset price, V_t the stochastic volatility of the stock price, and h_t the hazard rate of default of the counterparty, then $X_t = (r_t, \ln A_t, V_t, h_t)$ models these processes jointly. For financial applications this model is usually under the risk neutral measure. This implies (among other things) that the drift of $\ln A_t$ is $r_t - \frac{1}{2}V_t$.

For ease of notation, we write $Z_t = e_{S_t} \in \{0,1\}^h$, a vector of zeros with S_t-th entry one. $\mathcal{Z}$ is the state space of Z_t. Then by Elliott (1993),

$$\mathrm{d}Z_t = QZ_t\mathrm{d}t + \mathrm{d}M_t, \tag{3}$$

where M_t is a martingale. Without proof we assume throughout this text that for every $x \in \mathcal{X}$, $z \in \mathcal{Z}$ there exists a unique solution $(X, Z) = (X^x, Z^z)$ of (1) with $X_0 = x$ and $Z_0 = z$.

To ensure that the process does not escape $\mathcal{X}$ we need some admissibility conditions on the parameters in (2). In what follows, we denote $I = \{1, \dots, m\}$ and $J = \{m+1, \dots, d\}$. Also, for any sets of indices M and N, and vector v and matrix w, $v_M = [v_i]_{i\in M}$ and $w_{MN} = [w_{ij}]_{i\in M, j\in N}$ are the corresponding sub-vector and sub-matrix.

Definition 2.2 *We call* X *an MS-affine process with admissible parameters if* X *is MS-affine and*

$$\begin{aligned}
a_s, \alpha_i &\ \textit{are symmetric positive semi-definite},\\
a_{sII} &= 0 \textit{ for all } s \in \mathcal{S} \textit{ (and thus } a_{sIJ} = a_{sJI}^\top = 0\textit{)},\\
\alpha_j &= 0 \textit{ for all } j \in J,\\
\alpha_{i,kl} = \alpha_{i,lk} &= 0 \textit{ for } k \in I\backslash\{i\}, \textit{for all } i, l \in \{1, \dots, d\},\\
b_s &\in \mathcal{X} \textit{ for all } s \in \mathcal{S},\\
\mathcal{B}_{IJ} &= 0,\\
\mathcal{B}_{II} &\ \textit{has nonnegative off-diagonal elements}.
\end{aligned}$$

We now state our main contribution. $\mathrm{diag}(F_s)$ refers to the (block) diagonal matrix from the regime specific matrices $F_1, \dots, F_h$.

Theorem 2.1 *Let* X *be an MS-affine process with admissible parameters. Let* $u \in \mathrm{i}\mathbb{R}^d$*,* $t \le T$*,* $x \in \mathcal{X}$ *and* $z \in \mathcal{Z}$*. Then there exists unique solutions* $A(t,u) : \mathbb{R}_+ \times \mathrm{i}\mathbb{R}^d \to \mathbb{C}^{d\times d}$ *and* $B(t,u)$:

[1]Taking α_i and β_i regime dependent complicates further analysis and we are not sure whether ODE solutions to the characteristic function are possible in that case.

$\mathbb{R}_+ \times \mathrm{i}\mathbb{R}^d \to \mathbb{C}^d$ *to*

$$\begin{aligned}
\partial_t A(t,u) &= A(t,u)\left(\operatorname{diag}\left(\tfrac{1}{2}B_J(t,u)^\top a_{sJJ}B_J(t,u) + b_s^\top B(t,u)\right) + Q\right), \\
A(0,u) &= I_h, \\
\partial_t B_i(t,u) &= \tfrac{1}{2}B(t,u)^\top \alpha_i B(t,u) + \beta_i^\top B(t,u), \quad i \in I, \\
\partial_t B_J(t,u) &= \mathcal{B}_{JJ}^\top B_J(t,u), \\
B(0,u) &= u,
\end{aligned} \tag{4}$$

such that the $\mathcal{F}_t$-conditional regime specific characteristic function satisfies

$$\mathbb{E}\left[e^{u^\top X_T} Z_T \middle| \mathcal{F}_t\right] = A(T-t,u)e^{B^\top(T-t,u)X_t}Z_t. \tag{5}$$

Before proving the above theorem, we state (without proof) the following lemma, which is useful in an MS setting. $\otimes$ denotes the Kronecker product.

Lemma 2.2 *Let $F_{S_t} \in \mathbb{R}^{p\times q}$ be a set of d matrices with Markov switching index, then $(Z_t \otimes I_p)F_{S_t} = \operatorname{diag}(F_{S_t})(Z_t \otimes I_q)$.*

Also, we use the following lemma adapted from Filipović (2009, Lemma 10.1).

Lemma 2.3 *Consider the system of ODEs*

$$\partial_t y(t,y_0) = f(y(t,y_0)), \qquad y(0,y_0) = y_0, \tag{6}$$

where $f : \mathbb{C}^d \to \mathbb{C}^d$ is a locally Lipschitz continuous function. Then:

1. *For every $y_0 \in \mathbb{C}^d$ there exists a lifetime $t_+(y_0) \in (0,\infty]$ such that there exists a unique solution $y(\cdot,y_0) : [0,t_+(y_0)) \to \mathbb{C}^d$ of (6).*
2. *The domain $\mathcal{D} = \left\{(t,y_0) \in \mathbb{R}_+ \times \mathbb{C}^d \middle| t \le t_+(y_0)\right\}$ is open in $\mathbb{R}_+ \times \mathbb{C}^d$ and maximal in the sense that either $t_+(y_0) = \infty$ or $\lim_{t\uparrow t_+(y_0)} \|y(t,y_0)\| = \infty$, respectively, for all $y_0 \in \mathbb{C}^d$.*

Proof of Theorem 2.1. Define $\Phi_t = A(T-t,u)e^{B(T-t,u)^\top X_t}$. We prove that $\Phi_t Z_t$ is martingale because this implies that $\mathbb{E}\left[e^{u^\top X_T}Z_T \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\Phi_T Z_T | \mathcal{F}_t\right] = \Phi_t Z_t$, and then (5) is true. The dynamics of $\Phi_t Z_t$ follow from Itô's lemma and Lemma 2.2,

$$\begin{aligned}
\mathrm{d}(\Phi_t Z_t) = \mathrm{d}\Phi_t Z_t + \Phi_t \mathrm{d}Z_t &= \Big(\left(\partial_t A(T-t,u)\right) e^{B(T-t,u)^\top X_t} \\
&\quad + \Phi_t \left(\partial_t B(T-t,u)\right)^\top X_t + \Phi_t B(T-t,u)^\top \mu_{S_t}(X_t) \\
&\quad + \tfrac{1}{2}\Phi_t B(T-t,u)^\top \sigma_{S_t}(X_t)\sigma_{S_t}(X_t)^\top B(T-t,u)\Big) Z_t \mathrm{d}t + \Phi_t Q Z_t \\
&\quad + \Phi_t B(T-t,u)^\top \sigma_{S_t}(X_t)\mathrm{d}W_t Z_t + \Phi_t \mathrm{d}M_t \\
&= \Phi_t \left(B(T-t,u)^\top \sigma_{S_t}(X_t)\mathrm{d}W_t Z_t + \mathrm{d}M_t\right).
\end{aligned}$$

Therefore, $\Phi_t Z_t$ is a local martingale. The remaining part of the proof is showing that this local martingale is uniformly bounded, so it is also a martingale.

We know from Filipović (2009, proof of Theorem 10.2) that by admissibility, for any $u \in \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$, $t \in \mathbb{R}_+$ a unique solution $B(t,u) : \mathbb{R}_+ \times \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n \to \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$ exists with infinite lifetime, so $\Re\left(B(t,u)^\top x\right) \leq 0$ for all $x \in \mathcal{X}$.

Apply Lemma 2.3 to the vectorization of the ODE of $A(t,u)$ (4), so $y = \operatorname{vec}(A)$, $y_0 = \operatorname{vec}(I_h)$ and f the vectorization of the RHS of (4). f is differentiable by differentiability of $B(t,u)$ and thus locally Lipschitz continuous. Therefore a unique solution for A exists with lifetime $t_+(\operatorname{vec}(I_h)) \in (0,\infty]$. We prove by contradiction that $t_+(\operatorname{vec}(I_h)) = \infty$. Suppose $t_+(\operatorname{vec}(I_h)) < \infty$, then $\lim_{t\uparrow t_+(\operatorname{vec}(I_h))} \|\operatorname{vec}(A(t,u))\| = \infty$. Note that

$$\|\operatorname{vec}(A(t,u))\|^2 = \operatorname{vec}(A(t,u))^* \operatorname{vec}(A(t,u)) = \operatorname{tr}(A(t,u)^* A(t,u)).$$

Define $\lambda = \max_{i=1,\ldots,h}\{-\lambda_i, 0\}$, with λ_i the eigenvalues of $Q + Q^\top$, then

$$\begin{aligned}
\partial_t \|\operatorname{vec}(A(t,u))\|^2 &= \operatorname{tr}(\partial_t A(t,u)^* A(t,u) + A(t,u)^* \partial_t A(t,u)) \\
&= \operatorname{tr}\left(\left(Q + Q^\top\right) A(t,u)^* A(t,u)\right) \\
&\quad + 2\operatorname{tr}\left(\Re\left(\operatorname{diag}\left(\tfrac{1}{2} B_J(t,u)^\top a_{sJJ} B_J(t,u) + b_s^\top B(t,u)\right)\right) A(t,u)^* A(t,u)\right) \\
&\leq \operatorname{tr}\left(\left(Q + Q^\top\right) A(t,u)^* A(t,u)\right) + \lambda \operatorname{tr}\left(A(t,u)^* A(t,u)\right) \\
&\leq \operatorname{tr}\left(Q + Q^\top + \lambda I_h\right) \operatorname{tr}(A(t,u)^* A(t,u)) \\
&= \operatorname{tr}\left(Q + Q^\top + \lambda I_h\right) \|\operatorname{vec}(A(t,u))\|^2.
\end{aligned}$$

For the second equality we have substituted $\partial_t A(t,u)$ with (4). The first inequality follows from $\lambda \geq 0$ and the fact that for all $s \in \mathcal{S}$,

$$\begin{aligned}
&\Re\left(\tfrac{1}{2} B_J(t,u)^\top a_{sJJ} B_J(t,u) + b_s^\top B(t,u)\right) \\
&\quad = \tfrac{1}{2}\Re(B_J(t,u))^\top a_{sJJ} \Re(B_J(t,u)) - \tfrac{1}{2}\Im(B_J(t,u))^\top a_{sJJ} \Im(B_J(t,u)) + b_s^\top \Re(B(t,u)) \leq 0
\end{aligned}$$

by the admissibility restrictions on a_{sJJ} and b_s and the codomain of $B(t,u)$. The second inequality holds because $Q + Q^\top + \lambda I_h$ is positive semi-definite by construction and for any positive definite matrices C and D of the same size it holds that $\operatorname{tr}(CD) \leq \operatorname{tr}(C)\operatorname{tr}(D)$. Applying Gronwall's inequality gives $\|\operatorname{vec}(A(t,u))\|^2 \leq h e^{\operatorname{tr}\left(Q+Q^\top+\lambda I_h\right)t}$, for all $t < t_+(\operatorname{vec}(I_h))$. This yields $\lim_{t\uparrow t_+(\operatorname{vec}(I_h))} \|\operatorname{vec}(A(t,u))\| < \infty$, so by contradiction it follows that $t_+(\operatorname{vec}(I_h)) = \infty$; $A(t,u)$ has infinite lifetime for all $u \in \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$.

Combining these results we have that Φ_t and $B(t,u)$ are uniformly bounded for all $t \leq T$, so $\Phi_t Z_t$ is a martingale. ■

Theorem 2.1 is pivotal to derivatives pricing, but cannot be applied directly. Additionally, we need that (5) holds when $u \in \mathbb{R}^d$. Filipović (2009, Theorem 10.3 and Corollary 10.1) proves this for affine processes, and we conjecture that this result extends to MS-affine processes.

3. SIMPLE EXAMPLE

As an example on how to apply the above theorems to derivative pricing, we consider the bond price in a MS-Vasicek short rate model. Take the short rate model $\mathrm{d}r_t = \gamma\left(\mu_{S_t} - r_t\right)\mathrm{d}t + \sigma_{S_t}\mathrm{d}W_t$, where S_t is the continuous time Markov chain that switches between regimes and has generator Q. Introduce the integrator $\mathrm{d}R_t = r_t\mathrm{d}t$, $R_0 = 0$, then $X_t = (r_t, R_t)$ is a MS-affine process, and for $u = (0, -1)$ we have

$$1^\top \mathbb{E}\left[e^{u^\top X_T} Z_T \middle| \mathcal{F}_0\right] = \mathbb{E}\left[e^{-R_T} \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\exp\left(-\int_0^T r_t \mathrm{d}t\right) \middle| \mathcal{F}_0\right].$$

Using Theorem 2.1 we can solve the LHS and thus obtain the price of the bond (the RHS, if it is finite). More examples can be found in Filipović (2009, Chapter 10.3).

References

D. Duffie, D. Filipovic, and W. Schachermayer. Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3):984–1053, 2003.

Robert J. Elliott. New finite-dimensional filters and smoothers for noisily observed Markov chains. *IEEE Transactions on Information Theory*, 39(1):265–271, 1993.

Robert J. Elliott and Rogemar S. Mamon. An interest rate model with a Markovian mean reverting level. *Quantitative Finance*, 2(6):454–458, 2002.

Robert J. Elliott and Tak Kuen Siu. On Markov-modulated exponential-affine bond price formulae. *Applied Mathematical Finance*, 16(1):1–15, 2009.

Damir Filipović. *Term-Structure Models: A Graduate Course*. Springer Finance. Springer, 2009.

James D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57(2):357–384, 1989.

John C. Hull and Alan White. Collateral and credit issues in derivatives pricing. Rotman School of Management Working Paper, 2013.

De Koninklijke Vlaamse Academie van België voor Wetenschappen en Kunsten coördineert jaarlijks tot 25 wetenschappelijke bijeenkomsten, ook contactfora genoemd, in de domeinen van de natuurwetenschappen (inclusief de biomedische wetenschappen), menswetenschappen en kunsten. De contactfora hebben tot doel Vlaamse wetenschappers of kunstenaars te verenigen rond specifieke thema's.

De handelingen van deze contactfora vormen een aparte publicatiereeks van de Academie.

Contactforum "Actuarial and Financial Mathematics Conference" (6-7 February 2014, Prof. M. Vanmaele)

Deze handelingen van de "Actuarial and Financial Mathematics Conference 2014" geven een inkijk in een aantal onderwerpen die in de editie van 2014 van dit contactforum aan bod kwamen. Zoals de vorige jaren handelden de voordrachten over zowel actuariële als financiële onderwerpen en technieken met speciale aandacht voor de wisselwerking tussen beide. Deze internationale conferentie biedt een forum aan zowel experten als jonge onderzoekers om hun onderzoeksresultaten ofwel in een voordracht ofwel via een poster aan een ruim publiek voor te stellen bestaande uit academici uit binnen- en buitenland alsook collega's uit de bank- en verzekeringswereld.