DIVERGENCE-FREE WAVELET BASES ON THE HYPERCUBE: FREE-SLIP BOUNDARY CONDITIONS, AND APPLICATIONS FOR SOLVING THE INSTATIONARY STOKES EQUATIONS

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ABSTRACT. We construct wavelet Riesz bases for the usual Sobolev spaces of divergence free functions on $(0,1)^n$ that have vanishing normals at the boundary. We give a simultaneous space-time variational formulation of the instationary Stokes equations that defines a boundedly invertible mapping between a Bochner space and the dual of another Bochner space. By equipping these Bochner spaces by tensor products of temporal and divergence-free spatial wavelets, the Stokes problem is rewritten as an equivalent well-posed bi-infinite matrix vector equation. This equation can be solved with an adaptive wavelet method in linear complexity with best possible rate, that, under some mild Besov smoothness conditions, is nearly independent of the space dimension. For proving one of the intermediate results, we construct an eigenfunction basis of the stationary Stokes operator.

1. INTRODUCTION

Divergence-free wavelet bases have been advocated for solving Stokes and incompressible Navier-Stokes equations. Our main interest in such bases is to use them for solving the time-dependent Stokes equations. On the space of divergence-free velocities, these equations are of parabolic nature. We derive a simultaneous space-time variational formulation that defines a boundedly invertible operator between a Bochner space and the dual of another Bochner space. By equipping these spaces by tensor products of temporal and spatial wavelet bases – the latter being a basis of divergence-free wavelets – we arrive at an equivalent, well-posed bi-infinite matrix vector problem. By solving this problem with an adaptive wavelet method, the best possible convergence rate will be realized in linear complexity. Thanks to the use of tensor product bases, apart for some log-factors, this rate will be equal as when solving a one-dimensional problem with wavelets of the same order.

The construction of a divergence-free wavelet basis on $\mathbb{R}^n$ by Lemarié-Rieusset in [LR92] relies on the availability of two pairs of biorthogonal Riesz bases $(\Psi, \tilde{\Psi})$ and $(\Psi^+, \tilde{\Psi}^-)$ for $L_2(\mathbb{R})$, that for some invertible diagonal matrix $\mathbf{D}$, satisfy

$$\dot{\Psi}^+ = \mathbf{D}\Psi, \quad \dot{\tilde{\Psi}} = -\mathbf{D}\tilde{\Psi}^-. \tag{1.1}$$

Here we view bases formally as column vectors. Shift invariant wavelet bases that satisfy these conditions were constructed in [LR92].

Date: March 9, 2010.

1991 *Mathematics Subject Classification.* 35K99, 65T60, 65M12, 76D03.

Key words and phrases. Divergence free wavelets, simultaneous space-time variational formulation, instationary Stokes equations, adaptive wavelet scheme, eigenfunction basis for stationary Stokes operator, elliptic regularity.

Replacing $\mathbb{R}$ by $\mathrm{I} := (0,1)$, the same construction applies to yield divergence-free wavelet bases on I^n, *assuming* that two pairs of biorthogonal Riesz bases for $L_2(\mathrm{I})$ are available that satisfy (1.1). With the notation $\langle \Sigma, \Upsilon \rangle := [\langle \sigma, v \rangle]_{\sigma\in\Sigma, v\in\Upsilon}$, integration by parts shows that (1.1) implies that necessarily

$$\begin{aligned}\Psi^+(1)\tilde{\Psi}(1)^\top - \Psi^+(0)\tilde{\Psi}(0)^\top &= \langle \dot{\Psi}^+, \tilde{\Psi}\rangle_{L_2(\mathrm{I})} + \langle \Psi^+, \dot{\tilde{\Psi}}\rangle_{L_2(\mathrm{I})} \\ &= \langle \mathbf{D}\Psi, \tilde{\Psi}\rangle_{L_2(\mathrm{I})} - \langle \Psi^+, \mathbf{D}\tilde{\Psi}^-\rangle_{L_2(\mathrm{I})} = \mathbf{D}\circ \mathrm{Id} - \mathrm{Id}\circ \mathbf{D} = 0,\end{aligned}$$

i.e.,

$$\psi^+(1)\tilde{\psi}(1) - \psi^+(0)\tilde{\psi}(0) = 0 \quad (\psi^+ \in \Psi^+, \tilde{\psi} \in \tilde{\Psi}). \tag{1.2}$$

To obtain such vanishing boundary terms, one may consider $\Psi^+ \subset H_0^1(\mathrm{I})$. Then, any element of $\Psi = \mathbf{D}^{-1}\dot{\Psi}^+$ has a vanishing mean, so that Ψ cannot be a basis for $L_2(\mathrm{I})$, the reason being that the mean value is a nonzero, continuous functional on $L_2(\mathrm{I})$ (it is not continuous on $L_2(\mathbb{R})$, and therefore the latter space *can* be equipped with a Riesz basis of functions all having a vanishing mean). The collection Ψ can be arranged to be a basis for $L_{2,0}(\mathrm{I}) = \{u \in L_2(\mathrm{I}) : \int_{\mathrm{I}} u = 0\}$. In that case, however, for $n \geq 3$, the resulting divergence-free wavelets will not span the whole space of divergence-free vector fields, but only that space intersected with

$$\widehat{L_2(\mathrm{I}^n)^n} := L_2(\mathrm{I})\otimes L_{2,0}(\mathrm{I})\otimes\cdots\otimes L_{2,0}(\mathrm{I}) \times \quad \cdots \quad \times L_{2,0}(\mathrm{I})\otimes\cdots\otimes L_{2,0}(\mathrm{I})\otimes L_2(\mathrm{I}),$$

the co-dimension being *infinite*.

Alternatively, we may search $\tilde{\Psi}$ in $H_0^1(\mathrm{I})$. In this case, the same argument shows that $\tilde{\Psi}^-$ cannot be a basis for $L_2(\mathrm{I})$, and so neither can Ψ^+, and we end up with the same problem.

A third possibility is to impose periodic boundary conditions for both Ψ^+ and $\tilde{\Psi}$. In this case, any element from even both Ψ and $\tilde{\Psi}^-$ has vanishing mean, giving rise to the same problem as above.

In [Ste08], these consideration led us to search $\tilde{\Psi}$ and Ψ^+ such that the elements of $\tilde{\Psi}$ vanish at 1, and those of Ψ^+ vanish at 0. With this option we could satisfy (1.1), and with that we constructed divergence-free wavelet bases on I^n. To the best of our knowledge, this was the first time that on a bounded domain such a basis was constructed. Divergence free wavelets on bounded domains were constructed earlier, but due to the difficulties outlined above, in any case in three and more dimensions, they did not generate a basis.

A disadvantage of the construction from [Ste08] is that, due to the special boundary conditions satisfied by Ψ^+, the resulting divergence-free wavelets span the space of divergence-free vector fields subject to the unusual boundary conditions of having vanishing normal components on *half* of the boundary of I^n.

In this paper, we will construct divergence-free wavelets that span the space of divergence-free vector fields on I^n that have vanishing normals on the *whole* of the boundary, i.e., that satisfy the standard free-slip boundary conditions. In order to do so, we make an orthogonal decomposition of $L_2(\mathrm{I}^n)^n$ into $2^n - 1$ subspaces, each of them being (isomorphic to) $\widehat{L_2(\mathrm{I}^k)^k}$ for $k = 1, \ldots, n$. Each of these spaces $\widehat{L_2(\mathrm{I}^k)^k}$ can be orthogonally split into a space of gradients and a space of divergence-free vector fields with vanishing normals on the boundary (Helmholtz decomposition). We know how to equip the latter spaces with wavelet bases, and we conclude that the union of (isomorphic images of) these bases is a Riesz basis for the space of all divergence-free vector fields in $L_2(\mathrm{I}^n)^n$ with vanishing normals on the boundary.

The same procedure will be followed to construct Riesz bases for the spaces of all divergence free vector fields in $H^1(\mathrm{I}^n)^n$ or $H^2(\mathrm{I}^n)^n$ that satisfy certain boundary conditions. These spaces become relevant for deriving a well-posed simultaneous space-time variational formulation of the time-dependent Stokes problem.

This paper is organized as follows: In Sect. 2, we present the orthogonal decomposition of $L_2(\mathrm{I}^n)^n$ into finitely many subspaces each of them being (isomorphic to) $\widehat{L_2(\mathrm{I}^k)^k}$ for $k = 1, \ldots, n$. In Sect. 3, for each of these subspaces, we equip its subspace of divergence vector fields with vanishing normals at the boundary with a Riesz basis of wavelet type. In Sect. 4, we derive a simultaneous space-time variational formulation of the time-dependent Stokes problem. The operator that corresponds to this variational form is shown to be boundedly invertible between two pairs of Hilbert spaces. For one of these results, an elliptic regularity result for the stationary Stokes problem is used that is shown in Sect. 6. In Sect. 5, using the results concerning boundedly invertibility and the construction of divergence-free wavelet bases, the instationary Stokes problem is formulated as an equivalent well-posed bi-infinite matrix vector problem that can be solved with an adaptive wavelet method. To show the elliptic regularity result, in Sect. 6 an eigenfunction basis of the stationary Stokes operator is constructed which seems not be available yet. In Sect. 7, it is shown that also the problem for the pressure can be written as an equivalent well-posed bi-infinite matrix vector equation. Finally, in Sect. 8, it is discussed to which extend the results from this paper apply to the Stokes problem with no-slip boundary conditions.

In this paper, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Orthogonal space decompositions

2.1. Decomposition of $L_2(\mathrm{I}^n)^n$. Let $\mathrm{I} := (0,1)$ and, for $n \in \{2, 3, \ldots\}$,

$$\begin{aligned} \mathbf{H}(\mathrm{div};\mathrm{I}^n) &:= \{\mathbf{u} \in L_2(\mathrm{I}^n)^n : \mathrm{div}\,\mathbf{u} \in L_2(\mathrm{I}^n)\}, \\ \mathbf{H}_0(\mathrm{div};\mathrm{I}^n) &:= \{\mathbf{u} \in \mathbf{H}(\mathrm{div};\mathrm{I}^n) : \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\mathrm{I}^n\}, \\ \boxed{\boldsymbol{\mathcal{H}}(\mathrm{I}^n)} = \mathbf{H}_0(\mathrm{div}0;\mathrm{I}^n) &:= \{\mathbf{u} \in \mathbf{H}_0(\mathrm{div};\mathrm{I}^n) : \mathrm{div}\,\mathbf{u} = 0\}. \end{aligned}$$

The boundary conditions incorporated in $\mathbf{H}_0(\mathrm{div};\mathrm{I}^n)$, and so in $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$, are commonly called free-slip boundary conditions, since they do not restrict the tangential components of a flow.

Our goal is to construct a Riesz basis of wavelet type for $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$. By applying integration by parts, one directly verifies that the subspaces $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ and $\mathbf{grad}\, H^1(\mathrm{I}^n)$ of $L_2(\mathrm{I}^n)^n$ are orthogonal. Below, we will make an orthogonal decomposition of $L_2(\mathrm{I}^n)^n$ into a finite sum of subspaces such that each of them is either in $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ or in $\mathbf{grad}\, H^1(\mathrm{I}^n)$. Furthermore, we will be able to equip all aforementioned subspaces in $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ with a Riesz basis of wavelet type. We may conclude that the union of these bases is a Riesz basis for $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$.

Actually, the same procedure will be followed for the subspaces in $\mathbf{grad}\, H^1(\mathrm{I}^n)$, with which we end up with a Riesz basis for that space as well.

We set $L_2 := L_2(\mathrm{I})$, $L_{2,0} := \{u \in L_2 : \int_{\mathrm{I}} u = 0\}$, and for $n \in \mathbb{N}$, $s \geq 0$,

$$\widehat{L_2(\mathrm{I}^n)^n} := L_2 \otimes L_{2,0} \otimes \cdots \otimes L_{2,0} \times \quad \cdots \quad \times L_{2,0} \otimes \cdots \otimes L_{2,0} \otimes L_2,$$
$$\widehat{H}^s(\mathrm{I}^n) := H^s(\mathrm{I}^n) \cap (L_{2,0} \otimes \cdots \otimes L_{2,0}),$$

and for $n \geq 2$,

$$\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n) := \boldsymbol{\mathcal{H}}(\mathrm{I}^n) \cap \widehat{L_2(\mathrm{I}^n)^n}.$$

In Sect. 3, we will give a constructive proof of the following result:

Proposition 2.1. *There exist biorthogonal Riesz bases* $\mathbf{\Psi}^{(n)} = \mathbf{\Psi}^{(n)}_{\mathrm{df}} \cup \mathbf{\Psi}^{(n)}_{\mathrm{gr}}$ *and* $\tilde{\mathbf{\Psi}}^{(n)} = \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{df}} \cup \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{gr}}$ *for* $\widehat{L_2(\mathrm{I}^n)^n}$ (*of wavelet type*), *such that*

$$\mathbf{\Psi}^{(n)}_{\mathrm{df}} \subset \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n), \quad \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{gr}} \subset \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n).$$

Corollary 2.2. *It holds that* $\mathbf{\Psi}^{(n)}_{\mathrm{df}}$ *and* $\tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{gr}}$ *are Riesz bases for* $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$ *and* $\mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n)$, *respectively.*

Proof. For say $\mathbf{u} \in \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$, by $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n) \perp \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n)$ we have $\mathbf{u} = \langle \mathbf{u}, \tilde{\mathbf{\Psi}}^{(n)} \rangle_{L_2(\mathrm{I}^n)^n} \mathbf{\Psi}^{(n)}$ $= \langle \mathbf{u}, \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{df}} \rangle_{L_2(\mathrm{I}^n)^n} \mathbf{\Psi}^{(n)}_{\mathrm{df}}$ and $\|\mathbf{u}\|_{L_2(\mathrm{I}^n)^n} \eqsim \|\langle \mathbf{u}, \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{df}} \rangle_{L_2(\mathrm{I}^n)^n}\|_{\ell_2}$. □

Having biorthogonal Riesz bases $\mathbf{\Psi}^{(n)}$, $\tilde{\mathbf{\Psi}}^{(n)}$ as in Proposition 2.1 implies the following result:

Proposition 2.3. *The following Helmholtz decomposition holds:*

$$\widehat{L_2(\mathrm{I}^n)^n} = \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n) \oplus^{\perp} \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n),$$

that for $n = 1$ *has to be read as* $L_2(\mathrm{I})$ $(= \widehat{L_2(\mathrm{I})}) = (\widehat{H}^1(\mathrm{I}))'$.

Proof. Since $\mathbf{u} = \langle \mathbf{u}, \mathbf{\Psi}^{(n)}_{\mathrm{df}} \rangle_{L_2(\mathrm{I}^n)^n} \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{df}} + \langle \mathbf{u}, \mathbf{\Psi}^{(n)}_{\mathrm{gr}} \rangle_{L_2(\mathrm{I}^n)^n} \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{gr}} = \langle \mathbf{u}, \mathbf{\Psi}^{(n)}_{\mathrm{gr}} \rangle_{L_2(\mathrm{I}^n)^n} \tilde{\mathbf{\Psi}}^{(n)}_{\mathrm{gr}}$ for any $\mathbf{u} \in \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)^{\perp}$, we have $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)^{\perp} \subset \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n)$, and so, by $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)^{\perp} \supset \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n)$, $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)^{\perp} = \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n)$. Since $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$ is closed in $\widehat{L_2(\mathrm{I}^n)^n}$, we have $\widehat{L_2(\mathrm{I}^n)^n} = \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n) \oplus \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)^{\perp}$. □

Next, we make an orthogonal decomposition of $L_2(I^n)^n$ into $\widehat{L_2(\mathrm{I}^n)^n}$ and spaces of vector fields that have one or more zero coordinates and that are independent of the corresponding variables. Let us explain the idea for $n = 3$. Consider the orthogonal decompositions $L_2 = L_{2,0} \oplus^{\perp} \mathbb{1}$ and

$$\begin{aligned}
L_2 \otimes L_2 \otimes L_2 &= L_2 \otimes L_{2,0} \otimes L_{2,0} \oplus^{\perp} L_2 \otimes L_{2,0} \otimes \mathbb{1} \oplus^{\perp} L_2 \otimes \mathbb{1} \otimes L_{2,0} \oplus^{\perp} L_2 \otimes \mathbb{1} \otimes \mathbb{1} \\
L_2 \otimes L_2 \otimes L_2 &= L_{2,0} \otimes L_2 \otimes L_{2,0} \oplus^{\perp} \mathbb{1} \otimes L_2 \otimes L_{2,0} \oplus^{\perp} L_{2,0} \otimes L_2 \otimes \mathbb{1} \oplus^{\perp} \mathbb{1} \otimes L_2 \otimes \mathbb{1} \\
L_2 \otimes L_2 \otimes L_2 &= L_{2,0} \otimes L_{2,0} \otimes L_2 \oplus^{\perp} L_{2,0} \otimes \mathbb{1} \otimes L_2 \oplus^{\perp} \mathbb{1} \otimes L_{2,0} \otimes L_2 \oplus^{\perp} \mathbb{1} \otimes \mathbb{1} \otimes L_2.
\end{aligned}$$

Using a vector notation for Cartesian products, by collecting terms we find that

$$L_2(\mathrm{I}^3)^3 = \widehat{L_2(\mathrm{I}^3)^3} \tag{2.1}$$

$$\oplus^\perp \begin{bmatrix} L_2 \otimes L_{2,0} \otimes \mathbb{1} \\ L_{2,0} \otimes L_2 \otimes \mathbb{1} \\ 0 \end{bmatrix} \oplus^\perp \begin{bmatrix} L_2 \otimes \mathbb{1} \otimes L_{2,0} \\ 0 \\ L_{2,0} \otimes \mathbb{1} \otimes L_2 \end{bmatrix} \oplus^\perp \begin{bmatrix} 0 \\ \mathbb{1} \otimes L_2 \otimes L_{2,0} \\ \mathbb{1} \otimes L_{2,0} \otimes L_2 \end{bmatrix} \tag{2.2}$$

$$\oplus^\perp \begin{bmatrix} L_2 \otimes \mathbb{1} \otimes \mathbb{1} \\ 0 \\ 0 \end{bmatrix} \oplus^\perp \begin{bmatrix} 0 \\ \mathbb{1} \otimes L_2 \otimes \mathbb{1} \\ 0 \end{bmatrix} \oplus^\perp \begin{bmatrix} 0 \\ 0 \\ \mathbb{1} \otimes \mathbb{1} \otimes L_2 \end{bmatrix}. \tag{2.3}$$

By Proposition 2.3, it holds that

$$\widehat{L_2(\mathrm{I}^3)^3} = \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^3) \oplus^\perp \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^3),$$

$$\begin{bmatrix} L_2 \otimes L_{2,0} \otimes \mathbb{1} \\ L_{2,0} \otimes L_2 \otimes \mathbb{1} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^2) \otimes \mathbb{1} \\ 0 \end{bmatrix}}_{\subset \boldsymbol{\mathcal{H}}(\mathrm{I}^3)} \oplus^\perp \underbrace{\begin{bmatrix} \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^2) \otimes \mathbb{1} \\ 0 \end{bmatrix}}_{\subset \mathbf{grad}\, H^1(\mathrm{I}^3)},$$

and similarly for the other two terms in (2.2), and

$$\begin{bmatrix} L_2 \otimes \mathbb{1} \otimes \mathbb{1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (\widehat{H}^1(\mathrm{I}))' \otimes \mathbb{1} \otimes \mathbb{1} \\ 0 \\ 0 \end{bmatrix} \subset \mathbf{grad}\, H^1(\mathrm{I}^3),$$

and similarly for the other two terms in (2.3).

Having found an orthogonal decomposition of $L_2(\mathrm{I}^3)^3$ into subspaces that are either in $\boldsymbol{\mathcal{H}}(\mathrm{I}^3)$ or in $\mathbf{grad}\, H^1(\mathrm{I}^3)$, we conclude that $L_2(\mathrm{I}^3)^3 = \boldsymbol{\mathcal{H}}(\mathrm{I}^3) \oplus^\perp \mathbf{grad}\, H^1(\mathrm{I}^3)$.

To give a more formal description of above orthogonal decomposition, as well as to generalize it to $n \neq 3$, for $1 \leq k \leq n$, and $S = \{s_1, \ldots, s_k\} \subset \{1, \ldots, n\}$, we define the embedding

$$E_S^{(n)} : \widehat{L_2(\mathrm{I}^k)^k} \to L_2(\mathrm{I}^n)^n \quad \text{by} \quad (E_S^{(n)} \mathbf{v})(x) = \sum_{i=1}^k v_i(x_{s_1}, \ldots, x_{s_k}) \mathbf{e}_{s_i}.$$

So for $n = 3$, $\operatorname{Im} E_{\{1,2\}}^{(3)} = \begin{bmatrix} L_2 \otimes L_{2,0} \otimes \mathbb{1} & L_{2,0} \otimes L_2 \otimes \mathbb{1} & 0 \end{bmatrix}^\top$, $\operatorname{Im} E_{\{1\}}^{(3)} = \begin{bmatrix} L_2 \otimes \mathbb{1} \otimes \mathbb{1} & 0 & 0 \end{bmatrix}^\top$, and $E_{\{1,2,3\}}^{(3)} = I$.

With $(\,)^\top$ denoting the adjoint with respect to L_2 scalar products, it holds that $(E_S^{(n)})^\top E_{S'}^{(n)} = \begin{cases} 0 & \text{when } S \neq S', \\ \mathrm{Id} & \text{when } S = S. \end{cases}$ Furthermore,

$$L_2(\mathrm{I}^n)^n = \sum_{\emptyset \neq S \subset \{1,\ldots,n\}} \operatorname{Im} E_S^{(n)}, \tag{2.4}$$

which decomposition is thus orthogonal,

$$\operatorname{Im}(E_S^{(n)}|_{\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^{\#S})}) \subset \boldsymbol{\mathcal{H}}(\mathrm{I}^n), \quad \operatorname{Im}(E_S^{(n)}|_{\mathbf{grad}\, \widehat{H}^1(\mathrm{I}^{\#S})}) \subset \mathbf{grad}\, H^1(\mathrm{I}^n),$$

and thus, thanks to Proposition 2.3,

$$L_2(\mathrm{I}^n)^n = \sum_{S \subset \{1,\ldots,n\},\, \#S \geq 2} \operatorname{Im}(E_S^{(n)}|_{\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^{\#S})}) + \sum_{\emptyset \neq S \subset \{1,\ldots,n\}} \operatorname{Im}(E_S^{(n)}|_{\mathbf{grad}\, \widehat{H}^1(\mathrm{I}^{\#S})}),$$

which decomposition is orthogonal because of $\boldsymbol{\mathcal{H}}(\mathrm{I}^n) \perp \mathbf{grad}\, H^1(\mathrm{I}^n)$.

In view of Corollary 2.2, we end up with the following result:

Theorem 2.4. *It holds that*

$$L_2(\mathrm{I}^n)^n := \boldsymbol{\mathcal{H}}(\mathrm{I}^n) \oplus^{\perp} \mathbf{grad}\, H^1(\mathrm{I}^n),$$

that for $n = 1$ has to be read as $L_2(\mathrm{I}) = H^1(\mathrm{I})'$. In the situation of Proposition 2.1, the collections

$$\boldsymbol{\Psi}_{\mathrm{df}} := \bigcup_{S \subset \{1,\ldots,n\},\, \#S \geq 2} E_S^{(n)} \boldsymbol{\Psi}_{\mathrm{df}}^{(\#S)}, \quad \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}} := \bigcup_{\emptyset \neq S \subset \{1,\ldots,n\}} E_S^{(n)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(\#S)},$$

are Riesz bases for $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ and $\mathbf{grad}\, H^1(\mathrm{I}^n)$, respectively.

For $n = 3$, this theorem reads as the statements that

$$\boldsymbol{\Psi}_{\mathrm{df}}^{(3)} \cup E_{\{1,2\}}^{(3)} \boldsymbol{\Psi}_{\mathrm{df}}^{(2)} \cup E_{\{1,3\}}^{(3)} \boldsymbol{\Psi}_{\mathrm{df}}^{(2)} \cup E_{\{2,3\}}^{(3)} \boldsymbol{\Psi}_{\mathrm{df}}^{(2)}$$

is a Riesz basis for $\boldsymbol{\mathcal{H}}(\mathrm{I}^3)$, and that

$$\tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(3)} \cup E_{\{1,2\}}^{(3)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(2)} \cup E_{\{1,3\}}^{(3)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(2)} \cup E_{\{2,3\}}^{(3)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(2)} \cup E_{\{1\}}^{(3)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(1)} \cup E_{\{2\}}^{(3)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(1)} \cup E_{\{3\}}^{(3)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(1)}$$

is a Riesz basis for $\mathbf{grad}\, H^1(\mathrm{I}^3)$. For $n = 2$, it reads as the statements that $\tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(2)} \cup E_{\{1\}}^{(2)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(1)} \cup E_{\{2\}}^{(2)} \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}^{(1)}$ is a Riesz basis for $\mathbf{grad}\, H^1(\mathrm{I}^2)$, and that $\boldsymbol{\Psi}_{\mathrm{df}}^{(2)}$ is a Riesz basis for $\boldsymbol{\mathcal{H}}(\mathrm{I}^2)$, in particular meaning that $\boldsymbol{\mathcal{H}}(\mathrm{I}^2) = \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^2)$.

2.2. Decomposition of (subspaces of) $H^1(\mathrm{I}^n)^n$ and $H^2(\mathrm{I}^n)^n$.

With

$$\mathbf{H}_0^1(\mathrm{I}^n) := \{u \in H^1(\mathrm{I}^n)^n : \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\mathrm{I}^n\},$$
$$\mathbf{H}_0^2(\mathrm{I}^n) := \{u \in H^2(\mathrm{I}^n)^n : \mathbf{u} \cdot \mathbf{n} = 0,\ \tfrac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\tau}_i = 0 \text{ on } \partial\mathrm{I}^n,\ 1 \leq i \leq n-1\},$$

where $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ is an orthonormal set of tangent vectors, for solving the Stokes equations we like to construct Riesz bases for

$$\boxed{\boldsymbol{\mathcal{V}}(\mathrm{I}^n)} := \mathbf{H}_0^1(\mathrm{I}^n) \cap \boldsymbol{\mathcal{H}}(\mathrm{I}^n), \quad \boxed{\boldsymbol{\mathcal{W}}(\mathrm{I}^n)} := \mathbf{H}_0^2(\mathrm{I}^n) \cap \boldsymbol{\mathcal{H}}(\mathrm{I}^n).$$

Similarly to the construction of a Riesz basis for $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$, we will construct these bases by making an orthogonal splitting of $\mathbf{H}_0^1(\mathrm{I}^n)$ and $\mathbf{H}_0^2(\mathrm{I}^n)$.

Note that an equivalent formulation of the boundary conditions involved in the definition of $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$ is, for $1 \leq i \leq n$, $u_i = 0$ on $x_i \in \{0, 1\}$, and $\frac{\partial u_i}{\partial \mathbf{n}} = 0$ on the other faces of the boundary.

The following result will be verified in Sect. 3.

Proposition 2.5. *For $m \in \{1, 2\}$, the collection $\boldsymbol{\Psi}^{(n)}$ from Proposition 2.1 can be constructed such that, normalized in $H^m(\mathrm{I}^n)^n$, it is a Riesz basis for*

$$\widehat{\mathbf{H}}_0^m(\mathrm{I}^n) := \mathbf{H}_0^m(\mathrm{I}^n) \cap \widehat{L_2(\mathrm{I}^n)^n}.$$

Corollary 2.6. *It holds that $\boldsymbol{\Psi}_{\mathrm{df}}^{(n)}$, normalized in $H^m(\mathrm{I}^n)^n$, is a Riesz basis for*

$$\widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n) := \boldsymbol{\mathcal{V}}(\mathrm{I}^n) \cap \widehat{L_2(\mathrm{I}^n)^n} \quad (m = 1)$$

or for

$$\widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n) := \boldsymbol{\mathcal{W}}(\mathrm{I}^n) \cap \widehat{L_2(\mathrm{I}^n)^n} \quad (m = 2).$$

Proof. By Proposition 2.5 and $\tilde{\boldsymbol{\Psi}}^{(n)}_{\mathrm{gr}} \subset \mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n)$, for $\mathbf{u} \in \widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n)$ ($m = 1$) or $\mathbf{u} \in \widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n)$ ($m = 2$), we have $\mathbf{u} = \langle \mathbf{u}, \tilde{\boldsymbol{\Psi}}^{(n)} \rangle_{L_2(\mathrm{I}^n)^n} \boldsymbol{\Psi}^{(n)} = \langle \mathbf{u}, \tilde{\boldsymbol{\Psi}}^{(n)}_{\mathrm{df}} \rangle_{L_2(\mathrm{I}^n)^n} \boldsymbol{\Psi}^{(n)}_{\mathrm{df}}$ in $H^m(\mathrm{I}^n)^n$ with $\|\mathbf{u}\|^2_{H^m(\mathrm{I}^n)^n} \eqsim \sum_{\tilde{\boldsymbol{\psi}} \in \tilde{\boldsymbol{\Psi}}^{(n)}_{\mathbf{df}}} |\langle \mathbf{u}, \tilde{\boldsymbol{\psi}} \rangle|^2 \|\boldsymbol{\psi}_{\tilde{\boldsymbol{\psi}}}\|^2_{H^m(\mathrm{I}^n)^n}$, where $\boldsymbol{\psi}_{\tilde{\boldsymbol{\psi}}} \in \boldsymbol{\Psi}^{(n)}_{\mathrm{df}}$ denotes the primal wavelet corresponding to $\tilde{\boldsymbol{\psi}}$. □

For $m \in \{1, 2\}$, $\emptyset \neq S \subset \{1, \ldots, n\}$, we have that $E^{(n)}_S|_{\widehat{\mathbf{H}}^m_0(\mathrm{I}^{\#S})} : \widehat{\mathbf{H}}^m_0(\mathrm{I}^{\#S}) \to \mathbf{H}^m_0(\mathrm{I}^n)$ and

$$(E^{(n)}_S|_{\widehat{\mathbf{H}}^m_0(\mathrm{I}^{\#S})})^* (E^{(n)}_{S'}|_{\widehat{\mathbf{H}}^1_0(\mathrm{I}^{\#S'})}) = \begin{cases} 0 & \text{when } S \neq S', \\ \mathrm{Id} & \text{when } S = S', \end{cases}$$

where $(\ )^*$ denotes the adjoint with respect to the H^m inner products. Moreover,

$$\mathrm{Im}(E^{(n)}_S|_{\widehat{\mathbf{H}}^m_0(\mathrm{I}^{\#S})}) = \mathbf{H}^m_0(\mathrm{I}^n) \cap \mathrm{Im}\, E^{(n)}_S,$$

so that

$$\mathbf{H}^m_0(\mathrm{I}^n) = \sum_{\emptyset \neq S \subset \{1,\ldots,n\}} \mathrm{Im}(E^{(n)}_S|_{\widehat{\mathbf{H}}^m_0(\mathrm{I}^{\#S})}),$$

which decomposition is stable, since even orthogonal.

Using Proposition 2.5, we conclude that, normalized in $H^m(\mathrm{I}^n)^n$,

$$\sum_{\emptyset \neq S \subset \{1,\ldots,n\}} E^{(n)}_S \boldsymbol{\Psi}^{(\#S)} \quad \text{is a Riesz basis for} \quad \mathbf{H}^m_0(\mathrm{I}^n).$$

Recalling that $(E^{(n)}_S)^\top E^{(n)}_{S'} = \begin{cases} 0 & \text{when } S \neq S', \\ \mathrm{Id} & \text{when } S = S', \end{cases}$ we infer that its dual basis is $\sum_{\emptyset \neq S \subset \{1,\ldots,n\}} E^{(n)}_S \tilde{\boldsymbol{\Psi}}^{(\#S)}$. Splitting $\boldsymbol{\Psi}^{(\#S)} = \boldsymbol{\Psi}^{(\#S)}_{\mathrm{df}} \cup \boldsymbol{\Psi}^{(\#S)}_{\mathrm{gr}}$ and $\tilde{\boldsymbol{\Psi}}^{(\#S)} = \tilde{\boldsymbol{\Psi}}^{(\#S)}_{\mathrm{df}} \cup \tilde{\boldsymbol{\Psi}}^{(\#S)}_{\mathrm{gr}}$, and recalling that $E^{(n)}_S \boldsymbol{\Psi}^{(\#S)}_{\mathrm{df}} \subset \boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ and $E^{(n)}_S \tilde{\boldsymbol{\Psi}}^{(\#S)}_{\mathrm{gr}} \subset \mathbf{grad}\, H^1(\mathrm{I}^n)$, we end up with the following result:

Theorem 2.7. *In the situation of Propositions 2.1 and 2.5, for $m \in \{1, 2\}$ the collection $\boldsymbol{\Psi}_{\mathrm{df}} = \bigcup_{S \subset \{1,\ldots,n\}, \#S \geq 2} E^{(n)}_S \boldsymbol{\Psi}^{(\#S)}_{\mathrm{df}}$, normalized in $H^m(\mathrm{I}^n)^n$, is a Riesz basis for $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$* ($m = 1$) or for $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$ ($m = 2$).

Remark 2.8. In view of solving for the pressure component of the Stokes equations, we collect some additional facts about the wavelet basis construction: Let $L_{2,0}(\mathrm{I}^n) := \{u \in L_2(\mathrm{I}^n) : \int_{\mathrm{I}^n} u \, d\mathbf{x} = 0\}$. By Friedrich's inequality, $\mathbf{grad} : H^1(\mathrm{I}^n) \cap L_{2,0}(\mathrm{I}^n) \to \mathbf{grad}\, H^1(\mathrm{I}^n)$ is boundedly invertible. So, in the setting of Theorem 2.4, the unique collection $\tilde{\Pi} \subset H^1(\mathrm{I}^n) \cap L_{2,0}(\mathrm{I}^n)$ with $\mathbf{grad}\, \tilde{\Pi} = \tilde{\boldsymbol{\Psi}}_{\mathrm{gr}}$ is a Riesz basis of that space. Similarly, in the setting of Corollary 2.2, for $1 \leq k \leq n$, the unique collection $\tilde{\Pi}^{(k)} \subset \widehat{H}^1(\mathrm{I}^k)$, with $\mathbf{grad}\, \tilde{\Pi}^{(k)} = \tilde{\boldsymbol{\Psi}}^{(k)}_{\mathbf{gr}}$ is a Riesz basis for $\widehat{H}^1(\mathrm{I}^k)$.

For $1 \leq k \leq n$, and $S = \{s_1, \ldots, s_k\} \subset \{1, \ldots, n\}$, let us define the embeddding

$$F^{(n)}_S : \underbrace{L_{2,0} \otimes \cdots \otimes L_{2,0}}_{k\times} \to L_{2,0}(\mathrm{I}^n) \quad \text{by} \quad (F^{(n)}_S v)(\mathbf{x}) = v(x_{s_1}, \ldots, x_{s_k}).$$

Then $E^{(n)}_S \circ \mathbf{grad} = \mathbf{grad} \circ F^{(n)}_S$, and we infer that

$$\tilde{\Pi} = \cup_{\emptyset \neq S \subset \{1,\ldots,n\}} F^{(n)}_S \tilde{\Pi}^{(\#S)}.$$

We have $L_{2,0}(\mathrm{I}^n) = \sum_{\emptyset\neq S\subset\{1,\ldots,n\}} \operatorname{Im} F_S^{(n)}$, $(F_S^{(n)})^\top F_S^{(n)} = \begin{cases} 0 & \text{when } S \neq S', \\ \mathrm{Id} & \text{when } S = S. \end{cases}$ We conclude that for $\Pi^{(k)} \subset \underbrace{L_{2,0} \otimes \cdots \otimes L_{2,0}}_{k\times}$ being the (unique) collection that is dual to $\tilde{\Pi}^{(k)}$ (as we will see, a dual collection with this regularity exists), $\Pi = \cup_{\emptyset\neq S\subset\{1,\ldots,n\}} F_S^{(n)} \Pi^{(\#S)}$ is the unique collection in $L_{2,0}(\mathrm{I}^n)$ that is dual to $\tilde{\Pi}$.

For $m \in \mathbb{N}$, we have $H^m(\mathrm{I}^n) \cap L_{2,0}(\mathrm{I}^n) = \sum_{\emptyset\neq S\subset\{1,\ldots,n\}} \operatorname{Im}(F_S^{(n)}|_{\widehat{H}^m(\mathrm{I}^{\#S})})$, and $(F_S^{(n)}|_{\widehat{H}^m(\mathrm{I}^{\#S})})^*(F_S^{(n)}|_{\widehat{H}^m(\mathrm{I}^{\#S})}) = \begin{cases} 0 & \text{when } S \neq S', \\ \mathrm{Id} & \text{when } S = S. \end{cases}$ Therefore, $\tilde{\Pi}$ (Π) is, properly scaled, a Riesz basis for $H^m(\mathrm{I}^n) \cap L_{2,0}(\mathrm{I}^n)$ if and only if for all $1 \leq k \leq n$, $\tilde{\Pi}^{(k)}$ ($\Pi^{(k)}$) is, properly scaled, a Riesz basis for $\widehat{H}^m(\mathrm{I}^k)$. Furthermore, $\tilde{\Pi}$ (Π) is, properly scaled, a Riesz basis for $(H^m(\mathrm{I}^n) \cap L_{2,0}(\mathrm{I}^n))'$ if and only if, for $1 \leq k \leq n$, $\Pi^{(k)}$ ($\tilde{\Pi}^{(k)}$) is, properly scaled, a Riesz basis for $\widehat{H}^m(\mathrm{I}^k)$.

3. Riesz bases of wavelet type for $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$, $\widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n)$, and $\widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n)$

The construction presented in this section is similar to that from [Ste08]. For this reason the exposition is kept concise. In Subsect. 3.1, starting from a pair of biorthogonal univariate wavelet bases for $L_{2,0}(\mathrm{I})$, a new pair is constructed by integration/differentiation. This construction generalizes the one from [LR92] for the stationary multiresolution case on the line. Then, in Subsect. 3.2, the divergence-free wavelets on the hypercube are constructed as Cartesian products of tensor products of these univariate wavelets.

The idea to construct divergence-free wavelets by essentially tensorizing univariate wavelet bases was proposed in [DP06]. Compared to *isotropic* divergence-free wavelet bases as developed in [LR92], the construction of such *anisotropic* divergence-free wavelet bases is simpler, and the so-called "curse of dimensionality" is avoided as the convergence rates that are obtained with such bases are (nearly) dimension independent.

3.1. **Biorthogonal wavelets on the interval.** Let

$$\Psi = \{\psi_\lambda : \lambda \in \nabla\}, \quad \tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \nabla\}$$

be biorthogonal Riesz bases for $L_{2,0}(\mathrm{I})$ of wavelet type, such that

$$\{2^{-|\lambda|}\tilde{\psi}_\lambda : \lambda \in \nabla\} \text{ is a Riesz basis for } \widehat{H}^1(\mathrm{I}) \ (= H^1(\mathrm{I}) \cap L_{2,0}(\mathrm{I})).$$

Here $|\lambda| \in \mathbb{N}_0$ denotes the *level* of λ.

From this pair, by integration or differentiation, we define another pair of primal and dual wavelets. Define

$$\Psi^+ = \{\psi_\lambda^+ : \lambda \in \nabla\}, \quad \tilde{\Psi}^- = \{\tilde{\psi}_\lambda^- : \lambda \in \nabla\} \tag{3.1}$$

by

$$\psi_\lambda^+ : x \mapsto \int_0^x 2^{|\lambda|}\psi_\lambda(y)dy, \quad \tilde{\psi}_\lambda^- = -2^{-|\lambda|}\dot{\tilde{\psi}}_\lambda.$$

Proposition 3.1. $(\Psi^+, \tilde{\Psi}^-)$ *are biorthogonal Riesz bases for* $L_2(\mathrm{I})$, $\{2^{-|\lambda|}\psi_\lambda^+ : \lambda \in \nabla\}$ *is a Riesz basis for* $H_0^1(\mathrm{I})$, *and* $\operatorname{supp} \psi_\lambda^+ \subset \operatorname{convhull}(\operatorname{supp} \psi_\lambda)$, $\operatorname{supp} \psi_\lambda^- \subset \operatorname{supp} \tilde{\psi}_\lambda$.

Proof. The last statement is obvious, and the one but last follows from $\Psi \subset L_{2,0}(\mathrm{I})$. This property also implies that $\Psi^+ \subset H_0^1(\mathrm{I})$, from which it follows that

$$\langle \psi_\lambda^+, \tilde{\psi}_\mu^- \rangle_{L_2(\mathrm{I})} = \langle \psi_\lambda^+, -2^{-|\mu|}\dot{\tilde{\psi}}_\mu^- \rangle_{L_2(\mathrm{I})} = 2^{-|\mu|}\langle \dot{\psi}_\lambda^+, \tilde{\psi}_\mu \rangle_{L_2(\mathrm{I})} = 2^{|\lambda|-|\mu|}\langle \psi_\lambda, \tilde{\psi}_\mu \rangle_{L_2(\mathrm{I})},$$

and so that $(\Psi^+, \tilde{\Psi}^-)$ are biorthogonal.

Since $L_{2,0}(\mathrm{I}) \to H_0^1(\mathrm{I}) : g \mapsto (x \mapsto \int_0^x g(y)dy)$ is bounded, with bounded inverse $f \mapsto \dot{f}$, Ψ being a Riesz basis for $L_{2,0}(\mathrm{I})$ is equivalent to $\{2^{-|\lambda|}\psi_\lambda^+ : \lambda \in \nabla\}$ being a Riesz basis for $H_0^1(\mathrm{I})$.

Since $\widehat{H}^1(\mathrm{I}) \to L_2(\mathrm{I}) : g \mapsto \dot{g}$ is bounded, with bounded inverse $f \mapsto (x \mapsto \int_0^x f(y)dy - \int_0^1 \int_0^z f(y)dydz$, $\{2^{-|\lambda|}\tilde{\psi}_\lambda : \lambda \in \nabla\}$ being a Riesz basis for $\widehat{H}^1(\mathrm{I})$ is equivalent to $\tilde{\Psi}^-$ being a Riesz basis for $L_2(\mathrm{I})$. □

Arguments as applied in above proof can also be used to show that if, properly scaled, Ψ is a Riesz basis for a scale of Sobolev spaces, then so is Ψ^+ for a shifted scale. Below, however, we follow the alternative route of verifying Jackson and Bernstein estimates because of their own interest. We omit the proof since it may follow standard lines.

Proposition 3.2. *In addition to biorthogonality of $(\Psi, \tilde{\Psi}) \subset L_{2,0}(\mathrm{I})$, assume that for some $0 < \gamma < d \in \mathbb{N}$, $1 < \tilde{\gamma} < \tilde{d} \in \mathbb{N}$,*

$$\inf_{v \in \mathrm{span}\{\psi_\lambda : |\lambda| \leq \ell\}} \|u - v\|_{L_2(\mathrm{I})} \lesssim 2^{-\ell d}\|u\|_{H^d(\mathrm{I})} \quad (u \in \widehat{H}^d(\mathrm{I})), \tag{3.2}$$

$$\inf_{v \in \mathrm{span}\{\tilde{\psi}_\lambda : |\lambda| \leq \ell\}} \|u - v\|_{L_2(\mathrm{I})} \lesssim 2^{-\ell \tilde{d}}\|u\|_{H^{\tilde{d}}(\mathrm{I})} \quad (u \in \widehat{H}^{\tilde{d}}(\mathrm{I})), \tag{3.3}$$

$$\textit{for } s < \gamma, \ \|\cdot\|_{H^s(\mathrm{I})} \lesssim 2^{\ell s}\|\cdot\|_{L_2(\mathrm{I})} \ \textit{ on } \ \mathrm{span}\{\psi_\lambda : |\lambda| \leq \ell\}, \tag{3.4}$$

$$\textit{for } s < \tilde{\gamma}, \ \|\cdot\|_{H^s(\mathrm{I})} \lesssim 2^{\ell s}\|\cdot\|_{L_2(\mathrm{I})} \ \textit{ on } \ \mathrm{span}\{\tilde{\psi}_\lambda : |\lambda| \leq \ell\}. \tag{3.5}$$

Then

$$\inf_{v \in \mathrm{span}\{\psi_\lambda^+ : |\lambda| \leq \ell\}} \|u - v\|_{L_2(\mathrm{I})} \lesssim 2^{-\ell(d+1)}\|u\|_{H^{d+1}(\mathrm{I})} \quad (u \in H^{d+1}(\mathrm{I}) \cap H_0^1(\mathrm{I})),$$

$$\inf_{v \in \mathrm{span}\{\tilde{\psi}_\lambda^- : |\lambda| \leq \ell\}} \|u - v\|_{L_2(\mathrm{I})} \lesssim 2^{-\ell(\tilde{d}-1)}\|u\|_{H^{\tilde{d}-1}(\mathrm{I})} \quad (u \in H^{\tilde{d}-1}(\mathrm{I})),$$

$$\textit{for } s < \gamma + 1, \ \|\cdot\|_{H^s(\mathrm{I})} \lesssim 2^{\ell s}\|\cdot\|_{L_2(\mathrm{I})} \ \textit{ on } \ \mathrm{span}\{\psi_\lambda^+ : |\lambda| \leq \ell\},$$

$$\textit{for } s < \tilde{\gamma} - 1, \ \|\cdot\|_{H^s(\mathrm{I})} \lesssim 2^{\ell s}\|\cdot\|_{L_2(\mathrm{I})} \ \textit{ on } \ \mathrm{span}\{\tilde{\psi}_\lambda^- : |\lambda| \leq \ell\}.$$

The next consequence of the Jackson and Bernstein estimates from Proposition 3.2 and the fact that $(\Psi, \tilde{\Psi})$ and $(\Psi^+, \tilde{\Psi}^-)$ are biorthogonal Riesz bases for $L_{2,0}(\mathrm{I})$ and $L_2(\mathrm{I})$, respectively, is well-known, see e.g. [Dah96].

Corollary 3.3. *Under the conditions of Proposition 3.2, it holds that*

$$\{2^{-|\lambda|s}\psi_\lambda : \lambda \in \nabla\} \textit{ is a Riesz basis for } \widehat{H}^s(\mathrm{I}),\ s \in [0, \gamma),$$

$$\{2^{-|\lambda|s}\tilde{\psi}_\lambda : \lambda \in \nabla\} \textit{ is a Riesz basis for } \widehat{H}^s(\mathrm{I}),\ s \in [0, \tilde{\gamma}),$$

$$\{2^{-|\lambda|s}\psi_\lambda^+ : \lambda \in \nabla\} \textit{ is a Riesz basis for } \mathcal{H}_0^s(\mathrm{I}),\ s \in [0, \gamma + 1),$$

$$\{2^{-|\lambda|s}\tilde{\psi}_\lambda^- : \lambda \in \nabla\} \textit{ is a Riesz basis for } H^s(\mathrm{I}),\ s \in [0, \tilde{\gamma} - 1),$$

where $\mathcal{H}_0^s(\mathrm{I}) := \begin{cases} [L_2(\mathrm{I}), H_0^1(\mathrm{I})]_{s,2} & \text{when } s \in [0,1], \\ H^s(\mathrm{I}) \cap H_0^1(\mathrm{I}) & \text{when } s \geq 1. \end{cases}$

It is easy to construct biorthogonal Riesz bases Ψ and $\tilde{\Psi}$ for $L_2(\mathrm{I})$ that satisfy the above Jackson and Bernstein assumptions for whatever values of d, $\tilde{d}$, γ, and $\tilde{\gamma}$. Simply take standard biorthogonal wavelet bases for $L_2(\mathrm{I})$ that satisfy the assumptions (3.2)-(3.5) with $\widehat{H}^d(\mathrm{I})$ and $\widehat{H}^{\tilde{d}}(\mathrm{I})$ reading as $H^d(\mathrm{I})$ and $H^{\tilde{d}}(\mathrm{I})$. These bases can be organized so that they contain only one scaling function without vanishing moment. By removing these functions, collections are obtained that satisfy the conditions of Proposition 3.2.

Remark 3.4. With an appropriate generalization of the Jackson and Bernstein assumptions, the results of Proposition 3.2 and Corollary 3.3 for the Sobolev spaces measuring smoothness in $L_2(\mathrm{I})$ can be generalized to Sobolev or Besov spaces measuring smoothness in $L_p(\mathrm{I})$ for $p \neq 2$. Such results are particularly relevant in the context of nonlinear approximation.

The collections of univariate wavelets from Corollary 3.3 will be used to construct a collection of vector valued multivariate functions that, normalized in the corresponding norms, is a Riesz basis for $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$ and $\widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n)$. For constructing such a collection that, normalized in $H^2(\mathrm{I}^n)^n$, is also a Riesz basis for $\widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n)$, homogeneous Neumann boundary conditions have to be incorporated in the construction of Ψ.

Proposition 3.5. *Let* $(\Psi, \tilde{\Psi}) \subset L_{2,0}(\mathrm{I})$ *be biorthogonal collections with*

$$\Psi \subset \{v \in \widehat{H}^2(\mathrm{I}) : \dot{v}(0) = \dot{v}(1) = 0\},$$

that for some $1 < \tilde{\gamma} < \tilde{d}$, $2 < \gamma$, *satisfy* (3.3)–(3.5), *and that, instead of* (3.2), *for some* $d > \gamma$ *satisfy*

$$\inf_{v \in \mathrm{span}\{\psi_\lambda : |\lambda| \leq \ell\}} \|u - v\|_{L_2(\mathrm{I})} \lesssim 2^{-\ell d} \|u\|_{H^d(\mathrm{I})} \quad (u \in \{v \in \widehat{H}^d(\mathrm{I}) : \dot{v}(0) = \dot{v}(1) = 0\}).$$

Then $(\Psi, \tilde{\Psi})$ *and* $(\Psi^+, \tilde{\Psi}^-)$ *are biorthogonal Riesz bases for* $L_{2,0}(\mathrm{I})$ *and* $L_2(\mathrm{I})$, *respectively,* $\{4^{-|\lambda|}\psi_\lambda : \lambda \in \nabla\}$ *is a Riesz basis for* $\{v \in \widehat{H}^2(\mathrm{I}) : \dot{v}(0) = \dot{v}(1) = 0\}$, $\{2^{-|\lambda|}\psi_\lambda : \lambda \in \nabla\}$ *is a Riesz basis for* $\widehat{H}^1(\mathrm{I})$, *and* $\{4^{-|\lambda|}\psi_\lambda^+ : \lambda \in \nabla\}$ *is a Riesz basis for* $H_0^1(\mathrm{I}) \cap H^2(\mathrm{I})$.

Proof. Similar to the first two statements of Corollary 3.3, for $s \in [0, \tilde{\gamma})$, $\{2^{-|\lambda|s}\tilde{\psi}_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $\widehat{H}^s(\mathrm{I})$, and for $s \in [0, \gamma)$, $\{2^{-|\lambda|s}\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $\breve{H}^s(\mathrm{I}) := [L_{2,0}(\mathrm{I}), \{u \in \widehat{H}^d(\mathrm{I}) : \dot{u}(0) = \dot{u}(1) = 0\}]_{s/d,2}$. In particular, $(\Psi, \tilde{\Psi})$ and $(\Psi^+, \tilde{\Psi}^-)$ are biorthogonal Riesz bases for $L_{2,0}(\mathrm{I})$ and $L_2(\mathrm{I})$ (cf. proof of Proposition 3.1 for the second statement), and, since $\breve{H}^2(\mathrm{I}) = \{v \in \widehat{H}^2(\mathrm{I}) : \dot{v}(0) = \dot{v}(1) = 0\}$, $\{4^{-|\lambda|}\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for the latter space, and, since $\breve{H}^1(\mathrm{I}) = \widehat{H}^1(\mathrm{I})$, $\{2^{-|\lambda|}\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $\widehat{H}^1(\mathrm{I})$. Finally, since $\widehat{H}^1(\mathrm{I}) \to H_0^1(\mathrm{I}) \cap H^2(\mathrm{I}) : g \mapsto (x \mapsto \int_0^x g(y)dy)$ is boundedly invertible, with inverse $f \mapsto \dot{f}$, the latter fact is equivalent to $\{4^{-|\lambda|}\psi_\lambda^+ : \lambda \in \nabla\}$ being a Riesz basis for $H_0^1(\mathrm{I}) \cap H^2(\mathrm{I})$. $\square$

Remarks 3.6. In the setting of Proposition 3.5, we have $\ddot{\psi}_\lambda^+(0) = \ddot{\psi}_\lambda^+(1) = 0$. As a consequence, the error in the best approximation from $\mathrm{span}\{\psi_\lambda^+ : |\lambda| \leq \ell\}$ for a general smooth function in $H_0^1(\Omega)\cap H^2(\Omega)$ cannot be expected to smaller than of order $\sqrt{2^{-\ell}}$ (assuming locally supported $\tilde{\Psi}$ and thus $\tilde{\Psi}^-$). Although this low rate can be compensated by local refinements towards the boundary, for our goal it is irrelevant since $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$, and so the elements of its basis, will only be used as a *test space* or as *test functions*, respectively.

As far as we know, biorthogonal univariate wavelet collections $(\Psi,\tilde{\Psi})$ as in Proposition 3.5, where thus Ψ satisfies the unusual homogeneous Neumann boundary conditions, have not been constructed before. We envisage however that the procedures given in e.g. [DKU99, Dij09] apply here as well.

3.2. **Divergence-free wavelets.** Let $(\Psi,\tilde{\Psi})$ be wavelet collections as in Proposition 3.2. Then from that proposition together with Corollary 3.3 we have the following result:

Corollary 3.7. *For $0\leq s<\gamma$ and $0\leq\tilde{s}<\tilde{\gamma}-1$,*

$$\Big\{\big(\sum_{m=1}^n 4^{|\lambda_m|}\big)^{-\frac{s}{2}}\psi_{\lambda_1}\otimes\cdots\otimes\psi^+_{\lambda_k}\otimes\cdots\otimes\psi_{\lambda_n}:\boldsymbol{\lambda}\in\boldsymbol{\nabla}:=\nabla^n\Big\},$$
$$\Big\{\big(\sum_{m=1}^n 4^{|\lambda_m|}\big)^{-\frac{\tilde{s}}{2}}\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}^-_{\lambda_k}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}:\boldsymbol{\lambda}\in\boldsymbol{\nabla}\Big\}$$

are Riesz bases for

$$\begin{array}{rc}
 & \widehat{H}^s\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes\overset{\downarrow kth\ position}{L_2}\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\ \cap\\
 & \vdots\\
kth\ position\ \rightarrow & L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes\mathcal{H}^s_0\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\ \cap\\
 & \vdots\\
 & L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes L_2\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes\widehat{H}^s
\end{array}$$

or for

$$\begin{array}{rc}
 & \widehat{H}^{\tilde{s}}\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes L_2\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\ \cap\\
 & \vdots\\
kth\ position\ \rightarrow & L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes H^{\tilde{s}}\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\ \cap\\
 & \vdots\\
 & L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes L_2\otimes L_{2,0}\otimes\cdots\otimes L_{2,0}\otimes\widehat{H}^{\tilde{s}}
\end{array}$$

respectively. Here, we wrote $\widehat{H}^s$ and $\mathcal{H}^s_0$ for $\widehat{H}^s(\mathrm{I})$ and $\mathcal{H}^s_0(\mathrm{I})$, respectively.

For $s=\tilde{s}=0$, the collections are biorthogonal.

Proof. Since, normalized in the corresponding norms, the collections are Riesz bases for each of the spaces that form the intersection, normalized in the norm of the intersection space, they generate a Riesz basis for that space.

For completeness, for Hilbert spaces K_1 and K_2, $K_1\cap K_2$ is a Hilbert space with norm $\sqrt{\|\cdot\|^2_{K_1}+\|\cdot\|^2_{K_2}}$. □

From Corollary 3.7 and the definition of $\widehat{\mathbf{H}}_0^1(\mathrm{I}^n)$ we infer the following result.

Corollary 3.8. *Setting for $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$, $1 \leq k \leq n$, the vector-valued wavelets*

$$\begin{aligned}\underline{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},k} &:= \psi_{\lambda_1} \otimes \cdots \otimes \psi_{\lambda_{k-1}} \otimes \psi^{+}_{\lambda_k} \otimes \psi_{\lambda_{k+1}} \otimes \cdots \otimes \psi_{\lambda_n} \mathbf{e}_k,\\ \underline{\tilde{\boldsymbol{\psi}}}^{(n)}_{\boldsymbol{\lambda},k} &:= \tilde{\psi}_{\lambda_1} \otimes \cdots \otimes \tilde{\psi}_{\lambda_{k-1}} \otimes \tilde{\psi}^{-}_{\lambda_k} \otimes \tilde{\psi}_{\lambda_{k+1}} \otimes \cdots \otimes \tilde{\psi}_{\lambda_n} \mathbf{e}_k,\end{aligned} \tag{3.6}$$

for $0 \leq s < \gamma$ and $0 \leq \tilde{s} < \tilde{\gamma} - 1$, we have that

$$\Big\{\Big(\sum_{m=1}^{n} 4^{|\lambda_m|}\Big)^{-\frac{s}{2}} \underline{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},k} : 1 \leq k \leq n,\ \boldsymbol{\lambda} \in \boldsymbol{\nabla}\Big\}, \quad \Big\{\Big(\sum_{m=1}^{n} 4^{|\lambda_m|}\Big)^{-\frac{\tilde{s}}{2}} \underline{\tilde{\boldsymbol{\psi}}}^{(n)}_{\boldsymbol{\lambda},k} : 1 \leq k \leq n,\ \boldsymbol{\lambda} \in \boldsymbol{\nabla}\Big\}$$

are Riesz bases for $\begin{cases} [\widehat{L_2(\mathrm{I}^n)^n}, \widehat{\mathbf{H}}_0^1(\mathrm{I}^n)]_{s,2} & \textit{when } s \in [0,1], \\ \widehat{\mathbf{H}}_0^1(\mathrm{I}^n) \cap H^s(\mathrm{I}^n)^n & \textit{when } s \geq 1, \end{cases}$ *or for* $\widehat{L_2(\mathrm{I}^n)^n} \cap H^{\tilde{s}}(\mathrm{I}^n)^n$*, respectively.*

For $s = \tilde{s} = 0$, the collections are biorthogonal, and the primal collection is in $\mathbf{H}_0(\mathrm{div}; \mathrm{I}^n)$.

For any $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$, let us now select an orthogonal $\mathbf{A}^{(\boldsymbol{\lambda})} \in \mathbb{R}^{n \times n}$ with its nth row given by

$$\mathbf{A}^{(\boldsymbol{\lambda})}_{n\bullet} = \boldsymbol{\alpha}^{\top} \quad \text{where } \boldsymbol{\alpha}\ (= \boldsymbol{\alpha}_{\boldsymbol{\lambda}}) := [2^{|\lambda_1|} \cdots 2^{|\lambda_n|}]^{\top} / \Big(\sum_{m=1}^{n} 4^{|\lambda_m|}\Big)^{\frac{1}{2}}.$$

An example of such a matrix $\mathbf{A}^{(\boldsymbol{\lambda})}$ is given by the Householder transformation

$$\mathbf{A}^{(\boldsymbol{\lambda})} = I - \frac{2(\boldsymbol{\alpha} - \boldsymbol{e}_n)(\boldsymbol{\alpha} - \boldsymbol{e}_n)^{\top}}{(\boldsymbol{\alpha} - \boldsymbol{e}_n)^{\top}(\boldsymbol{\alpha} - \boldsymbol{e}_n)},$$

that for $n = 2, 3$ reads as

$$\begin{bmatrix} -\alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \quad \begin{bmatrix} 1 - \frac{\alpha_1^2}{1-\alpha_3} & -\frac{\alpha_1 \alpha_2}{1-\alpha_3} & \alpha_1 \\ -\frac{\alpha_1 \alpha_2}{1-\alpha_3} & 1 - \frac{\alpha_2^2}{1-\alpha_3} & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix},$$

respectively.

We use the matrices $\mathbf{A}^{(\boldsymbol{\lambda})}$ to orthogonally transform the bases from Corollary 2.2: We define $\boldsymbol{\Psi}^{(n)} = \{\boldsymbol{\psi}^{(n)}_{\boldsymbol{\lambda},k} : \boldsymbol{\lambda} \in \boldsymbol{\nabla},\ 1 \leq k \leq n\}$, $\tilde{\boldsymbol{\Psi}}^{(n)} = \{\tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},k} : \boldsymbol{\lambda} \in \boldsymbol{\nabla},\ 1 \leq k \leq n\}$ by setting for any $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$,

$$\begin{bmatrix} \boldsymbol{\psi}^{(n)}_{\boldsymbol{\lambda},1} \\ \vdots \\ \boldsymbol{\psi}^{(n)}_{\boldsymbol{\lambda},n} \end{bmatrix} := \mathbf{A}^{(\boldsymbol{\lambda})} \begin{bmatrix} \underline{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},1} \\ \vdots \\ \underline{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},n} \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},n} \\ \vdots \\ \tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},n} \end{bmatrix} := \mathbf{A}^{(\boldsymbol{\lambda})} \begin{bmatrix} \underline{\tilde{\boldsymbol{\psi}}}^{(n)}_{\boldsymbol{\lambda},1} \\ \vdots \\ \underline{\tilde{\boldsymbol{\psi}}}^{(n)}_{\boldsymbol{\lambda},n} \end{bmatrix}. \tag{3.7}$$

Now we are ready to verify the claims made in Propositions 2.1 and 2.5:

Proposition 3.9. (a). *For $0 \leq s < \gamma$ and $0 \leq \tilde{s} < \tilde{\gamma} - 1$,*

$$\Big\{\Big(\sum_{m=1}^{n} 4^{|\lambda_m|}\Big)^{-\frac{s}{2}} \boldsymbol{\psi}^{(n)}_{\boldsymbol{\lambda},k} : 1 \leq k \leq n,\ \boldsymbol{\lambda} \in \boldsymbol{\nabla}\Big\}, \Big\{\Big(\sum_{m=1}^{n} 4^{|\lambda_m|}\Big)^{-\frac{\tilde{s}}{2}} \tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda},k} : 1 \leq k \leq n,\ \boldsymbol{\lambda} \in \boldsymbol{\nabla}\Big\}$$

are Riesz bases for $\begin{cases} [\widehat{L_2(\mathrm{I}^n)^n}, \widehat{\mathbf{H}}_0^1(\mathrm{I}^n)]_{s,2} & \textit{when } s \in [0,1], \\ \widehat{\mathbf{H}}_0^1(\mathrm{I}^n) \cap H^s(\mathrm{I}^n)^n & \textit{when } s \geq 1, \end{cases}$ *or for* $\widehat{L_2(\mathrm{I}^n)^n} \cap H^{\tilde{s}}(\mathrm{I}^n)^n$*, respectively. Furthermore* $\langle \boldsymbol{\Psi}^{(n)}, \tilde{\boldsymbol{\Psi}}^{(n)} \rangle_{L_2(\mathrm{I}^n)^n} = \mathrm{Id}$, *and* $\boldsymbol{\Psi}^{(n)} \subset \mathbf{H}_0(\mathrm{div}; \mathrm{I}^n)$.

(b). *With* $\mathbf{\Psi}_{\rm df}^{(n)} := \{\boldsymbol{\psi}_{\boldsymbol{\lambda},k}^{(n)} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}, 1 \leq k \leq n-1\}$, $\tilde{\mathbf{\Psi}}_{\rm gr}^{(n)} := \{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda},n}^{(n)} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}$, *and* $\mathbf{\Psi}_{\rm gr}^{(n)} := \mathbf{\Psi}^{(n)} \setminus \mathbf{\Psi}_{\rm df}^{(n)}$ *and* $\tilde{\mathbf{\Psi}}_{\rm df}^{(n)} := \tilde{\mathbf{\Psi}}^{(n)} \setminus \tilde{\mathbf{\Psi}}_{\rm gr}^{(n)}$, *we have*

$$\mathbf{\Psi}_{\rm df}^{(n)} \subset \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n), \quad \tilde{\mathbf{\Psi}}_{\rm gr}^{(n)} = \mathbf{grad}\, \tilde{\Pi}^{(n)},$$

where $\tilde{\Pi}^{(n)} = \{-\tilde{\psi}_{\lambda_1} \otimes \cdots \otimes \tilde{\psi}_{\lambda_n} / (\sum_{m=1}^n 4^{|\lambda_m|})^{\frac{1}{2}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\} \subset \widehat{H}^1(\mathrm{I}^n)$.

Proof. (a). This part is a consequence of Corollary 3.8. Biorthogonality of the collections from Corollary 3.8 is preserved because $\mathbf{A}^{(\boldsymbol{\lambda})}$ is orthogonal. The remainder follows from the fact that the scaling factors $(\sum_{m=1}^n 4^{|\lambda_m|})^{-s/2}$ and $(\sum_{m=1}^n 4^{|\lambda_m|})^{-\tilde{s}/2}$ in the statement of Corollary 3.8 are independent of k.

(b). By definition of the nth row of the orthogonal $\mathbf{A}^{(\boldsymbol{\lambda})}$, and $\dot{\psi}_\lambda^+ = 2^{|\lambda|}\psi_\lambda$, for $1 \leq k \leq n-1$,

$$\operatorname{div} \boldsymbol{\psi}_{\boldsymbol{\lambda},k}^{(n)} = \sum_{m=1}^n \mathbf{A}_{km}^{(\boldsymbol{\lambda})} \operatorname{div} \underline{\boldsymbol{\psi}}_{\boldsymbol{\lambda},m}^{(n)} = \Big(\sum_{m=1}^n \mathbf{A}_{km}^{(\boldsymbol{\lambda})} 2^{|\lambda_m|}\Big) \psi_{\lambda_1} \otimes \cdots \otimes \psi_{\lambda_n} = 0.$$

The last statement follows from $\tilde{\psi}_\lambda^- = -2^{-|\lambda|}\dot{\tilde{\psi}}_\lambda$. □

Having this result, from Corollaries 2.2 and 2.6 recall that $\mathbf{\Psi}_{\rm df}^{(n)}$ and $\tilde{\mathbf{\Psi}}_{\rm gr}^{(n)}$ *are Riesz bases for* $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$ *and* $\mathbf{grad}\, \widehat{H}^1(\mathrm{I}^n)$, respectively, and that, when $\gamma > 1$, for $s \in [1, \gamma)$, $\{(\sum_{m=1}^n 4^{|\lambda_m|})^{-s/2}\boldsymbol{\psi}_{\boldsymbol{\lambda},k}^{(n)} : 1 \leq k \leq n-1, \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}$ *is a Riesz basis for* $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n) \cap H^s(\mathrm{I}^n)^n$, *and so, for* $s = 1$, *in particular for* $\widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n)$.

With a biorthogonal collection $(\Psi, \tilde{\Psi})$ as in Proposition 3.5, the analogous construction of divergence-free wavelets can be followed.

Proposition 3.10. (a). *For* $0 \leq s \leq 2$ *and* $0 \leq \tilde{s} < \tilde{\gamma} - 1$,

$$\Big\{\Big(\sum_{m=1}^n 4^{|\lambda_m|}\Big)^{-\frac{s}{2}} \boldsymbol{\psi}_{\boldsymbol{\lambda},k}^{(n)} : 1 \leq k \leq n, \boldsymbol{\lambda} \in \boldsymbol{\nabla}\Big\}, \Big\{\Big(\sum_{m=1}^n 4^{|\lambda_m|}\Big)^{-\frac{\tilde{s}}{2}} \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda},k}^{(n)} : 1 \leq k \leq n, \boldsymbol{\lambda} \in \boldsymbol{\nabla}\Big\}$$

are Riesz bases for $[\widehat{L_2(\mathrm{I}^n)^n}, \widehat{\mathbf{H}}_0^2(\mathrm{I}^n)]_{s,2}$ *or for* $\widehat{L_2(\mathrm{I}^n)^n} \cap H^s(\mathrm{I}^n)^n$, *respectively. Furthermore* $\langle \mathbf{\Psi}^{(n)}, \tilde{\mathbf{\Psi}}^{(n)}\rangle_{L_2(\mathrm{I}^n)^n} = \mathrm{Id}$, *and* $\mathbf{\Psi}^{(n)} \subset \mathbf{H}_0(\mathrm{div}; \mathrm{I}^n)$.

(b). *Equal to Part* (b) *of Proposition 3.9.*

As was shown in Corollaries 2.2 and 2.6, with this result we have that for $s = 0, 1, 2$, $\{(\sum_{m=1}^n 2^{|\lambda_m|})^{-s/2}\boldsymbol{\psi}_{\boldsymbol{\lambda},k}^{(n)} : 1 \leq k \leq n-1, \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}$ *is a Riesz basis for* $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$, $\widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n)$, *and* $\widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n)$, respectively.

4. Simultaneous space-time variational formulation of the time-dependent Stokes equations

For given functions $\mathbf{q}$, g_i and a divergence-free vector field $\mathbf{u}_0$, we consider the instationary Stokes problem of finding the velocities $\mathbf{u}$ and pressure p that satisfy

$$\left\{\begin{array}{rll} \frac{\partial \mathbf{u}}{\partial t} - \boldsymbol{\Delta}\mathbf{u} - \mathbf{grad}\, p = \mathbf{q} & \text{on } [0,T] \times \mathrm{I}^n, \\ \operatorname{div} \mathbf{u} = 0 & \text{on } [0,T] \times \mathrm{I}^n, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } [0,T] \times \partial\mathrm{I}^n, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\tau}_i = g_i & \text{on } [0,T] \times \partial\mathrm{I}^n,\ 1 \leq i \leq n-1, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0 & \text{on } \mathrm{I}^n, \end{array}\right. \tag{4.1}$$

where $\boldsymbol{\tau}_1, \ldots \boldsymbol{\tau}_{n-1}$ is an orthonormal set of tangent vectors.

By taking the standard scalar product of the first equation with smooth test functions $\mathbf{v}$ of t and $\mathbf{x}$, that as function of $\mathbf{x}$ are divergence-free and have vanishing normals at the boundary, and that as function of t vanish at T, and by applying integration by parts in space and time, we arrive at the corresponding variational formulation

$$b(\mathbf{u},\mathbf{v}) = \mathbf{f}(\mathbf{v}),$$

for all such $\mathbf{v}$, where

$$b(\mathbf{w},\mathbf{v}) := -\int_0^T \int_{\mathrm{I}^n} \mathbf{w} \cdot \frac{\partial \mathbf{v}}{\partial t}\, d\mathbf{x} dt + \int_0^T a(\mathbf{w},\mathbf{v})\, dt,$$

$$a(\mathbf{w},\mathbf{v}) := \int_{\mathrm{I}^n} \mathbf{grad}_{\mathbf{x}} \mathbf{w} : \mathbf{grad}_{\mathbf{x}} \mathbf{v}\, d\mathbf{x}, \tag{4.2}$$

$$\mathbf{f}(\mathbf{v}) := \int_0^T \int_{\mathrm{I}^n} \mathbf{q} \cdot \mathbf{v} dt d\mathbf{x} + \int_0^T \int_{\partial \mathrm{I}^n} \sum_{i=1}^{n-1} (\mathbf{v} \cdot \boldsymbol{\tau}_i) g_i\, dt d\mathbf{x} + \int_{\mathrm{I}^n} \mathbf{u}_0 \cdot \mathbf{v}(0,\cdot)\, d\mathbf{x}. \tag{4.3}$$

It holds that $\boldsymbol{\mathcal{V}}(\mathrm{I}^n) \hookrightarrow \boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ with dense embedding, and a is bounded and coercive bilinear form on $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$. This dense embedding will determine the interpretation of $\int_{\mathrm{I}^n} \mathbf{w} \cdot \frac{\partial \mathbf{v}}{\partial t} d\mathbf{x}$ for $\mathbf{w} \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ and $\frac{\partial \mathbf{v}}{\partial t} \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)'$.

Theorem 4.1. *With*

$$\boldsymbol{\mathcal{X}}_0 := L_2((0,T);\boldsymbol{\mathcal{V}}(\mathrm{I}^n)),\ \boldsymbol{\mathcal{Y}}_0 := L_2((0,T);\boldsymbol{\mathcal{V}}(\mathrm{I}^n)) \cap H^1_{0,\{T\}}((0,T);\boldsymbol{\mathcal{V}}(\mathrm{I}^n)'),$$

the operator $\mathbf{B} \in \mathcal{L}(\boldsymbol{\mathcal{X}}_0, \boldsymbol{\mathcal{Y}}_0')$ *defined by* $(\mathbf{B}\mathbf{w})(\mathbf{v}) = b(\mathbf{w},\mathbf{v})$ *is boundedly invertible.*

Here $H^1_{0,\{T\}}(0,T)$ denotes the closure in $H^1(0,T)$ of the space of smooth functions on $(0,T)$ that vanish at $t = T$.

Theorem 4.1 is proved by checking the following three conditions:

$$\sup_{0\neq \mathbf{w}\in\boldsymbol{\mathcal{X}}_0,\, 0\neq\mathbf{v}\in\boldsymbol{\mathcal{Y}}_0} \frac{|b(\mathbf{w},\mathbf{v})|}{\|\mathbf{w}\|_{\boldsymbol{\mathcal{X}}_0} \|\mathbf{v}\|_{\boldsymbol{\mathcal{Y}}_0}} < \infty \quad (\textit{continuity}), \tag{4.4}$$

$$\inf_{0\neq\mathbf{v}\in\boldsymbol{\mathcal{Y}}_0} \sup_{0\neq \mathbf{w}\in\boldsymbol{\mathcal{X}}_0} \frac{|b(\mathbf{w},\mathbf{v})|}{\|\mathbf{w}\|_{\boldsymbol{\mathcal{X}}_0} \|\mathbf{v}\|_{\boldsymbol{\mathcal{Y}}_0}} > 0 \quad (\textit{inf sup-condition}), \tag{4.5}$$

$$\forall 0 \neq \mathbf{w} \in \boldsymbol{\mathcal{X}}_0,\ \sup_{0\neq\mathbf{v}\in\boldsymbol{\mathcal{Y}}_0} |b(\mathbf{w},\mathbf{v})| > 0 \quad (\textit{surjectivity}), \tag{4.6}$$

This can be done similarly to [SS09, Appendix A], cf. also e.g. [Tem79, Ch.3,§1.2] where, however, the aim was not to establish bounded invertibility of $\mathbf{B}$ between suitable Hilbert spaces, but only existence and uniqueness of the solution of the variational problem. In [SS09, Tem79], the bilinear form b and so the spaces $\boldsymbol{\mathcal{X}}_0$ and $\boldsymbol{\mathcal{Y}}_0$ were different from here, since there no integration by parts with respect to time was applied. With the current approach, the condition $\mathbf{u}(0) = \mathbf{u}_0$ is incorporated in the variational formulation as a natural boundary condition instead of an essential one.

For solving the operator equation $\mathbf{B}\mathbf{u} = \mathbf{f}$ with an adaptive wavelet scheme, the spaces $\boldsymbol{\mathcal{X}}_0$ and $\boldsymbol{\mathcal{Y}}_0$ have to be equipped with Riesz bases of wavelets type. To equip $\boldsymbol{\mathcal{Y}}_0$ with a Riesz basis, we need a collection of spatial functions that normalized in the corresponding norm is a Riesz basis for both $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ *and* $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)'$. In the previous sections, see Theorem 2.7, we constructed a Riesz basis $\boldsymbol{\Psi}_{\mathrm{df}}$ of wavelet type for $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$. It is, properly scaled, a Riesz basis for $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)'$ if and only if its dual collection is, properly scaled, a Riesz basis for $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$. In the current setting, the dual collection of $\boldsymbol{\Psi}_{\mathrm{df}}$ is the unique collection $\tilde{\boldsymbol{\Psi}}_{\mathrm{df}}$ *in* $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ with $\langle \boldsymbol{\Psi}_{\mathrm{df}}, \tilde{\boldsymbol{\Psi}}_{\mathrm{df}} \rangle_{L_2(\mathrm{I}^n)^n} = \mathrm{Id}$, i.e., $\tilde{\boldsymbol{\Psi}}_{\mathrm{df}} = \langle \boldsymbol{\Psi}_{\mathrm{df}}, \boldsymbol{\Psi}_{\mathrm{df}} \rangle^{-1}_{L_2(\mathrm{I}^n)^n} \boldsymbol{\Psi}_{\mathrm{df}}$. Unfortunately, we do not know whether this collection is, properly scaled, a Riesz basis for $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$.

For this reason, below we prove bounded invertibility of $\mathbf{B} : \boldsymbol{\mathcal{X}}_1 \to \boldsymbol{\mathcal{Y}}_1$ for other spaces $\boldsymbol{\mathcal{X}}_1$ and $\boldsymbol{\mathcal{Y}}_1$, created from $\boldsymbol{\mathcal{X}}_0$ and $\boldsymbol{\mathcal{Y}}_0$ by applying a shift in the spatial smoothness indices.

Theorem 4.2. *With*

$$\boldsymbol{\mathcal{X}}_1 := L_2((0,T); \boldsymbol{\mathcal{H}}(\mathrm{I}^n)),\ \boldsymbol{\mathcal{Y}}_1 := L_2((0,T); \boldsymbol{\mathcal{W}}(\mathrm{I}^n)) \cap H^1_{0,\{T\}}((0,T); \boldsymbol{\mathcal{H}}(\mathrm{I}^n)),$$

the operator $\mathbf{B} \in \mathcal{L}(\boldsymbol{\mathcal{X}}_1, \boldsymbol{\mathcal{Y}}_1')$ *defined by* $(\mathbf{B}\mathbf{w})(\mathbf{v}) = b(\mathbf{w}, \mathbf{v})$ *is boundedly invertible.*

Proof. From $a(\mathbf{w}, \mathbf{v}) = -\int_{\mathrm{I}^n} \mathbf{w} \cdot \boldsymbol{\Delta} \mathbf{v}\, d\mathbf{x}$ for $\mathbf{w} \in \boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ and $\mathbf{v} \in \boldsymbol{\mathcal{W}}(\mathrm{I}^n)$, it follows easily that $\mathbf{B} \in \mathcal{L}(\boldsymbol{\mathcal{X}}_1, \boldsymbol{\mathcal{Y}}_1')$.

Membership of $\mathbf{B}^{-1} \in \mathcal{L}(\boldsymbol{\mathcal{Y}}_1', \boldsymbol{\mathcal{X}}_1)$ is equivalent to $(\mathbf{B}')^{-1} \in \mathcal{L}(\boldsymbol{\mathcal{X}}_1', \boldsymbol{\mathcal{Y}}_1)$. To demonstrate the latter, we have to show that for any $\mathbf{f} \in \boldsymbol{\mathcal{X}}_1 = \boldsymbol{\mathcal{X}}_1'$, the variational problem of finding $\mathbf{z}$ such that

$$\int_0^T \int_{\mathrm{I}^n} -\mathbf{w} \cdot \frac{\partial \mathbf{z}}{\partial t}\, d\mathbf{x} dt + \int_0^T a(\mathbf{w}, \mathbf{z})\, dt = \int_0^T \int_{\mathrm{I}^n} \mathbf{f} \cdot \mathbf{w}\, d\mathbf{x} dt \quad (\mathbf{w} \in \boldsymbol{\mathcal{X}}_1), \tag{4.7}$$

has a unique solution $\mathbf{z} \in \boldsymbol{\mathcal{Y}}_1$ with $\|\mathbf{z}\|_{\boldsymbol{\mathcal{Y}}_1} \lesssim \|\mathbf{f}\|_{\boldsymbol{\mathcal{X}}_1}$.

Although this result may follow from the theory of analytic semigroups, e.g. see [MM09], we give a more elementary derivation. From Theorem 4.1 we know that for $\mathbf{f} \in \boldsymbol{\mathcal{X}}_0' \supset \boldsymbol{\mathcal{X}}_1'$, (4.7), with test space $\boldsymbol{\mathcal{X}}_0$, has a unique solution $\mathbf{z} \in \boldsymbol{\mathcal{Y}}_0$. Below, we will show that for a subspace of sufficiently smooth $\mathbf{f}$, this solution is in $\boldsymbol{\mathcal{Y}}_1$, and thus that (4.7) holds for all $\mathbf{w} \in \boldsymbol{\mathcal{X}}_1$, and moreover that $\|\mathbf{z}\|_{\boldsymbol{\mathcal{Y}}_1} \lesssim \|\mathbf{f}\|_{\boldsymbol{\mathcal{X}}_1}$. Since the subspace of these smooth $\mathbf{f}$ will be dense in $\boldsymbol{\mathcal{X}}_1$, this will complete the proof.

Equation (4.7) is the variational formulation of the problem of finding, for $t \in [0,T]$, $\mathbf{z}(t,\cdot) \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ that satisfies

$$\begin{cases} \int_{\mathrm{I}^n} -\frac{\partial \mathbf{z}}{\partial t}(t,\cdot) \cdot \mathbf{w}\, d\mathbf{x} + a(\mathbf{w}, \mathbf{z}(t,\cdot)) = \int_{\mathrm{I}^n} \mathbf{f}(t,\cdot) \cdot \mathbf{w}\, d\mathbf{x} \quad (\mathbf{w} \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)), \\ \mathbf{z}(T,\cdot) = 0. \end{cases} \tag{4.8}$$

Note that as function of $\tilde{t} = T - t$, $\mathbf{z}$ satisfies a standard parabolic *initial* value problem. As shown in [Wlo82, Ch.IV,§27], if $\mathbf{f} \in H^2((0,T); \boldsymbol{\mathcal{V}}(\mathrm{I}^n)')$ with $\mathbf{f}(T,\cdot) \in \boldsymbol{\mathcal{W}}(\mathrm{I}^n)$ and $\frac{\partial \mathbf{f}}{\partial t}(T,\cdot) \in \boldsymbol{\mathcal{H}}(\mathrm{I}^n)$, then its solution $\mathbf{z} \in H^2((0,T); \boldsymbol{\mathcal{V}}(\mathrm{I}^n))$.

By substituting $\mathbf{w} = -\frac{\partial \mathbf{z}}{\partial t}(t,\cdot) \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ in (4.8), we obtain

$$\|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\mathrm{I}^n)^n} - \frac{1}{2}\frac{\partial}{\partial t} a(\mathbf{z}(t,\cdot), \mathbf{z}(t,\cdot)) = -\int_{\mathrm{I}^n} \mathbf{f}(t,\cdot) \cdot \frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\, d\mathbf{x}.$$

By integrating this equality over time, applying $\mathbf{z}(T,\cdot) = 0$ and Cauchy-Schwarz' inequality, and by additionally assuming that $\mathbf{f} \in L_2((0,T); \boldsymbol{\mathcal{H}}(\mathrm{I}^n))$, we arrive at

$$\int_0^T \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\mathrm{I}^n)^n} dt \le \frac{1}{2}\int_0^T \|\mathbf{f}(t,\cdot)\|^2_{L_2(\mathrm{I}^n)^n} dt + \frac{1}{2}\int_0^T \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\mathrm{I}^n)^n} dt,$$

or

$$\int_0^T \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\mathrm{I}^n)^n} dt \le \int_0^T \|\mathbf{f}(t,\cdot)\|^2_{L_2(\mathrm{I}^n)^n} dt. \tag{4.9}$$

By additionally assuming that $\mathbf{f}(t,\cdot) \in \mathcal{H}(\mathrm{I}^n)$, from $\frac{\partial \mathbf{z}}{\partial t}(t,\cdot) \in \mathcal{V}(\mathrm{I}^n) \subset \mathcal{H}(\mathrm{I}^n)$ and the *elliptic regularity* result from the forthcoming Theorem 6.1, (4.8) shows that $\mathbf{z}(t,\cdot) \in \mathcal{W}(\mathrm{I}^n)$ with $\|\mathbf{z}(t,\cdot)\|_{H^2(\mathrm{I}^n)^n} \lesssim \|\mathbf{f}(t,\cdot)\|_{L_2(\mathrm{I}^n)^n} + \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|_{L_2(\mathrm{I}^n)^n}$. By integrating this inequality over time and applying (4.9), we obtain that

$$\|\mathbf{z}\|_{L_2((0,T);\mathcal{W}(\mathrm{I}^n))} \lesssim \|\mathbf{f}\|_{L_2((0,T);L_2(\mathrm{I}^n)^n)} \tag{4.10}$$

By combining (4.9) and (4.10), we have $\|\mathbf{z}\|_{\mathcal{Y}_1} \lesssim \|\mathbf{f}\|_{\mathcal{X}_1}$ and the proof is completed. □

Theorem 4.2 shows that the instationary Stokes problem can be formulated as

$$\mathbf{B}\mathbf{u} = \mathbf{f} \tag{4.11}$$

where $\mathbf{f}$ is defined in (4.3) and $\mathbf{B} \in \mathcal{L}(\mathcal{X}_1, \mathcal{Y}_1')$ is boundedly invertible.

Remark 4.3. Let us collect some sufficient conditions for $\mathbf{f} \in \mathcal{Y}_1'$. Because of $\mathcal{V}(\mathrm{I}^n) = [\mathcal{H}(\mathrm{I}^n), \mathcal{W}(\mathrm{I}^n)]_{1/2}$, for $\mathbf{v} \in \mathcal{Y}_1$, $\mathbf{v}(0,\cdot) \in \mathcal{V}(\mathrm{I}^n)$ with $\|\mathbf{v}(0,\cdot)\|_{H^1(\mathrm{I}^n)^n} \lesssim \|\mathbf{v}\|_{\mathcal{Y}_1}$ ([DL92, Ch. XVIII, §1.3]). So for $\mathbf{u}_0 \in \mathcal{V}(\mathrm{I}^n)'$, and, say, $\mathbf{q} \in L_2((0,T);\mathcal{W}(\mathrm{I}^n)')$ and $g_i \in L_2((0,T);H^{-3/2}(\partial I^n))$ $(i = 1,\ldots,n-1)$, we have $\mathbf{f} \in \mathcal{Y}_1'$ with $\|\mathbf{f}\|_{\mathcal{Y}_1'} \lesssim \|\mathbf{u}_0\|_{\mathcal{V}(\mathrm{I}^n)'} + \|\mathbf{q}\|_{L_2((0,T);\mathcal{W}(\mathrm{I}^n)')} + \sum_{i=0}^{n-1} \|g_i\|_{L_2((0,T);H^{-3/2}(\partial I^n))}$.

5. The time-dependent Stokes equations as a bi-infinite matrix vector equation, and the adaptive wavelet solver

Let $\boldsymbol{\Sigma}^{\mathcal{X}_1} = \{\sigma^{\mathcal{X}_1}\}$ and $\boldsymbol{\Sigma}^{\mathcal{Y}_1} = \{\sigma^{\mathcal{Y}_1}\}$ be *Riesz bases* for $\mathcal{X}_1$ and $\mathcal{Y}_1$, respectively. Then (4.11) can be equivalently formulated as

$$\vec{\mathbf{B}}\vec{\mathbf{u}} = \vec{\mathbf{f}}, \tag{5.1}$$

where $\vec{\mathbf{u}}$ is the vector of coefficients of $\mathbf{u}$ with respect to the basis $\boldsymbol{\Sigma}^{\mathcal{X}_1}$, $\vec{\mathbf{f}} := [\mathbf{f}(\sigma^{\mathcal{Y}_1})]_{\sigma^{\mathcal{Y}_1} \in \boldsymbol{\Sigma}^{\mathcal{Y}_1}}$ and $\vec{\mathbf{B}} := (\mathbf{B}\boldsymbol{\Sigma}^{\mathcal{X}_1})(\boldsymbol{\Sigma}^{\mathcal{Y}_1}) := [(\mathbf{B}\sigma^{\mathcal{X}_1})(\sigma^{\mathcal{Y}_1})]_{\sigma^{\mathcal{Y}_1} \in \boldsymbol{\Sigma}^{\mathcal{Y}_1}, \sigma^{\mathcal{X}_1} \in \boldsymbol{\Sigma}^{\mathcal{X}_1}}$. For $\mathbf{f} \in \mathcal{Y}_1'$, the vector $\vec{\mathbf{f}}$ is in $\ell_2(\boldsymbol{\Sigma}^{\mathcal{Y}_1})$, and by Theorem 4.2, $\vec{\mathbf{B}} \in \mathcal{L}(\ell_2(\boldsymbol{\Sigma}^{\mathcal{X}_1}), \ell_2(\boldsymbol{\Sigma}^{\mathcal{Y}_1}))$ is boundedly invertible.

Since $\mathcal{X}_1$ and $\mathcal{Y}_1$ are (intersections of) tensor products of spaces in time and space, a natural construction of Riesz bases for these spaces is as follows: Let $\Theta^{\mathcal{X}_1}$, $\Theta^{\mathcal{Y}_1}$ and $\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}$, $\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{Y}_1}$ be collections of temporal and spatial trial and test wavelets such that, normalized in the corresponding norms, $\Theta^{\mathcal{X}_1}$ is a Riesz basis for $L_2(0,T)$, $\Theta^{\mathcal{Y}_1}$ is a Riesz basis for $L_2(0,T)$ and for $H^1_{0,\{T\}}(0,T)$, $\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}$ is a Riesz basis for $\mathcal{H}(\mathrm{I}^n)$, and $\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{Y}_1}$ is a Riesz basis for $\mathcal{H}(\mathrm{I}^n)$ and for $\mathcal{W}(\mathrm{I}^n)$. Such temporal wavelets can be constructed easily, e.g. see [CS10], whereas Sections 2 and 3 of the current paper were devoted to the construction of such a collections $\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}$ and $\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{Y}_1}$, see Theorems 2.4 and 2.7.

Having such collections, normalized in the corresponding norms, $\Theta^{\mathcal{X}_1} \otimes \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}$ is a Riesz basis for $\mathcal{X}_1 = L_2(0,T) \otimes \mathcal{H}(\mathrm{I}^n)$, and $\Theta^{\mathcal{Y}_1} \otimes \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{Y}_1}$ is a Riesz basis for $L_2(0,T) \otimes \mathcal{W}(\mathrm{I}^n)$ and for $H^1_{0,\{T\}}(0,T) \otimes \mathcal{H}(\mathrm{I}^n)$, and so for $\mathcal{Y}_1$. With, for $\mathcal{Z} \in \{\mathcal{X}_1, \mathcal{Y}_1\}$, $\mathbf{D}_{\mathcal{Z}} := \operatorname{diag}\{\|\theta^{\mathcal{Z}} \otimes \boldsymbol{\psi}^{\mathcal{Z}}\|_{\mathcal{Z}} : \theta^{\mathcal{Z}} \in \Theta^{\mathcal{Z}}, \boldsymbol{\psi}^{\mathcal{Z}} \in \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{Z}}\}$, it holds that $\vec{\mathbf{f}} =$

$\vec{\mathbf{D}}_{\mathcal{Y}_1}^{-1}[\mathbf{f}(\theta^{\mathcal{Y}_1} \otimes \boldsymbol{\psi}^{\mathcal{Y}_1})]_{\theta \in \Theta^{\mathcal{Y}_1}, \boldsymbol{\psi}^{\mathcal{Y}_1} \in \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{Y}_1}}$ and

$$\vec{\mathbf{B}} = \vec{\mathbf{D}}_{\mathcal{Y}_1}^{-1}\big[- \langle \Theta^{\mathcal{X}_1}, \dot{\Theta}^{\mathcal{Y}_1} \rangle_{L_2(0,T)} \otimes \langle \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}, \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{Y}_1} \rangle_{L_2(\mathrm{I}^n)^n} + \langle \Theta^{\mathcal{X}_1}, \Theta^{\mathcal{Y}_1} \rangle_{L_2(0,T)} \otimes a(\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}, \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1})\big]\vec{\mathbf{D}}_{\mathcal{X}_1}^{-1}.$$

When the temporal and spatial wavelets are sufficiently smooth and have sufficiently many vanishing moments, then the adaptive wavelet scheme applied to $\vec{\mathbf{B}}\vec{\mathbf{u}} = \vec{\mathbf{f}}$ converges with the best possible rate in $\mathcal{X}_1$ in linear complexity. The best possible rate is that of the best N-term approximation from the span of $\Theta^{\mathcal{X}_1} \otimes \boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}$. We refer to [Ste09] and the references cited there.

When the univariate wavelet basis Ψ, that is used as a building block to construct $\boldsymbol{\Psi}_{\mathrm{df}}^{\mathcal{X}_1}$, and the temporal wavelets $\Theta^{\mathcal{X}_1}$ are of order d, i.e., when (3.2) and a similar estimate is valid for $\Theta^{\mathcal{X}_1}$, then the error in the best N-term approximation is of order $N^{-d}(\log N)^{n(\frac{1}{2}+d)}$. So, thanks to the use of tensor product bases, up to some log factors, this rate is *independent of the total dimension* $n+1$ of the time-space cylinder $[0,T] \times \mathrm{I}^n$. This result can be demonstrated using non-adaptive sparse-grid approximations, assuming sufficient smoothness of $\mathbf{u}$. By considering adaptive approximations however, the smoothness conditions can be largely reduced, being the reason to consider an adaptive scheme in the first place. For any $s < d$, a characterization of the space of vector fields that can be approximated at rate s can be given in terms of Besov smoothness, see [Nit06].

Remark 5.1. As we have seen in Subsect. 3.1, for constructing a Riesz basis for $\mathcal{W}(\mathrm{I}^n)$, we have to start with a collection of univariate wavelets Ψ that satisfies homogeneous Neumann boundary conditions. Renormalized in the corresponding norms, this $\boldsymbol{\Psi}_{\mathrm{df}}$ is also a Riesz basis for $\mathcal{V}(\mathrm{I}^n)$ and $\mathcal{H}(\mathrm{I}^n)$. Since all its elements, however, satisfy the boundary conditions incorporated in the definition of $\mathcal{W}(\mathrm{I}^n)$, locally at the boundary, $\boldsymbol{\Psi}_{\mathrm{df}}$ has strongly reduced approximation properties for a velocity field that does not satisfy these boundary conditions, i.e., when one or more g_i's are not identically zero (Ψ satisfies (3.2) only for $d=1$). Although this can be compensated by local refinements at the boundary (the non-linear approximation classes can be shown to be only marginal smaller as a consequence of the low order at the boundary, see [CS10, §6.7]), it can be expected to be quantitatively disadvantageous. Concluding we can say that when not all g_i's are zero, for the construction of the basis for $\mathcal{H}(\mathrm{I}^n)$ and thus for the trial space $\mathcal{X}_1$, it is better to use a collection Ψ that satisfies (3.2) for a d that in any case is larger than 1, and which thus not satisfy the boundary conditions required for the construction of a basis for $\mathcal{W}(\mathrm{I}^n)$ and thus for the test space $\mathcal{Y}_1$.

6. Elliptic regularity of the stationary Stokes problem

For a given function $\mathbf{q}$, we consider the *stationary* Stokes problem of finding the velocities $\mathbf{u}$ and pressure p that satisfy

$$\begin{cases} -\boldsymbol{\Delta}\mathbf{u} - \mathbf{grad}\, p = \mathbf{q} & \text{on } \mathrm{I}^n, \\ \operatorname{div} \mathbf{u} = 0 & \text{on } \mathrm{I}^n, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \partial\mathrm{I}^n, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\tau}_i = 0 & \text{on } \partial\mathrm{I}^n,\ 1 \le i \le n-1, \end{cases} \tag{6.1}$$

or, concerning the velocities, its variational problem of finding, for given $\mathbf{q} \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)'$, $\mathbf{u} \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)$, such that

$$a(\mathbf{u},\mathbf{v}) = \mathbf{q}(\mathbf{v}) \quad (\mathbf{v} \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)). \tag{6.2}$$

Our aim in this section is to prove the following regularity result that was used in the proof of Theorem 4.2:

Theorem 6.1. *If* $\mathbf{q} \in L_2(\mathrm{I}^n)^n$, *then* $\mathbf{u} \in \boldsymbol{\mathcal{W}}(\mathrm{I}^n)$, *with*

$$\|\mathbf{u}\|_{H^2(\mathrm{I}^n)^n} \lesssim \|\mathbf{q}\|_{L_2(\mathrm{I}^n)^n}.$$

Remark 6.2. In case of no-slip boundary conditions, i.e., $\mathbf{u} = 0$ at the boundary, Theorem 6.1 with $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$ reading as $\{\mathbf{v} \in (H_0^1(\Omega) \cap H^2(\Omega))^n : \operatorname{div} \mathbf{v} = 0\}$ has been shown for 2-dimensional convex polygons Ω in [KO76]. This result has been generalized to three dimensional convex polyhedrons in [Dau89]. For domains with non-smooth boundaries, we could not find, however, corresponding results for free-slip boundary conditions.

Using the fact that we work on the hypercube, we give an elementary proof of Theorem 6.1 by constructing an orthogonal basis of eigenfunctions for the Stokes problem. This basis can be used to construct an explicit expression for the solution of the stationary and unstationary Stokes problem as an infinite series.

Remark 6.3. Having such an expansion for the exact solution, one may think of constructing an approximation by collecting the N largest "Fourier" coefficients. Since the eigenfunctions will be globally supported, this does not not allow for local refinements. As a consequence, even for a smooth but for the rest general right hand side $\mathbf{q}$, the convergence rate of this approximation can be expected to be (much) lower than that of an adaptive wavelet scheme. See [DS09, Remark 6.5], [Dij09] for an analysis of the corresponding issue for the Poisson problem on the hypercube.

The eigenfunctions of the Stokes operator are studied intensively in the literature. Only in rare cases explicit expressions are available (cf. [LR02]). We could not find such expressions for the case that is studied here.

Lemma 6.4. *With*

$$\Phi^+ = \{\phi_p^+ := x \mapsto \sqrt{2}\sin p\pi x : p \in \mathbb{N}\}$$
$$\Phi = \{\phi_p := x \mapsto \sqrt{2}\cos p\pi x : p \in \mathbb{N}\}$$

we have

$$\dot{\phi}_p^+ = p\pi\phi_p, \quad \phi_p^+ = -\tfrac{1}{p\pi}\dot{\phi}_p \tag{6.3}$$

and the following collections are orthonormal bases:

$$
\begin{aligned}
&\Phi^+ \text{ for } L_2(0,T),\\
&\Phi \text{ for } L_{2,0}(0,T),\\
&\left\{\tfrac{\phi_p^+}{p\pi} : p \in \mathbb{N}\right\} \text{ for } H_0^1(0,T), \text{ equipped with } |\cdot|_{H^1(0,T)},\\
&\left\{\tfrac{\phi_p}{p\pi} : p \in \mathbb{N}\right\} \text{ for } \widehat{H}^1(0,T), \text{ equipped with } |\cdot|_{H^1(0,T)},\\
&\left\{\tfrac{\phi_p^+}{(p\pi)^2} : p \in \mathbb{N}\right\} \text{ for } H_0^1(0,T)\cap H^2(0,T), \text{ equipped with } |\cdot|_{H^2(0,T)},\\
&\left\{\tfrac{\phi_p}{(p\pi)^2} : p \in \mathbb{N}\right\} \text{ for } \{v \in \widehat{H}^2(0,T) : \dot v(0) = \dot v(T) = 0\}, \text{ equipped with } |\cdot|_{H^2(0,T)}.
\end{aligned}
$$

Proof. Orthonormality of all collections is easily verified. It is well known that Φ^+ and Φ are orthonormal bases for $L_2(0,T)$ and $L_{2,0}(0,T)$.

For $u \in H_0^1(0,T)$, let $u = \sum_{p\in\mathbb{N}} c_p\phi_p^+$ in $L_2(0,T)$. Then $c_p = \int_0^T u\phi_p^+ = -\int_0^T u\frac{\dot\phi_p}{p\pi} = \frac{1}{p\pi}\int_0^T \dot u\phi_p$, or $(p\pi c_p)_p \in \ell_2(\mathbb{N})$ and so $u = \sum_{p\in\mathbb{N}} p\pi c_p \frac{\phi_p^+}{p\pi}$ in $H^1(0,T)$. Similarly, for $u \in H_0^1(0,T)\cap H^2(0,T)$, $c_p = \frac{1}{p\pi}\int_0^T \dot u\frac{\dot\phi_p^+}{p\pi} = \frac{-1}{(p\pi)^2}\int_0^T \ddot u\phi_p^+$. Writing $u = \sum_{p\in\mathbb{N}} d_p\phi_p$, for $u \in \widehat{H}^1(0,T)$, $d_p = \int_0^T u\phi_p = \int_0^T u\frac{\dot\phi_p^+}{p\pi} = -\frac{1}{p\pi}\int_0^T \dot u\phi_p^+$, and for $u \in \widehat{H}^2(0,T)$ with $\dot u(0) = \dot u(T) = 0$, $d_p = \frac{1}{p\pi}\int_0^T \dot u\frac{\dot\phi_p}{p\pi} = \frac{-1}{(p\pi)^2}\int_0^T \ddot u\phi_p$. $\square$

The property (6.3) for the *orthonormal* collections Φ^+ and Φ is similar to (3.1) for the *bi*orthogonal wavelet collections $(\Psi^+, \tilde\Psi^-)$ and $(\Psi, \tilde\Psi)$. Therefore, using Φ^+ and Φ as univariate building blocks, below a procedure analogous to that from Sect. 3 will be followed to construct bases for $\widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n)$, $\widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n)$, and $\widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n)$. Then, as in Sect. 2, by making an orthogonal decomposition of $L_2(\mathrm{I}^n)^n$ into 2^n-1 subspaces isomorphic to $\widehat{L_2(\mathrm{I}^n)^k}$ for $k = 1,\ldots,n$, and by applying a Helmholtz decomposition on each of these subspaces, a union of (isomorphic images) of the aforementioned bases will be, properly scaled, a basis $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$, $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$, and $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$. Other than in Sect. 2-3, however, in the current setting, these bases will be *orthonormal*.

Similar to (3.6) and (3.7), for $\mathbf{p} \in \mathbb{N}^n$, $1 \le k \le n$, we set

$$\underline{\boldsymbol{\phi}}_{\mathbf{p},k}^{(n)} := \phi_{p_1}\otimes\cdots\otimes\phi_{p_{k-1}}\otimes\phi_{p_k}^+\otimes\phi_{p_{k-1}}\cdots\otimes\phi_{p_n}\mathbf{e}_k$$

and

$$\begin{bmatrix}\boldsymbol{\phi}_{\mathbf{p},1}^{(n)}\\ \vdots\\ \boldsymbol{\phi}_{\mathbf{p},n}^{(n)}\end{bmatrix} := \mathbf{A}^{(\mathbf{p})}\begin{bmatrix}\underline{\boldsymbol{\phi}}_{\mathbf{p},1}^{(n)}\\ \vdots\\ \underline{\boldsymbol{\phi}}_{\mathbf{p},n}^{(n)}\end{bmatrix},$$

where $\mathbf{A}^{(\mathbf{p})} \in \mathbb{R}^{n\times}$ is orthogonal with its last row given by

$$\mathbf{A}_{n\bullet}^{(\mathbf{p})} = [p_1\pi\cdots p_n\pi]/\|\pi\mathbf{p}\|_2.$$

Similar to Corollary 3.8, Proposition 3.9, and Corollaries 2.2 and 2.6, we have

Proposition 6.5. *The following collections are orthonormal bases:*

$$\begin{aligned}
\mathbf{\Phi}_{\rm df}^{(n)} &:= \big\{\boldsymbol{\phi}_{\mathbf{p},k}^{(n)} : 1 \leq k \leq n-1,\, \mathbf{p} \in \mathbb{N}^n\big\} \text{ for } \widehat{\boldsymbol{\mathcal{H}}}(\mathrm{I}^n),\\
\widehat{\mathbf{\Phi}}_{\rm df}^{(n)} &:= \Big\{\frac{\boldsymbol{\phi}_{\mathbf{p},k}^{(n)}}{\|\pi\mathbf{p}\|_2} : 1 \leq k \leq n-1,\, \mathbf{p} \in \mathbb{N}^n\Big\} \text{ for } \widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n) \text{ with } |\cdot|_{H^1(\mathrm{I}^n)^n},\\
\widehat{\widehat{\mathbf{\Phi}}}_{\rm df}^{(n)} &:= \Big\{\frac{\boldsymbol{\phi}_{\mathbf{p},k}^{(n)}}{\|\pi\mathbf{p}\|_2^2} : 1 \leq k \leq n-1,\, \mathbf{p} \in \mathbb{N}^n\Big\} \text{ for } \widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n) \text{ with } |\cdot|_{H^2(\mathrm{I}^n)^n}.
\end{aligned}$$

Proof. The sets $\big\{\boldsymbol{\phi}_{\mathbf{p},k}^{(n)} : 1 \leq k \leq n,\, \mathbf{p} \in \mathbb{N}^n\big\}$, $\Big\{\frac{\boldsymbol{\phi}_{\mathbf{p},k}^{(n)}}{\|\pi\mathbf{p}\|_2} : 1 \leq k \leq n,\, \mathbf{p} \in \mathbb{N}^n\Big\}$, $\Big\{\frac{\boldsymbol{\phi}_{\mathbf{p},k}^{(n)}}{\|\pi\mathbf{p}\|_2^2} : 1 \leq k \leq n,\, \mathbf{p} \in \mathbb{N}^n\Big\}$ are orthonormal bases for $\widehat{L_2(\mathrm{I}^n)^n}$, $\widehat{L_2(\mathrm{I}^n)^n} \cap \mathbf{H}_0^1(\mathrm{I}^n)$ equipped with $|\cdot|_{H^1(\mathrm{I}^n)^n}$, and for $\widehat{L_2(\mathrm{I}^n)^n} \cap \mathbf{H}_0^2(\mathrm{I}^n)$ equipped with $|\cdot|_{H^2(\mathrm{I}^n)^n}$, respectively.

Now use that $\big\{\boldsymbol{\phi}_{\mathbf{p},k}^{(n)} : 1 \leq k \leq n,\, \mathbf{p} \in \mathbb{N}^n\big\} \subset \mathbf{H}_0(\mathrm{div};\mathrm{I}^n)$, for $1 \leq k \leq n-1$

$$\mathrm{div}\, \boldsymbol{\phi}_{\mathbf{p},k}^{(n)} = \sum_{m=1}^{n} \mathbf{A}_{km}^{(\mathbf{p})} \mathrm{div}\, \underline{\boldsymbol{\phi}}_{\mathbf{p},m}^{(n)} = \Big(\sum_{m=1}^{n} \mathbf{A}_{km}^{(\mathbf{p})} p_m \pi\Big) \phi_{p_1} \otimes \cdots \otimes \phi_{p_n} = 0,$$

and

$$\boldsymbol{\phi}_{\mathbf{p},n}^{(n)} = -\mathbf{grad}\, \phi_{p_1} \otimes \cdots \otimes \phi_{p_n} / \|\pi\mathbf{p}\|_2. \quad \square$$

Similar to Theorems 2.4 and 2.7, we conclude the following:

Corollary 6.6. *The following collections are orthonormal bases:*

$$\begin{aligned}
\mathbf{\Phi}_{\rm df} &:= \bigcup_{S \subset \{1,\ldots,n\}, \#S \geq 2} E_S^{(n)} \mathbf{\Phi}_{\rm df}^{(\#S)} \text{ for } \boldsymbol{\mathcal{H}}(\mathrm{I}^n),\\
\widehat{\mathbf{\Phi}}_{\rm df} &:= \bigcup_{S \subset \{1,\ldots,n\}, \#S \geq 2} E_S^{(n)} \widehat{\mathbf{\Phi}}_{\rm df}^{(\#S)} \text{ for } \boldsymbol{\mathcal{V}}(\mathrm{I}^n) \text{ with } |\cdot|_{H^1(\mathrm{I}^n)^n},\\
\widehat{\widehat{\mathbf{\Phi}}}_{\rm df} &:= \bigcup_{S \subset \{1,\ldots,n\}, \#S \geq 2} E_S^{(n)} \widehat{\widehat{\mathbf{\Phi}}}_{\rm df}^{(\#S)} \text{ for } \boldsymbol{\mathcal{W}}(\mathrm{I}^n) \text{ with } |\cdot|_{H^2(\mathrm{I}^n)^n}.
\end{aligned}$$

Proof of Theorem 6.1. Since $\widehat{\mathbf{\Phi}}_{\rm df}$ is an orthonormal basis for $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ equipped with $|\cdot|_{H^1(\mathrm{I}^n)^n} = a(\cdot,\cdot)^{\frac{1}{2}}$, for $\mathbf{q} \in \boldsymbol{\mathcal{V}}(\mathrm{I}^n)'$, the solution of (6.2) is

$$\mathbf{u} = \langle \mathbf{q}, \widehat{\mathbf{\Phi}}_{\rm df}\rangle_{L_2(\mathrm{I}^n)^n}^{\top} \widehat{\mathbf{\Phi}}_{\rm df}.$$

From Proposition 6.5 we know that for some invertible diagonal matrix $\vec{D}$, $\widehat{\mathbf{\Phi}}_{\rm df} = \vec{D}\mathbf{\Phi}_{\rm df}$ and $\widehat{\widehat{\mathbf{\Phi}}}_{\rm df} = \vec{D}\widehat{\mathbf{\Phi}}_{\rm df}$, and so for $\mathbf{q} \in \boldsymbol{\mathcal{H}}(\mathrm{I}^n)$,

$$\mathbf{u} = \langle \mathbf{q}, \mathbf{\Phi}_{\rm df}\rangle_{L_2(\mathrm{I}^n)^n}^{\top} \widehat{\widehat{\mathbf{\Phi}}}_{\rm df}.$$

Since $\mathbf{\Phi}_{\rm df}$ is an orthonormal basis for $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$, and $\widehat{\widehat{\mathbf{\Phi}}}_{\rm df}$ is an orthonormal basis for $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$ equipped with $|\cdot|_{H^2(\mathrm{I}^n)^n}$, we conclude that

$$|\mathbf{u}|_{H^2(\mathrm{I}^n)^n} = \|\langle \mathbf{q}, \mathbf{\Phi}_{\rm df}\rangle_{L_2(\mathrm{I}^n)^n}\|_{\ell_2} = \|\mathbf{Q}_{\rm df}\mathbf{q}\|_{L_2(\mathrm{I}^n)^n},$$

where $\mathbf{Q}_{\rm df}$ is the $L_2(\mathrm{I}^n)^n$-orthogonal projector onto $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$. $\square$

Remark 6.7. For $\mathbf{q} \in \mathbf{H}_0^1(\mathrm{I})'$, the pressure component p of the solution of (6.1) can be found from $-\int_{\mathrm{I}^n} \mathbf{grad}\, p \cdot \mathbf{v}\, d\mathbf{x} = \mathbf{q}(\mathbf{v}) - a(\mathbf{u}, \mathbf{v})$ ($\mathbf{v} \in \mathbf{H}_0^1(\mathrm{I}^n)$). Assuming that $\mathbf{q} \in L_2(\mathrm{I}^n)^n$, as we will see we can test with $\mathbf{v} \in L_2(\mathrm{I}^n)^n$ and obtain $p \in H^1(\mathrm{I}^n) \cap L_{2,0}(\mathrm{I}^n)$.

Indeed, using Remark 2.8, let us equip $H^1(\mathrm{I}^n) \cap L_{2,0}(\mathrm{I}^n)$ with the orthonormal basis (with respect to $|\cdot|_{H^1(\mathrm{I}^n)}$)

$$\Pi = \bigcup_{\emptyset \neq S \subset \{1,\ldots,n\}} F_S^{(n)} \Big\{ \frac{\phi_{p_1} \otimes \cdots \otimes \phi_{p_{\#S}}}{\|\pi \mathbf{p}\|_2} : \mathbf{p} \in \mathbb{N}^{\#S} \Big\}$$

We decompose $L_2(\mathrm{I}^n)^n = \boldsymbol{\mathcal{H}}(\mathrm{I}^n) \oplus^{\perp} \mathbf{grad}\, H^1(\mathrm{I}^n)$, and equip $\mathbf{grad}\, H^1(\mathrm{I}^n)$ with the $L_2(\mathrm{I}^n)^n$-orthonormal basis

$$\boldsymbol{\Phi}_{\mathrm{gr}} := -\mathbf{grad}\, \Pi = \bigcup_{\emptyset \neq S \subset \{1,\ldots,n\}} E_S^{(n)} \{ \boldsymbol{\phi}^{(\#S)}_{\mathbf{p},\#S} : \mathbf{p} \in \mathbb{N}^{\#S} \}.$$

Since the right-hand side $\mathbf{q}(\mathbf{v}) - a(\mathbf{u}, \mathbf{v}) = \mathbf{q}(\mathbf{v}) + \int_{\mathrm{I}^n} \boldsymbol{\Delta}\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x}$ vanishes for $\mathbf{v} \in \boldsymbol{\mathcal{H}}(\mathrm{I}^n)$, we conclude that

$$p = \langle \mathbf{q}, \boldsymbol{\Phi}_{\mathrm{gr}} \rangle^{\top}_{L_2(\mathrm{I}^n)^n} \Pi,$$

so that

$$|p|_{H^1(\mathrm{I}^n)} = \|\langle \mathbf{q}, \boldsymbol{\Phi}_{\mathrm{gr}} \rangle_{L_2(\mathrm{I}^n)^n}\|_{\ell_2} = \|(\mathrm{Id} - \mathbf{Q}_{\mathrm{df}})\mathbf{q}\|_{L_2(\mathrm{I}^n)^n}.$$

7. Solving for the pressure

Having discussed how to solve for the velocity field that satisfies the *instationary* Stokes equations (4.1), in this section we discuss the remaining problem how to find the pressure.

Taking into account the initial and boundary conditions on $\mathbf{u}$, by taking the standard scalar product of the first equation of (4.1) with smooth test functions $\mathbf{v}$ of t and $\mathbf{x}$, that, as function of $\mathbf{x}$, at the boundary have vanishing normals and normal derivatives with vanishing tangential components, and that, as function of t, vanish at T, we arrive at the problem of finding p such that

$$c(p, \mathbf{v}) := -\int_0^T \int_{\mathrm{I}^n} \mathbf{grad}\, p \cdot \mathbf{v}\, d\mathbf{x} dt = \int_0^T \int_{\mathrm{I}^n} \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\Delta}\mathbf{v}\, d\mathbf{x} dt + \mathbf{f}(\mathbf{v}), \tag{7.1}$$

where $\mathbf{f}$ is defined in (4.3).

Lemma 7.1. *With*

$$\check{\boldsymbol{\mathcal{Y}}}_1 := L_2((0,T); \mathbf{H}_0^2(\mathrm{I}^n)) \cap H^1_{0,\{T\}}((0,T), L_2(\mathrm{I}^n)^n),$$

let $\mathbf{f} \in \check{\boldsymbol{\mathcal{Y}}}_1'$ *and let* $b(\mathbf{u}, \mathbf{v}) = \mathbf{f}(\mathbf{v})$ ($\mathbf{v} \in \boldsymbol{\mathcal{Y}}_1$). *Then, as functional on* $\mathbf{v}$, *the right-hand side of* (7.1) *is in* $\check{\boldsymbol{\mathcal{Y}}}_1'$, *vanishes on* $\boldsymbol{\mathcal{Y}}_1$, *and has norm less than or equal to some absolute multiple of* $\|\mathbf{f}\|_{\check{\boldsymbol{\mathcal{Y}}}_1'}$.

Proof. Since $\boldsymbol{\mathcal{Y}}_1 \hookrightarrow \check{\boldsymbol{\mathcal{Y}}}_1$, $\check{\boldsymbol{\mathcal{Y}}}_1' \hookrightarrow \boldsymbol{\mathcal{Y}}_1'$, and so by Theorem 4.2, $\mathbf{u} \in L_2((0,T); \boldsymbol{\mathcal{H}}(\mathrm{I}^n)) \hookrightarrow L_2((0,T); L_2(\mathrm{I}^n)^n)$ with $\|\mathbf{u}\|_{L_2((0,T);L_2(\mathrm{I}^n)^n)} \lesssim \|\mathbf{f}\|_{\check{\boldsymbol{\mathcal{Y}}}_1'}$. Since the bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto \int_0^T \int_{\mathrm{I}^n} \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\Delta}\mathbf{v}\, d\mathbf{x} dt$ is bounded on $L_2((0,T); L_2(\mathrm{I}^n)^n) \times \check{\boldsymbol{\mathcal{Y}}}_1$, the proof is completed. □

Remark 7.2. Similar to Remark 4.3, sufficient conditions for $\mathbf{f} \in \breve{\boldsymbol{\mathcal{Y}}}_1'$ are $\mathbf{u}_0 \in \mathbf{H}_0^1(\mathrm{I}^n)'$, and, say, $\mathbf{q} \in L_2((0,T);\mathbf{H}_0^2(\mathrm{I}^n)')$ and $g_i \in L_2((0,T);H^{-3/2}(\partial I^n))$ $(i = 1,\ldots,n-1)$.

Theorem 7.3. *With*

$$\mathcal{P} := \Big(L_2\big((0,T);H^1(\mathrm{I}^n)\cap L_{2,0}(\mathrm{I}^n)\big)\cap H^1_{0,\{T\}}\big((0,T);(H^1(\mathrm{I}^n)\cap L_{2,0}(\mathrm{I}^n))'\big)\Big)',$$

c is a bounded bilinear form on $\mathcal{P}\times\breve{\boldsymbol{\mathcal{Y}}}_1$. For any any $\mathbf{g}\in\breve{\boldsymbol{\mathcal{Y}}}_1'$ with $\mathbf{g}(\mathbf{v})=0$ for $\mathbf{v}\in\boldsymbol{\mathcal{Y}}_1$, there exists a unique $p\in\mathcal{P}$ such that

$$c(p,\mathbf{v}) = \mathbf{g}(\mathbf{v}) \quad (\mathbf{v}\in\breve{\boldsymbol{\mathcal{Y}}}_1), \tag{7.2}$$

and $\|p\|_{\mathcal{P}} \lesssim \|\mathbf{g}\|_{\breve{\boldsymbol{\mathcal{Y}}}_1'}$.

Proof. Let Θ be a Riesz basis for $L_2(0,T)$ with dual basis $\tilde{\Theta}=\{\tilde{\theta}_\theta:\theta\in\Theta\}$, such that $\{\frac{\theta}{\|\theta\|_{H^1(0,T)}}:\theta\in\Theta\}$ is a Riesz basis for $H^1_{0,\{T\}}(0,T)$.

Let $\Psi=\{\psi_\lambda:\lambda\in\nabla\}$, $\tilde{\Psi}=\{\tilde{\psi}_\lambda:\lambda\in\nabla\}$ be biorthogonal collections in $L_{2,0}(\mathrm{I})$ as in Proposition 3.5. Then, as shown in Proposition 3.10(a) together with Sect. 2, for $s=0$ and $s=2$,

$$\bigcup_{\emptyset\neq S\subset\{1,\ldots,n\}}\Big\{\big(\sum_{m=1}^{\#S}4^{|\lambda_m|}\big)^{-\frac{s}{2}}E_S^{(n)}\boldsymbol{\psi}^{(\#S)}_{\boldsymbol{\lambda},k} : 1\leq k\leq \#S,\, \boldsymbol{\lambda}\in\nabla^{\#S}\Big\}$$

as defined by (3.6) and (3.7) is a Riesz basis for $L_2(\mathrm{I}^n)^n$ and $\mathbf{H}_0^2(\mathrm{I}^n)$, respectively. Consequently,

$$\bigcup_{\emptyset\neq S\subset\{1,\ldots,n\}}\Big\{\frac{\theta\otimes E_S^{(n)}\boldsymbol{\psi}^{(\#S)}_{\boldsymbol{\lambda},k}}{\sqrt{(\sum_{m=1}^{\#S}4^{|\lambda_m|})^2+\|\theta\|^2_{H^1(0,T)}}} : 1\leq k\leq \#S,\, \boldsymbol{\lambda}\in\nabla^{\#S},\,\theta\in\Theta\Big\}$$

is a Riesz basis for $\breve{\boldsymbol{\mathcal{Y}}}_1$.

As we have seen before, all basis functions with indices $k<\#S$ are, as function of the spatial variables, divergence free and have vanishing normals at the boundary, and they form a Riesz basis for $\boldsymbol{\mathcal{Y}}_1$. By applying integration by parts, we conclude that $c(\cdot,\cdot)$ vanishes on $\mathcal{P}\times\boldsymbol{\mathcal{Y}}_1$. Denoting the space of the remaining basis functions as $\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$, we have $\breve{\boldsymbol{\mathcal{Y}}}_1=\boldsymbol{\mathcal{Y}}_1\oplus\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$.

As follows from Proposition 3.10(b) and Remark 2.8, a collection that is dual to the basis for $\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$ is obtained by applying $-\mathbf{grad}_{\mathbf{x}}$ to

$$\bigcup_{\emptyset\neq S\subset\{1,\ldots,n\}}\Big\{\Big(\textstyle\sum_{m=1}^{\#S}4^{|\lambda_m|}+\frac{\|\theta\|^2_{H^1(0,T)}}{\sum_{m=1}^{\#S}4^{|\lambda_m|}}\Big)^{\frac{1}{2}}\tilde{\theta}_\theta\otimes F_S^{(n)}\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}_{\lambda_{\#S}} : \boldsymbol{\lambda}\in\nabla^{\#S},\,\theta\in\Theta\Big\}, \tag{7.3}$$

which is in $L_2((0,T);L_{2,0}(\mathrm{I}^n))$. So by searching p from the span of this collection and by testing (7.2) with all basis functions for $\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$, the system matrix is the identity. We infer that the proof is completed once we have shown that (7.3) is a Riesz basis for $\mathcal{P}$.

Again Remark 2.8 shows that the collection in $L_2((0,T);L_{2,0}(\mathrm{I}^n))$ that is dual to (7.3) is

(7.4)
$$\bigcup_{\emptyset\neq S\subset\{1,\ldots,n\}}\Big\{\Big(\textstyle\sum_{m=1}^{\#S}4^{|\lambda_m|}+\frac{\|\theta\|^2_{H^1(0,T)}}{\sum_{m=1}^{\#S}4^{|\lambda_m|}}\Big)^{-\frac{1}{2}}\theta\otimes F_S^{(n)}\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_{\#S}}:\boldsymbol{\lambda}\in\nabla^{\#S},\,\theta\in\Theta\Big\}.$$

The univariate collections Ψ and $\{2^{-|\lambda|}\psi_\lambda:\lambda\in\nabla\}$ are Riesz bases for $L_{2,0}(\mathrm{I})$ and $\widehat{H}^1(\mathrm{I})$, respectively, and so $\{(\sum_{m=1}^{\#S}4^{|\lambda_m|})^{-\frac{1}{2}}\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_{\#S}}:\boldsymbol{\lambda}\in\nabla^{\#S}\}$ is a Riesz basis for $\widehat{H}^1(\mathrm{I}^{\#S})$. Since the same statements are valid with (Ψ,ψ_λ) reading as $(\tilde{\Psi},\tilde{\psi}_\lambda)$, using Remark 2.8 we infer that for $m\in\{-1,1\}$,

$$\bigcup_{\emptyset\neq S\subset\{1,\ldots,n\}}\Big\{\big(\textstyle\sum_{m=1}^{\#S}4^{|\lambda_m|}\big)^{-\frac{m}{2}}F_S^{(n)}\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_{\#S}}:\boldsymbol{\lambda}\in\nabla^{\#S}\Big\}$$

is a Riesz basis for $H^1(\mathrm{I}^n)\cap L_{2,0}(\mathrm{I}^n)$ $(m=1)$ or for $(H^1(\mathrm{I}^n)\cap L_{2,0}(\mathrm{I}^n))'$ $(m=-1)$. Because of the conditions on Θ, these results imply that (7.4) is a Riesz basis for $L_2\big((0,T);H^1(\mathrm{I}^n)\cap L_{2,0}(\mathrm{I}^n)\big)\cap H^1_{0,\{T\}}\big((0,T);(H^1(\mathrm{I}^n)\cap L_{2,0}(\mathrm{I}^n))'\big)$, and thus that (7.3) is a Riesz basis for its dual, being the space $\mathcal{P}$, which was left to show. □

The proof of theorem 7.3 already suggested a way to compute the pressure by the application of (an adaptive) numerical method. Let $\breve{\boldsymbol{\mathcal{Y}}}_1=\boldsymbol{\mathcal{Y}}_1\oplus\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$ be some stable decomposition. Then the canonical mapping between the space of $\mathbf{g}\in\breve{\boldsymbol{\mathcal{Y}}}_1'$ with $\mathbf{g}(\mathbf{v})=0$ for $\mathbf{v}\in\boldsymbol{\mathcal{Y}}_1$ and $(\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}})'$ is boundedly invertible. The pressure p solves (7.2) if and only if it solves

$$c(p,\mathbf{v})=\mathbf{g}(\mathbf{v})\quad(\mathbf{v}\in\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}).\tag{7.5}$$

The bilinear form c is bounded on $\mathcal{P}\times\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$; for any $\mathbf{v}\in\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$ there exists a $q\in\mathcal{P}$ such that $c(q,\mathbf{v})\neq0$; and, using Theorem 7.3, $c(\mathcal{P},\boldsymbol{\mathcal{Y}}_1)=0$, and the stability of the decomposition $\breve{\boldsymbol{\mathcal{Y}}}_1=\boldsymbol{\mathcal{Y}}_1\oplus\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$, we have

$$\|q\|_{\mathcal{P}}\lesssim\sup_{0\neq\mathbf{v}\in\breve{\boldsymbol{\mathcal{Y}}}_1}\frac{|c(q,\mathbf{v})|}{\|\mathbf{v}\|_{\breve{\boldsymbol{\mathcal{Y}}}_1}}\lesssim\sup_{0\neq\mathbf{v}\in\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}}\frac{|c(q,\mathbf{v})|}{\|\mathbf{v}\|_{\breve{\boldsymbol{\mathcal{Y}}}_1}},\quad(q\in\mathcal{P}).$$

We conclude that

$$(\mathrm{Id}\otimes(-\mathbf{grad})):\mathcal{P}\to(\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}})':q\mapsto(\mathbf{v}\mapsto c(q,\mathbf{v}))$$

is boundedly invertible, so that by equipping $\mathcal{P}$ and $\boldsymbol{\mathcal{Y}}_1^{\mathrm{c}}$ with Riesz bases of wavelet type, (7.5) with right-hand side from (7.1) can be solved with an adaptive wavelet scheme.

8. No-slip boundary conditions?

Let us discuss whether the material from this paper can be extended to the case of having no-slip boundary conditions, i.e., when the boundary conditions of the (in)stationary Stokes problem read as $\mathbf{u}=0$ on $([0,T]\times)\ \partial\mathrm{I}^n$. In this case, the definitions of $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ and $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$ have to be replaced by $\boldsymbol{\mathcal{V}}(\mathrm{I}^n):=\boldsymbol{\mathcal{H}}(\mathrm{I}^n)\cap H_0^1(\mathrm{I}^n)^n$ and $\boldsymbol{\mathcal{W}}(\mathrm{I}^n):=\boldsymbol{\mathcal{V}}(\mathrm{I}^n)\cap H^2(\mathrm{I}^n)^n$, and $\widehat{\boldsymbol{\mathcal{V}}}(\mathrm{I}^n):=\boldsymbol{\mathcal{V}}(\mathrm{I}^n)\cap\widehat{L_2(\mathrm{I}^n)^n}$, $\widehat{\boldsymbol{\mathcal{W}}}(\mathrm{I}^n):=\boldsymbol{\mathcal{W}}(\mathrm{I}^n)\cap\widehat{L_2(\mathrm{I}^n)^n}$.

The construction of wavelet Riesz bases for $\boldsymbol{\mathcal{H}}(\mathrm{I}^n)$ and $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ can follow the same lines as with free-slip boundary conditions. The difference is that the collection of univariate wavelets Ψ now has to satisfy (lowest order) homogeneous Dirichlet boundary conditions. Theorems 4.1 and 4.2 dealing with bounded invertibility of the operator defined by the simultaneous space-time variational formulation of the instationary Stokes problem are still valid. In the most relevant cases of $n = 2$ and $n = 3$, the elliptic regularity result similar to Theorem 6.1 that is needed for Theorem 4.2 is now shown in [KO76] and [Dau89], respectively. Actually, in this case, we do not know how to construct an eigenvector basis for the stationary Stokes operator, but because of these regularity results from the literature, we do not need it here.

Problems arise with the transformation of the instationary Stokes equations as a well-posed operator equation to a bi-infinite matrix vector equation. In the setting of Theorem 4.1, we need a wavelet collection $\boldsymbol{\Psi}_{\mathrm{df}}$ that, properly scaled, is a Riesz basis for both $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ *and* $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)'$. As in the free-slip boundaryconditions case, we do not not know how to construct such a collection. In the setting of Theorem 4.1, the problem is how to construct a Riesz basis for $\boldsymbol{\mathcal{W}}(\mathrm{I}^n)$. As we wrote before, the collection of univariate wavelets Ψ that is used as building block of the collection $\boldsymbol{\Psi}_{\mathrm{df}}$ now has to satisfy (lowest order) homogeneous Dirichlet boundary conditions. But then the collection Ψ^+ has vanishing function values and first order derivatives at the boundary, and thus cannot span $H^1_0(\mathrm{I}) \cap H^2(\mathrm{I})$ as is required.

Remark 8.1. Considering the *stationary* Stokes problem, it is easy to handle both free-slip and no-slip boundary conditions. Indeed for both $\boldsymbol{\mathcal{V}}(\mathrm{I}^n) := \boldsymbol{\mathcal{H}}(\mathrm{I}^n) \cap \mathbf{H}^1_0(\mathrm{I})$ or $\boldsymbol{\mathcal{V}}(\mathrm{I}^n) := \boldsymbol{\mathcal{H}}(\mathrm{I}^n) \cap H^1_0(\mathrm{I}^n)^n$, the bilinear form a from (4.2), (6.2) defines a boundedly invertible operator between $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ and its dual. By equipping $\boldsymbol{\mathcal{V}}(\mathrm{I}^n)$ with a Riesz basis, the corresponding variational problem can be equivalently written as well-posed bi-infinite matrix vector equation to which the adaptive wavelet scheme can be applied.

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