SPACE-TIME VARIATIONAL SADDLE POINT FORMULATIONS OF STOKES AND NAVIER–STOKES EQUATIONS [*]

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Abstract. The instationary Stokes and Navier−Stokes equations are considered in a simultaneously space-time variational saddle point formulation, so involving both velocities $\mathbf{u}$ and pressure p. For the instationary Stokes problem, it is shown that the corresponding operator is a *boundedly invertible* linear mapping between H_1 and H_2', both Hilbert spaces H_1 and H_2 being Cartesian products of (intersections of) Bochner spaces, or duals of those. Based on these results, the operator that corresponds to the Navier−Stokes equations is shown to map H_1 into H_2', with a Fréchet derivative that, at any $(\mathbf{u}, p) \in H_1$, is boundedly invertible. These results are essential for the numerical solution of the combined pair of velocities and pressure as function of simultaneously space and time. Such a numerical approach allows for the application of (adaptive) approximation from tensor products of spatial and temporal trial spaces, with which the instationary problem can be solved at a computational complexity that is of the order as for a corresponding stationary problem.

1. INTRODUCTION

1.1. Background and motivation

The classical approach to the existence of weak solutions of the instationary, incompressible Navier−Stokes equations views these equations as an infinite-dimensional dynamical system (see, *e.g.*, [23], Chap. III and the references there). In line with this view, most methods for the *numerical solution* of the instationary (Navier–) Stokes equations are *time marching* methods: assuming that some approximate solution on time t is available, for a sufficiently small time increment $\Delta t > 0$, an approximate solution on time $t + \Delta t$ is computed by solving a corresponding stationary problem.

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Because of the generally lacking global smoothness of the solution, efficient numerical schemes have to be adaptive. With suitable time-marching schemes, it is possible to adapt both the spatial "mesh", and the time step Δt depending on t. We refer to [2] for an *a posteriori* error analysis of such an approach.

Combined space-time adaptivity, where Δt is adapted also depending on the spatial location, are not easily accommodated by classical time stepping schemes, although some studies on local time stepping have appeared, see *e.g.* [8, 9, 17].

In any case, due to the character of time marching, it seems hard to guarantee a kind of quasi-optimal distribution of the 'grid-points' over space and time, and no mathematical results in this direction are presently known to us.

To develop an alternative for time marching schemes, in [4, 18] we studied *simultaneously space-time variational formulations* of linear *parabolic* evolution equations. The operators defined by such variational formulations were shown to be *boundedly invertible* between a Hilbert space H_1 and the dual of another Hilbert space H_2, both H_1 and H_2 being Cartesian products of Bochner spaces or intersections of those.

By equipping these Bochner spaces with Riesz bases, being *tensor products* of temporal and spatial wavelet collections, the space-time variational problem was written as an *equivalent* well-posed, bi-infinite, symmetric positive definite matrix-vector system by forming normal equations. By running on this system an adaptive wavelet scheme, in its original form being proposed in [3], a sequence of approximations is produced in linear computational complexity that converges with the best possible nonlinear approximation rate from the basis, *i.e.*, the rate of the so-called best N-term approximations.

Because of the application of tensorized wavelet collections in space and time, under mild (Besov) smoothness conditions the latter rate is *equal* (in some situations up to log-factors) to the best possible approximation rate for the solution of a corresponding stationary problem from the spatial wavelet basis, *i.e.*, there is (hardly) any increase in the order of computational complexity as a consequence of the additional time dimension. Numerical results illustrating this fact are given in [4].

Besides the computational realization of the best possible nonlinear approximation rate, the latter property is a major advantage when the approximate solution is needed as function of simultaneously space and time, as it is the case for example in time-dependent optimal control problems, see [10]. Indeed, with time marching schemes this would require the availability of the approximate solutions simultaneously at all discrete times, requiring a huge amount of memory.

The results concerning the adaptive wavelet solution method generalize to simultaneously space-time variational formulations of *nonlinear* parabolic evolution equations when they define a (two times continuously differentiable) mapping from H_1 into H_2', and the Fréchet derivative at the solution is *boundedly invertible* between H_1 and H_2' (see [20]). The latter condition is satisfied for example for a semi-linear equation with a time-independent spatial operator.

Aiming at the application of space-time variational formulations to the incompressible instationary (*Navier–*) *Stokes* equations, there are *two* possibilities.

The *first* one is to reduce these equations to problems for the divergence-free velocities only. Then the Stokes or linearized Navier−Stokes equations read as a linear parabolic evolution problems, and the aforementioned results concerning well-posed space-time variational formulations apply. The reduction to divergence-free velocities is also the standard approach followed in the literature for demonstrating existence and uniqueness of solutions (see *e.g.* [23], Chap. III).

In [21], we theoretically investigated the application of the adaptive wavelet scheme to the space-time variational divergence-free velocities formulation of the instationary Stokes problem. Wavelets suitable for this formulation were constructed for *rectangular* domains in [21, 22].

1.2. This paper

The approach to tackle the instationary (Navier–) Stokes equations by a reduction to equations for the divergence-free velocities has the obvious disadvantage that no results for the pressure are obtained. Moreover,

the numerical solution of these equations by an adaptive wavelet scheme requires a divergence-free wavelet basis, that seems to be realizable in restricted settings only.

Therefore, in this paper as the *second* possibility, we study *simultaneously space-time variational saddle point formulations* of the (Navier–) Stokes equations for the combined pair of velocities and pressure. For both free-slip and no-slip boundary conditions, we prove that the Stokes operator defined by this variational formulation is *boundedly invertible* between a Hilbert space H_1 and the dual H_2' of another Hilbert space H_2. In order to be able to arrive at this result, we have to assume H^2-regularity of the Poisson or of the stationary Stokes operator, which imposes certain smoothness or convexity conditions on the spatial domain.

In the space-times variational formulation of the present paper, both trial- and test-spaces H_1 and H_2 are Cartesian products of (intersections of) of Bochner spaces for velocities and pressure. Based on the results [18] for the space-time variational formulations of parabolic evolution equations, the velocity components of the test- and trial-spaces are as expected, and so are the corresponding pressure components of either test- or trial-space. The space for the remaining pressure component, now being fully determined by the instationary Stokes operator, is less standard being the dual of the intersection of two Bochner spaces. In any case for polytopal spatial domains, countable tensor product wavelet bases can be constructed for these spaces, which are separable Hilbert spaces (although separability is not used *a fortiori* in the present paper).

To the best of our knowledge, *well-posedness, i.e., bounded invertibility* of the instationary Stokes operator for the combined pair of velocities and pressure has not been established before. Compare the discussion at the end of ([23], Chap. III, Sect. 1.5), where regularity of the pair of velocities and pressure is established only under *additional* smoothness conditions on the right-hand side.

With the spaces H_1 and H_2 as above, additionally it will be shown that the instationary Navier–Stokes operator maps H_1 into H_2' (for no-slip conditions on two- and three-dimensional domains, and for free-slip conditions on two-dimensional domains). A generalization of the results for the instationary Stokes operator to the linearized instationary Navier–Stokes operator – the difference being a lower order spatial differential operator– shows that the latter, at any $(\mathbf{u}, p) \in H_1$, is a boundedly invertible operator between H_1 and H_2'. A first consequence is that any solution $(\mathbf{u}, p) \in H_1$ of the instationary Navier−Stokes equations is locally unique. Secondly, assuming that a sufficiently accurate initial approximation is available, it shows that the adaptive wavelet solver can be used to approximate the solution with the *best possible nonlinear approximation rate in space-time tensorized bases.*

Finally, since also Lipschitz continuity of the instationary Navier–Stokes operator will be shown, using a fixed-point argument we conclude existence of a space-time variational Navier–Stokes solution in H_1, albeit under a small data hypothesis.

This paper is organized as follows: in Section 2, necessary and sufficient conditions are recalled for bounded invertibility of generalized linear saddle point problems. In Sections 3 and 4, these conditions are verified for space-time variational formulations of the instationary Stokes problem with free- and no-slip boundary conditions, respectively. In Section 5, the aforementioned mapping properties of the instationary Navier–Stokes operator with homogeneous initial datum are verified.

Throughout, for positive constants $c_1, c_2, c_1 \lesssim c_2$ we will mean that c_1 can be bounded by a multiple of c_2, independently of parameters on which c_1 and c_2 may depend. Obviously, $c_1 \gtrsim c_2$ is defined as $c_2 \lesssim c_1$, and $c_1 \eqsim c_2$ as $c_1 \lesssim c_2$ and $c_1 \gtrsim c_2$.

2. Generalized saddle point problems

For reflexive Banach spaces U, V, P, and Q, and for bounded bilinear forms $a : U \times V \to \mathbb{R}$, $b : P \times V \to \mathbb{R}$, and $c : U \times Q \to \mathbb{R}$, we consider the problem of finding $(u, p) \in U \times P$ that, for given $f \in V'$, $g \in Q'$, satisfy

$$a(u, v) + b(p, v) + c(u, q) = f(v) + g(q) \quad (v \in V, q \in Q). \tag{2.1}$$

In this section, we collect sufficient and necessary conditions for the corresponding $L : (u, p) \mapsto (f, g) \in \mathcal{L}(U \times P, V' \times Q')$ to be boundedly invertible. These conditions can already be found in [1], and a Hilbert space

setting, in [15]. Since some intermediate results will be used in the following sections, we have chosen to include the short arguments.

$$(Bv)(p) = b(p, v) = (B'p)(v) \quad \text{and} \quad (Cu)(q) = c(u, q) = (C'q)(u).$$

We recall that for a closed subspace Z of a Banach space X the polar set $Z^0 \subset X'$ is defined by $\{f \in X' : f(Z) = 0\}$.

Theorem 2.1. *For given, bounded bilinear forms a, b and c as in* (2.1)*, the variational problem* (2.1) *defines a boundedly invertible linear mapping $U \times P \to V' \times Q'$ if and only if the following three conditions are satisfied:*

(i) *for all $f \in (\ker B)'$, there exists a unique $u \in \ker C$ such that $a(u, v) = f(v)$ ($v \in \ker B$),*
(ii) *for all $g \in Q'$, there exists some $u \in U$ such that $c(u, q) = g(q)$ ($q \in Q$),*
(iii) *for all $f \in (\ker B)^0$, there exists a unique $p \in P$ such that $b(p, v) = f(v)$ ($v \in V$).*

Proof. Suppose (i)–(iii) are satisfied, and let $f \in V'$, $g \in Q'$. By condition (ii), there exists a $\bar{u} \in U$ with $c(\bar{u}, q) = g(q)$ ($q \in Q$). Condition (i) shows that there exists a $u_0 \in \ker C$ with $a(u_0, v) = f(v) - a(\bar{u}, v)$ ($v \in \ker B$). So $u := u_0 + \bar{u}$ solves

$$a(u, v) + c(u, q) = f(v) + g(q) \quad (v \in \ker B, \ q \in Q).$$

This u is unique. Indeed, let $u_1, u_2 \in U$ be two solutions, then

$$a(u_1 - u_2, v) = c(u_2 - u_2, q) \quad (v \in \ker B, \ q \in Q).$$

By taking $v = 0$, we find $u_2 - u_1 \in \ker C$. Now by taking $q = 0$, we infer that $u_1 = u_2$ by (i). Finally, condition (iii) shows that there exists a unique $p \in P$ that solves

$$b(p, v) = f(v) - a(u, v) \quad (v \in V, \ q \in Q), \tag{2.2}$$

so that (2.1) has a unique solution $(u, p) \in U \times P$. An application of the *open mapping theorem* shows that (2.1) defines boundedly invertible linear mapping $U \times P \to V' \times Q'$.

Conversely, let (2.1) define a boundedly invertible linear mapping. Then condition (ii) follows easily. From $\begin{bmatrix} A & B' \\ C & 0 \end{bmatrix} : U \times P \to V' \times Q'$ being boundedly invertible, we have that $\operatorname{ran} B' \times \{0\} = \begin{bmatrix} A & B' \\ C & 0 \end{bmatrix}\Big|_{\{0\} \times P}$ is closed, and thus that $\operatorname{ran} B'$ is closed. By an application of the *closed range theorem*, we conclude that $(\ker B)^0 = \operatorname{ran} B'$, which is (iii). Now let $f \in (\ker B)'$. By an application of *Hahn−Banach's theorem* extend it to $f \in V'$, and take $g = 0$. Then for the solution (u, p) of (2.1), it holds that $u \in \ker C$, and $a(u, v) = f(v)$ ($v \in \ker B$). Now suppose that the last problem has two solutions $u_1, u_2 \in \ker C$. For $i = 1, 2$, define $p_i \in P$ as the solution of $b(p, v) = f(v) - a(u_i, v)$ ($v \in V$). Then both (u_1, p_1) and (u_2, p_2) solve (2.1), and we conclude $u_1 = u_2$, which completes the proof of (i). □

Proposition 2.2. *Having bounded b and c, conditions equivalent to (ii) and (iii) are, respectively,*

(ii)$'$ $\inf_{0 \neq q \in Q} \sup_{0 \neq u \in U} \frac{c(u, q)}{\|q\|_Q \|u\|_U} > 0$,
(iii)$'$ $\inf_{0 \neq p \in P} \sup_{0 \neq v \in V} \frac{b(p, v)}{\|p\|_P \|v\|_V} > 0$.

Proof. The equivalence of (ii) and (ii)$'$ follows from the equivalence of (a) and (e) in Lemma 2.3 stated below. Another application of Lemma 2.3 shows that (iii)$'$ implies that $B' \in \mathcal{L}(P, V')$ is a homeomorphism onto $(\ker B)^0$, which implies (iii). Conversely, since $\operatorname{ran} B' \subset (\ker B)^0$ by definition, (iii) implies that $(\ker B)^0 = \operatorname{ran} B'$ and that B' is injective, and so, by Lemma 2.3, that (iii)$'$ is valid. □

Lemma 2.3. *For reflexive Banach spaces X and Y, and for $T \in \mathcal{L}(X, Y')$, the following statements are equivalent:*

(a) $\inf_{0 \neq y \in Y} \sup_{0 \neq x \in X} \frac{(Tx)(y)}{\|x\|_X \|y\|_Y} > 0$,

(b) $T' \in \mathcal{L}(Y, X')$ *is a homeomorphism onto its range,*

(c) T' *injective and* $\operatorname{ran} T'$ *is closed,*

(d) T' *injective and* $\operatorname{ran} T' = (\ker T)^0$,

(e) T *is surjective.*

Proof. (a)⇔(b) and (b)⇒(c) follow easily.

(c)⇒(b) is a consequence of the *open mapping theorem.*

(c)⇔(d) follows from the *closed range theorem.*

(e)⇒(c): Since $\operatorname{ran} T$ is closed, the *closed range theorem* shows that $\operatorname{ran} T'$ is closed, and that $(\ker T')^0 = \operatorname{ran} T = Y'$, so that, by an application of the *Hahn−Banach theorem*, $\ker T' = \emptyset$.

(c)⇒(e): Since $\operatorname{ran} T'$ is closed, the *closed range theorem* shows that $\operatorname{ran} T = (\ker T')^0 = Y'$ because T' is injective. □

In the following, we shall the above existence results to verify the well-posedness of space-time variational saddle-point formulations of the Navier−Stokes equations. All the ensuing developments will require the preceding results in the particular setting of Hilbert spaces.

3. The instationary Stokes problem with free-slip boundary conditions, as a well-posed operator equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Given vector fields $\tilde{\mathbf{f}}$ on $(0, T) \times \Omega$ and $\mathbf{u}_0$ on Ω, and functions g on $(0, T) \times \Omega$, and g_i $(1 \leq i \leq n-1)$ on $(0, T) \times \partial\Omega$, we consider the instationary inhomogeneous Stokes problem with *free-slip* boundary conditions of finding for some $\nu > 0$ a velocity field $\mathbf{u}$ and corresponding pressure p that satisfy

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \tilde{\mathbf{f}} & \text{on } (0, T) \times \Omega, \\
\operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } (0, T) \times \Omega, \\
\mathbf{u} \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\Omega, \\
\frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\tau}_i = g_i & \text{on } (0, T) \times \partial\Omega,\ 1 \leq i \leq n-1, \\
\mathbf{u}(0, \cdot) = \mathbf{u}_0 & \text{on } \Omega,
\end{cases}
\tag{3.1}
$$

where $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ is an orthonormal set of tangent vectors.

Integrating the first equation against smooth vector fields $\mathbf{v}$, that as function of $\mathbf{x}$ have vanishing normals at $\partial\Omega$, and that as function of t vanish at $t = T$, and by applying integration by parts in space and time, and by integrating the second equation against smooth functions q, we arrive at a variational problem of the form (2.1), where

$$
\begin{cases}
a(\mathbf{u}, \mathbf{v}) = -\displaystyle\int_0^T \int_\Omega \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t}\, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v}\, \mathrm{d}\mathbf{x}\mathrm{d}t, \\
b(p, \mathbf{v}) = -\displaystyle\int_0^T \int_\Omega p \operatorname{div} \mathbf{v}\, \mathrm{d}\mathbf{x}\mathrm{d}t, \\
c(\mathbf{u}, q) = \displaystyle\int_0^T \int_\Omega q \operatorname{div} \mathbf{u}\, \mathrm{d}\mathbf{x}\mathrm{d}t, \\
\mathbf{f}(\mathbf{v}) = \displaystyle\int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\partial\Omega} \sum_{i=1}^{n-1} (\mathbf{v} \cdot \boldsymbol{\tau}_i) g_i \mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega \mathbf{u}_0 \cdot \mathbf{v}(0, \cdot)\, \mathrm{d}\mathbf{x}, \\
g(q) = \displaystyle\int_0^T \int_\Omega g\, q\, \mathrm{d}\mathbf{x}\mathrm{d}t.
\end{cases}
\tag{3.2}
$$

We will need the following assumption on the domain Ω concerning H^2-regularity of the Poisson problem with homogeneous Neumann boundary conditions.

Assumption 3.1. The bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is such that for any $h \in L_{2,0}(\Omega) := L_2(\Omega)/\mathbb{R}$, the solution $u \in H^1(\Omega)/\mathbb{R}$ of

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega hv \, \mathrm{d}x \quad (v \in H^1(\Omega)/\mathbb{R}),$$

is in $H^2(\Omega)$, with $\|u\|_{H^2(\Omega)} \lesssim \|h\|_{L_2(\Omega)}$.

This assumption is known to be satisfied when Ω is convex, or when it has a C^2-boundary.

Theorem 3.2. *With the spaces* $\tilde{H}^2(\Omega) := \{p \in H^2(\Omega) : \frac{\partial p}{\partial n} = 0 \text{ on } \partial\Omega\}/\mathbb{R}$, $\mathbf{H}^1(\Omega) := \{\mathbf{w} \in H^1(\Omega)^n : \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$ *and*

$$\mathbf{U} := L_2(0, T; \mathbf{H}^1(\Omega)),$$
$$P := \left(L_2(0, T; L_{2,0}(\Omega)) \cap H^1_{0,\{T\}}\left(0, T; \tilde{H}^2(\Omega)'\right) \right)',$$
$$\mathbf{V} := L_2(0, T; \mathbf{H}^1(\Omega)) \cap H^1_{0,\{T\}}(0, T; \mathbf{H}^1(\Omega)'),$$
$$Q := L_2(0, T; L_{2,0}(\Omega)),$$

and under Assumption 3.1, the mapping $\mathbf{L} : (\mathbf{u}, p) \mapsto (\mathbf{f}, g)$ *as in* (2.1) *with bilinear forms from* (3.2) *defines a boundedly invertible linear mapping* $\mathbf{U} \times P \to \mathbf{V}' \times Q'$.

(Here and in what follows, we denote for a Banach space B and a summability index $1 \leq p < \infty$ by $L_p(0, T; B)$ the space of strongly measurable functions $u : (0, T) \mapsto B$ such that $(0, T) \ni t \mapsto \|u(t)\|_B \in L_p(0, T)$. As usual, dual spaces should be interpreted with respect to the identifications $L_2(\Omega)' \simeq L_2(\Omega)$, $L_{2,0}(\Omega)' \simeq L_{2,0}(\Omega)$, or $L_2(0, T; L_{2,0}(\Omega))' \simeq L_2(0, T; L_{2,0}(\Omega))$, respectively. For $\Gamma \subset \{0, T\}$, $H^1_{0,\Gamma}(0, T)$ denotes the closure in $H^1(0, T)$ of the set of $w \in C^\infty(0, T) \cap H^1(0, T)$ with supp $w \cap \Gamma = \emptyset$.)

To prove this theorem, in the following, we will verify the conditions of the abstract existence and uniqueness result, Theorem 2.1.

The bilinear forms $a : \mathbf{U} \times \mathbf{V} \to \mathbb{R}$, $b : P \times \mathbf{V} \to \mathbb{R}$, and $c : \mathbf{U} \times Q \to \mathbb{R}$ are bounded. For b, this follows from div $\in \mathcal{L}(\mathbf{H}^1(\Omega), L_{2,0}(\Omega))$ and div $\in \mathcal{L}(\mathbf{H}^1(\Omega)', \tilde{H}^2(\Omega)')$, the latter, because of the density of $\mathcal{D}(\Omega)$ in $\mathbf{H}^1(\Omega)'$, being equivalent to $\nabla \in \mathcal{L}(\tilde{H}^2(\Omega), \mathbf{H}^1(\Omega))$. We conclude that $I \otimes \mathrm{div}_\mathbf{x} \in \mathcal{L}(\mathbf{V}, P')$, being equivalent to $b : \mathbf{V} \times P \to \mathbb{R}$ is bounded.

Knowing the boundedness of a, b, and c, next we verify the conditions (i)–(iii) of Theorem 2.1. We start with condition (ii). For $\mathbf{u} \in \mathbf{U}$, $q \in Q$, one has $c(\mathbf{u}, q) = -\int_0^T \int_\Omega \nabla_\mathbf{x} q \cdot \mathbf{u} \, \mathrm{d}\mathbf{x}\mathrm{d}t$. Since Ω is a bounded Lipschitz domain,

$$\nabla \in \mathcal{L}(L_{2,0}(\Omega), (H^1_0(\Omega)^n)') \text{ is a homeomorphism onto its range} \tag{3.3}$$

([14], *cf.* [23], Chap. 1, Rem. 1.4(ii)). By an application of Lemma 2.3, this means that

$$\inf_{0 \neq q \in L_{2,0}(\Omega)} \sup_{0 \neq \mathbf{u} \in H^1_0(\Omega)^n} \frac{\int_\Omega q \, \mathrm{div} \, \mathbf{u} \mathrm{d}\mathbf{x}}{\|q\|_{L_{2,0}(\Omega)} \|\mathbf{u}\|_{H^1(\Omega)^n}} > 0,$$

and so also that

$$\inf_{0 \neq q \in L_{2,0}(\Omega)} \sup_{0 \neq \mathbf{u} \in \mathbf{H}^1(\Omega)} \frac{\int_\Omega q \, \mathrm{div} \, \mathbf{u} \, \mathrm{d}\mathbf{x}}{\|q\|_{L_{2,0}(\Omega)} \|\mathbf{u}\|_{H^1(\Omega)^n}} > 0.$$

Since, additionally, $(\mathbf{u}, q) \mapsto \int_\Omega q \, \mathrm{div} \, \mathbf{u} \, \mathrm{d}\mathbf{x}$ is bounded on $\mathbf{H}^1(\Omega) \times L_{2,0}(\Omega)$, one has that $\nabla \in \mathcal{L}(L_{2,0}(\Omega), \mathbf{H}^1(\Omega)')$, and so $I \otimes \nabla_\mathbf{x} \in \mathcal{L}(Q, \mathbf{U}')$ are homeomorphisms onto their ranges by Lemma 2.3. Knowing the boundedness of $c : \mathbf{U} \times Q \to \mathbb{R}$, the latter is equivalent to condition (ii) of Theorem 2.1.

To show condition (i) of Theorem 2.1, we give a characterization of the kernels of

$$\mathbf{B} := I \otimes \operatorname{div} \in \mathcal{L}(\mathbf{V}, P'), \quad \mathbf{C} := I \otimes \operatorname{div} \in \mathcal{L}(\mathbf{U}, Q').$$

We set

$$\boldsymbol{\mathcal{H}}^1(\Omega) := \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \operatorname{div}\mathbf{u} = 0\}, \ \boldsymbol{\mathcal{H}}^0(\Omega) := \operatorname{clos}_{L_2(\Omega)^n} \boldsymbol{\mathcal{H}}^1(\Omega), \ \boldsymbol{\mathcal{H}}^{-1}(\Omega) := \boldsymbol{\mathcal{H}}^1(\Omega)'$$

(to be interpreted with respect to the identification $\boldsymbol{\mathcal{H}}^0(\Omega)' \simeq \boldsymbol{\mathcal{H}}^0(\Omega)$).

Since $\operatorname{div}\mathbf{H}^1(\Omega) \subset L_{2,0}(\Omega)$, it holds that

$$\ker\mathbf{C} = L_2(0, T; \boldsymbol{\mathcal{H}}^1(\Omega)). \tag{3.4}$$

Lemma 3.3. *With $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) := \operatorname{clos}_{\mathbf{H}^1(\Omega)'} \boldsymbol{\mathcal{H}}^1(\Omega)$, it holds that*

$$\ker\mathbf{B} = L_2(0, T; \boldsymbol{\mathcal{H}}^1(\Omega)) \cap H^1_{0,\{T\}}(0, T; \tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)).$$

Proof. Since both $L_2(0, T; L_{2,0}(\Omega))$ and $H^1_{0,\{T\}}(0, T; \tilde{H}^2(\Omega)')$ are continuously embedded in, *e.g.*, $L_2(0, T; \tilde{H}^2(\Omega)')$, these two spaces form a so-called *Banach couple*, also known as a compatible couple of Banach spaces. Since moreover their intersection is dense in both spaces, the dual of their intersection is isomorphic to the sum of their duals, see *e.g.* ([12], Chap. 1, Thm. 3.1), *i.e.*,

$$P = \left(L_2(0, T; L_{2,0}(\Omega)) \cap H^1_{0,\{T\}}(0, T; \tilde{H}^2(\Omega)')\right)' \simeq L_2(0, T; L_{2,0}(\Omega)) + H^1_{0,\{T\}}(0, T; \tilde{H}^2(\Omega)')'.$$

So $\mathbf{v} \in \ker\mathbf{B}$ if and only if $\mathbf{v} \in L_2(0, T; \mathbf{H}^1(\Omega)) \cap H^1_{0,\{T\}}(0, T; \mathbf{H}^1(\Omega)')$ and $(\mathbf{Bv})(p) = 0$ for all $p \in L_2(0, T; L_{2,0}(\Omega)) + H^1_{0,\{T\}}(0, T; \tilde{H}^2(\Omega)')'$. This is equivalent to $\mathbf{v} \in L_2(0, T; \mathbf{H}^1(\Omega))$ and $((I \otimes \operatorname{div})\mathbf{v})(p) = 0$ for all $p \in L_2(0, T; L_{2,0}(\Omega))$, *i.e.*, $\mathbf{v} \in L_2(0, T; \boldsymbol{\mathcal{H}}^1(\Omega))$ by (3.4), together with $\mathbf{v} \in H^1_{0,\{T\}}(0, T; \mathbf{H}^1(\Omega)')$ and $((I \otimes \operatorname{div})\mathbf{v})(p) = 0$ for all $p \in H^1_{0,\{T\}}(0, T; \tilde{H}^2(\Omega)')'$. The second condition means that with $\boldsymbol{\mathcal{N}} := \ker(\operatorname{div} \in \mathcal{L}(\mathbf{H}^1(\Omega)', \tilde{H}^2(\Omega)'))$, it holds that $\mathbf{v} \in H^1_{0,\{T\}}(0, T; \boldsymbol{\mathcal{N}})$, so what is left to show is that $\boldsymbol{\mathcal{N}} = \tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$.

By $\operatorname{div} \in \mathcal{L}(\mathbf{H}^1(\Omega)', \tilde{H}^2(\Omega)')$, $\boldsymbol{\mathcal{N}}$ contains $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$. To prove that $\boldsymbol{\mathcal{N}} \subset \tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$, it suffices to show the reversed inclusion for their polar sets

$$\{\mathbf{u} \in \mathbf{H}^1(\Omega) : \langle\mathbf{u}, \mathbf{w}\rangle_{L_2(\Omega)} = 0, \ \mathbf{w} \in \boldsymbol{\mathcal{N}}\} \supset \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \langle\mathbf{u}, \mathbf{w}\rangle_{L_2(\Omega)} = 0, \ \mathbf{w} \in \tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)\}. \tag{3.5}$$

The set on the right is contained in $\{\mathbf{u} \in \mathbf{H}^1(\Omega) : \langle\mathbf{u}, \mathbf{w}\rangle_{L_2(\Omega)} = 0, \ \mathbf{w} \in \mathcal{D}(\Omega), \ \operatorname{div}\mathbf{w} = 0\}$. As shown by De Rham ([7], *cf.* [23], Chap. 1, Prop. 1.1), a distribution $\mathbf{u}$ that vanishes on all divergence-free test functions is a gradient of another distribution. If, additionally $\mathbf{u} \in \mathbf{H}^1(\Omega)$, then necessarily $\mathbf{u} \in \nabla\tilde{H}^2(\Omega)$.

The adjoint of $\operatorname{div} \in \mathcal{L}(\mathbf{H}^1(\Omega)', \tilde{H}^2(\Omega)')$ is $-\nabla \in \mathcal{L}(\tilde{H}^2(\Omega), \mathbf{H}^1(\Omega))$. The latter operator is bounded, and so closed, and it is an homeomorphism with its image, so which in particular is closed. The *closed range theorem* now implies that the space on the left in (3.5) is equal to $\nabla\tilde{H}^2(\Omega)$. This completes the proof. □

Next, we show that, under conditions, the space $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$ in the characterization of $\ker\mathbf{B}$ can be replaced by $\boldsymbol{\mathcal{H}}^{-1}(\Omega)$.

Lemma 3.4. *If the $L_2(\Omega)^n$-orthogonal projector onto $\boldsymbol{\mathcal{H}}^0(\Omega)$ is bounded on $\mathbf{H}^1(\Omega)$, then $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) = \boldsymbol{\mathcal{H}}^{-1}(\Omega)$.*

Proof. As shown in, *e.g.*, ([23], Chapt. 1, Thm. 1.4), the closure of the set of divergence-free test functions in $L_2(\Omega)^n$ is $\{\mathbf{u} \in L_2(\Omega)^n : \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$, and so this space is contained in $\boldsymbol{\mathcal{H}}^0(\Omega)$. On the other hand, if for $(\mathbf{u}_k)_k \subset \boldsymbol{\mathcal{H}}^1(\Omega)$, $\mathbf{u}_k \to \mathbf{u}$ in $L_2(\Omega)^n$, and so in $\mathcal{D}(\Omega)'$, then $\operatorname{div} \mathbf{u} = 0$, and so $\mathbf{u}_k \to \mathbf{u}$ in $H(\operatorname{div}; \Omega)$, in particular meaning that $\mathbf{u} \cdot \mathbf{n} = \lim_{k\to\infty} \mathbf{u}_k \cdot \mathbf{n} = 0$ on $\partial\Omega$. We conclude that

$$\boldsymbol{\mathcal{H}}^0(\Omega) = \{\mathbf{u} \in L_2(\Omega)^n : \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Let Π denote the $L_2(\Omega)^n$-orthogonal projector onto $\boldsymbol{\mathcal{H}}^0(\Omega)$. From $\boldsymbol{\mathcal{H}}^1(\Omega) \subset \boldsymbol{\mathcal{H}}^0(\Omega) \cap \mathbf{H}^1(\Omega)$, we have $\boldsymbol{\mathcal{H}}^1(\Omega) \subset \operatorname{im} \Pi|_{\mathbf{H}^1(\Omega)}$. On the other hand, if Π is bounded on $\mathbf{H}^1(\Omega)$, then $\operatorname{im} \Pi|_{\mathbf{H}^1(\Omega)} \subset \boldsymbol{\mathcal{H}}^0(\Omega) \cap \mathbf{H}^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \operatorname{div} \mathbf{u} = 0\} = \boldsymbol{\mathcal{H}}^1(\Omega)$, *i.e.*,

$$\boldsymbol{\mathcal{H}}^1(\Omega) = \operatorname{im} \Pi|_{\mathbf{H}^1(\Omega)} . \tag{3.6}$$

If, for some $(f_n)_n \subset \boldsymbol{\mathcal{H}}^1(\Omega)$, $f_n \to f$ in $\mathbf{H}^1(\Omega)'$, then, viewed as functionals on $\boldsymbol{\mathcal{H}}^1(\Omega)$, $f_n \to f$ in $\boldsymbol{\mathcal{H}}^{-1}(\Omega)$, *i.e.*, $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) \subset \boldsymbol{\mathcal{H}}^{-1}(\Omega)$.

Conversely, let $f \in \boldsymbol{\mathcal{H}}^{-1}(\Omega)$. Then there exists a $(f_n)_n \subset \boldsymbol{\mathcal{H}}^1(\Omega)$ with $f_n \to f$ in $\boldsymbol{\mathcal{H}}^{-1}(\Omega)$. For any $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $f_n((I - \Pi)\mathbf{u}) = \langle f_n, (I - \Pi)\mathbf{u}\rangle_{L_2(\Omega)^n} = \langle (I - \Pi)f_n, \mathbf{u}\rangle_{L_2(\Omega)^n} = 0$. So, after trivially extending f to a functional on $\mathbf{H}^1(\Omega)$ by means of $f(\operatorname{im}(I - \Pi)) = 0$, by the boundedness of Π on $\mathbf{H}^1(\Omega)$ and (3.6) we have $\|f - f_n\|_{\mathbf{H}^1(\Omega)'} = \sup_{0 \neq \mathbf{u} \in \mathbf{H}^1(\Omega)} \frac{|(f - f_n)(\Pi u)|}{\|\mathbf{u}\|_{H^1(\Omega)^n}} \lesssim \sup_{0 \neq \mathbf{u} \in \mathbf{H}^1(\Omega)} \frac{|(f - f_n)(\Pi \mathbf{u})|}{\|\Pi \mathbf{u}\|_{H^1(\Omega)^n}} = \|f - f_n\|_{\boldsymbol{\mathcal{H}}^{-1}(\Omega)}$, or $\boldsymbol{\mathcal{H}}^{-1}(\Omega) \subset \tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$. $\square$

Theorem 3.5. *Under Assumption 3.1, we have* $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) = \boldsymbol{\mathcal{H}}^{-1}(\Omega)$.

Proof. As shown in, *e.g.*, ([23], Chap. 1, Thm. 1.4), one has the following *Helmholtz decomposition*

$$L_2(\Omega)^n = \boldsymbol{\mathcal{H}}^0(\Omega) \oplus^\perp \nabla(H^1(\Omega)/\mathbb{R}). \tag{3.7}$$

The $L_2(\Omega)^n$-orthogonal projector Π onto $\boldsymbol{\mathcal{H}}^0(\Omega)$ is known as the Leray projector. Given $\mathbf{u} \in L_2(\Omega)^n$, $\nabla z = (I - \Pi)\mathbf{u}$ is the solution of $\langle \mathbf{u} - \nabla z, \nabla w\rangle_{L_2(\Omega)^n} = 0$ $(w \in H^1(\Omega)/\mathbb{R})$.

When $\mathbf{u} \in \mathbf{H}^1(\Omega)$, this z solves the *Poisson problem* with *Neumann boundary conditions*

$$\begin{cases} -\Delta z = \operatorname{div} \mathbf{u} \text{ on } \Omega, \\ \dfrac{\partial z}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega, \\ \int_\Omega z \, dx = 0. \end{cases}$$

Under Assumption 3.1, this Poisson problem is $H^2(\Omega)$-*regular*, and so

$$\|\nabla z\|_{H^1(\Omega)^n} \lesssim \|z\|_{H^2(\Omega)} \lesssim \|\operatorname{div} \mathbf{u}\|_{L_2(\Omega)^n} \lesssim \|\mathbf{u}\|_{H^1(\Omega)^n},$$

i.e., $I - \Pi$ and thus Π is bounded on $\mathbf{H}^1(\Omega)$. Now the result follows from Lemma 3.4. $\square$

Using that on $\boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$, $(\mathbf{w}, \mathbf{v}) \mapsto \int_\Omega \nu\nabla \mathbf{w} : \nabla \mathbf{v} \, d\mathbf{x}$ is *bounded* and satisfies a *Gårding inequality*, we have the following result about the well-posedness of the variational formulation of the parabolic problem that results from the reduction of the instationary Stokes problem, with the homogeneous constraint $\operatorname{div}_{\mathbf{x}} \mathbf{u} = 0$, to a system of equations for the divergence-free velocities only:

Theorem 3.6. *With*

$$\boldsymbol{\mathcal{X}} := L_2(0, T; \boldsymbol{\mathcal{H}}^1(\Omega)), \quad \boldsymbol{\mathcal{Y}} := L_2(0, T; \boldsymbol{\mathcal{H}}^1(\Omega)) \cap H^1_{0,\{T\}}(0, T; \boldsymbol{\mathcal{H}}^{-1}(\Omega)),$$

$\mathbf{A} := \mathbf{u} \mapsto (\mathbf{v} \mapsto a(\mathbf{u}, \mathbf{v}))$ *is a boundedly invertible linear mapping from* $\boldsymbol{\mathcal{X}}$ *to* $\boldsymbol{\mathcal{Y}}'$.

Proof. The statement is equivalent to $\mathbf{A}'$ being boundedly invertible from $\boldsymbol{\mathcal{Y}}$ to $\boldsymbol{\mathcal{X}}'$, which in turn, by making the change of variable $T - t$ to t, is equivalent to the statement that

$$\mathbf{u} \mapsto \left(\mathbf{v} \mapsto \int_0^T \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega \nu \nabla_\mathbf{x} \mathbf{u} : \nabla_\mathbf{x} \mathbf{v} \, \mathrm{d}\mathbf{x}\mathrm{d}t\right),$$

from $L_2(0, T; \boldsymbol{\mathcal{H}}^1(\Omega)) \cap H^1_{0,\{0\}}(0, T; \boldsymbol{\mathcal{H}}^{-1}(\Omega))$ to $L_2(0, T; \boldsymbol{\mathcal{H}}^{-1}(\Omega))$ is boundedly invertible. The boundedness of this mapping follows easily. The mapping corresponds to a variational formulation of a parabolic problem with homogeneous initial datum in the space of divergence-free velocities. The boundedness of the inverse is a consequence of $(\mathbf{u}, \mathbf{v}) \mapsto \int_\Omega \nu \nabla_\mathbf{x} \mathbf{u} : \nabla_\mathbf{x} \mathbf{v} \, \mathrm{d}\mathbf{x}$ being bounded and coercive on $\boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$, and it is shown, *e.g.*, as a special case of ([18], Thm. 4.1), where a possible inhomogeneous initial condition is imposed weakly. $\qquad\square$

The characterizations of the kernels given by (3.4), Lemma 3.3, and Theorem 3.5, together with Theorem 3.6 imply condition (i) of Theorem 2.1.

Condition (iii) of Theorem 2.1 is equivalent to (iii)′ which, by Lemma 2.3, is equivalent to

$$\mathbf{B} = I \otimes \mathrm{div}_\mathbf{x} : \mathcal{L}\Big(L_2(0, T; \mathbf{H}^1(\Omega)) \cap H^1_{0,\{T\}}(0, T; \mathbf{H}^1(\Omega)'),$$
$$L_2(0, T; L_{2,0}(\Omega)) \cap H^1_{0,\{T\}}(0, T; \tilde{H}^2(\Omega)')\Big) \text{ is surjective.} \qquad (3.8)$$

Note that since $I \otimes \mathrm{div}_\mathbf{x}$ is not injective, to prove (3.8) it is generally *not* sufficient to show that $I \otimes \mathrm{div}_\mathbf{x}$ is surjective both as mapping in $\mathcal{L}(L_2(0, T; \mathbf{H}^1(\Omega)), L_2(0, T; L_{2,0}(\Omega)))$ and as mapping in $\mathcal{L}(H^1_{0,\{T\}}(0, T; \mathbf{H}^1(\Omega)'), H^1_{0,\{T\}}(0, T; \tilde{H}^2(\Omega)'))$.

Below, we will construct a mapping div^+ with $\mathrm{div} \circ \mathrm{div}^+ = I$, such that

$$\mathrm{div}^+ \in \mathcal{L}(L_{2,0}(\Omega), \mathbf{H}^1(\Omega)), \quad \mathrm{div}^+ \in \mathcal{L}(\tilde{H}^2(\Omega)', \mathbf{H}^1(\Omega)'). \qquad (3.9)$$

Since, consequently, $I \otimes \mathrm{div}_\mathbf{x}^+$ is a right-inverse for the mapping from (3.8), this will imply the surjectivity of the latter mapping.

We define $\mathrm{div}^+ : g \mapsto \mathbf{u}$ by

$$\begin{cases} \mathbf{u} + \nabla p = \mathbf{f} & \text{on } \Omega, \\ \mathrm{div}\,\mathbf{u} = g & \text{on } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{f} = 0$, or, more precisely, by its variational formulation to find $(\mathbf{u}, p) \in L_2(\Omega)^n \times H^1(\Omega)/\mathbb{R}$ such that

$$\int_\Omega \mathbf{u} \cdot \mathbf{v} + \int_\Omega \nabla p \cdot \mathbf{v} + \int_\Omega \nabla q \cdot \mathbf{u} = \mathbf{f}(\mathbf{v}) + g(q) \quad ((\mathbf{v}, q) \in L_2(\Omega)^n \times H^1(\Omega)/\mathbb{R}). \qquad (3.10)$$

From the fact that $\nabla \in \mathcal{L}(H^1(\Omega)/\mathbb{R}, L_2(\Omega)^n)$ is a homeomorphism onto its range, applications of Lemma 2.3, Proposition 2.2, and Theorem 2.1 confirm the well-known fact that this variational problem, for general $\mathbf{f} \in L_2(\Omega)^n$, defines a boundedly invertible operator from $L_2(\Omega)^n \times H^1(\Omega)/\mathbb{R}$ to its dual.

Under Assumption 3.1, for $\mathbf{f} \in \mathbf{H}^1(\Omega)$ and $g \in L_{2,0}(\Omega)$, the solution p of

$$\begin{cases} -\Delta p = g - \mathrm{div}\,\mathbf{f} & \text{on } \Omega, \\ \dfrac{\partial p}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

is in $\tilde{H}^2(\Omega)$, and $\mathbf{u} := \mathbf{f} - \nabla p \in \mathbf{H}^1(\Omega)$. We infer that the mapping $(\mathbf{f}, g) \mapsto (\mathbf{u}, p)$ defined by (3.10) is in $\mathcal{L}(\mathbf{H}^1(\Omega) \times L_{2,0}(\Omega), \mathbf{H}^1(\Omega) \times \tilde{H}^2(\Omega))$, and so, by considering the adjoint and using the symmetry of the left-hand side of (3.10) in $(\mathbf{u}, p)$ and $(\mathbf{v}, q)$, it is in $\mathcal{L}(\mathbf{H}^1(\Omega)' \times \tilde{H}^2(\Omega)', \mathbf{H}^1(\Omega)' \times L_{2,0}(\Omega))$. We conclude that (3.9) and thus condition (iii) of Theorem 2.1 are valid.

> *Having verified all conditions of Theorem 2.1, the proof of Theorem 3.2 is now completed.*

Finally in this section, we derive well-posedness of an alternative variational formulation. The variational formulation (3.2) of our Stokes problem (3.1) was derived by applying integration by parts over time. This has the advantage that the initial condition $\mathbf{u}(0, \cdot) = \mathbf{u}_0$ enters the variational formulation as a natural boundary condition, *i.e.*, in the right-hand side. In any case for a *homogeneous initial condition*, *i.e.*, $\mathbf{u}(0, \cdot) = \mathbf{0}$, an alternative variational formulation is obtained by not applying integration by parts over time. It reads as a variational formulation of the form (2.1), where

$$
\begin{cases}
a(\mathbf{u}, \mathbf{v}) = \int_0^T \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega \nu \nabla_\mathbf{x} \mathbf{u} : \nabla_\mathbf{x} \mathbf{v} \, \mathrm{d}\mathbf{x}\mathrm{d}t, \\
b(p, \mathbf{v}) = -\int_0^T \int_\Omega p \operatorname{div} \mathbf{v} \, \mathrm{d}\mathbf{x}\mathrm{d}t, \\
c(\mathbf{u}, q) = \int_0^T \int_\Omega q \operatorname{div} \mathbf{u} \, \mathrm{d}\mathbf{x}\mathrm{d}t, \\
\mathbf{f}(\mathbf{v}) = \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\partial\Omega} \sum_{i=1}^{n-1} (\mathbf{v} \cdot \boldsymbol{\tau}_i) g_i \mathrm{d}\mathbf{x}\mathrm{d}t \\
g(q) = \int_0^T \int_\Omega g \, q \, \mathrm{d}\mathbf{x}\mathrm{d}t.
\end{cases}
\tag{3.11}
$$

Theorem 3.7. *With*

$$
\begin{aligned}
\mathbf{U} &:= L_2(0, T; \mathbf{H}^1(\Omega)) \cap H_{0,\{0\}}^1(0, T; \mathbf{H}^1(\Omega)'), \\
P &:= L_2(0, T; L_{2,0}(\Omega)), \\
\mathbf{V} &:= L_2(0, T; \mathbf{H}^1(\Omega)), \\
Q &:= \left(L_2(0, T; L_{2,0}(\Omega)) \cap H_{0,\{0\}}^1(0, T; \tilde{H}^2(\Omega)') \right)',
\end{aligned}
$$

and under Assumption 3.1, the mapping $\mathbf{L} : (\mathbf{u}, p) \mapsto (\mathbf{f}, g)$ as in (2.1) with bilinear forms from (3.11) defines a boundedly invertible linear mapping $\mathbf{U} \times P \to \mathbf{V}' \times Q'$.

Proof. Denoting the spaces $\mathbf{U}$, $\mathbf{V}$, P, and Q, and operator $\mathbf{L}$ from Theorem 3.2 here as $\bar{\mathbf{U}}$, $\bar{\mathbf{V}}$, $\bar{P}$, $\bar{Q}$, and $\bar{\mathbf{L}}$, and defining $(Rw)(t, x) = w(T - t, x)$, we have

$$
(\mathbf{L}(\mathbf{u}, p))(\mathbf{v}, q) = (\bar{\mathbf{L}}(R\mathbf{v}, -Rq))(R\mathbf{u}, -Rp) = (\bar{\mathbf{L}}'(R\mathbf{u}, -Rp))(R\mathbf{v}, -Rq).
$$

From $\bar{\mathbf{L}}' \in \mathcal{L}(\bar{\mathbf{V}} \times \bar{Q}, \bar{\mathbf{U}}' \times \bar{P}')$ being a boundedly invertible, and $R\mathbf{U} = \bar{\mathbf{V}}$, $RP = \bar{Q}$, $R\mathbf{V} = \bar{\mathbf{U}}$, and $RQ = \bar{P}$, the proof is completed. $\qquad\square$

4. The instationary Stokes problem, with no-slip boundary conditions, as a well-posed operator equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Given vector fields $\tilde{\mathbf{f}}$ on $(0, T) \times \Omega$ and $\mathbf{u}_0$ on Ω, and a function g on $(0, T) \times \Omega$, we consider the instationary inhomogeneous Stokes problem with *no-slip boundary conditions* to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_\mathbf{x} \mathbf{u} + \nabla_\mathbf{x} p = \tilde{\mathbf{f}} & \text{on } (0, T) \times \Omega, \\
\operatorname{div}_\mathbf{x} \mathbf{u} = g & \text{on } (0, T) \times \Omega, \\
\mathbf{u} = 0 & \text{on } (0, T) \times \partial\Omega, \\
\mathbf{u}(0, \cdot) = \mathbf{u}_0 & \text{on } \Omega.
\end{cases}
$$

By integrating the first equation against smooth vector fields $\mathbf{v}$, that as function of $\mathbf{x}$ vanish at $\partial\Omega$, and that as function of t vanish at $t = T$, and by applying integration by parts in space and time, and by integrating the second equation against smooth functions q, and by applying integration by parts, we arrive at a variational problem of the form (2.1), where

$$
\begin{cases}
a(\mathbf{u}, \mathbf{v}) = -\int_0^T \int_\Omega \mathbf{u} \cdot \dfrac{\partial \mathbf{v}}{\partial t}\, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega \nu \nabla_\mathbf{x} \mathbf{u} : \nabla_\mathbf{x} \mathbf{v}\, \mathrm{d}\mathbf{x}\mathrm{d}t, \\[2mm]
b(p, \mathbf{v}) = \int_0^T \int_\Omega \mathbf{v} \cdot \nabla p\, \mathrm{d}\mathbf{x}\mathrm{d}t, \\[2mm]
c(\mathbf{u}, q) = -\int_0^T \int_\Omega \mathbf{u} \cdot \nabla q\, \mathrm{d}\mathbf{x}\mathrm{d}t, \\[2mm]
\mathbf{f}(\mathbf{v}) = \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega \mathbf{u}_0 \cdot \mathbf{v}(0, \cdot)\, \mathrm{d}\mathbf{x}, \\[2mm]
g(q) = \int_0^T \int_\Omega g\, q\, \mathrm{d}\mathbf{x}\mathrm{d}t.
\end{cases}
\tag{4.1}
$$

Remark 4.1. With $\hat{H}^2(\Omega) := \{p \in H^2(\Omega) : \nabla p = 0 \text{ on } \partial\Omega\}/\mathbb{R}$, following the exposition in Section 3, an obvious choice for the spaces $\mathbf{U}, P$ and $\mathbf{V}, Q$ for the variables $\mathbf{u}, p$ and $\mathbf{v}, q$, would be

$$
L_2(0, T; H_0^1(\Omega)^n), \quad \left(L_2(0, T; L_{2,0}(\Omega)) \cap H_{0,\{T\}}^1 \big(0, T; \hat{H}^2(\Omega)' \big) \right)',
$$
$$
L_2(0, T; H_0^1(\Omega)^n) \cap H_{0,\{T\}}^1(0, T; H^{-1}(\Omega)^n), \quad L_2(0, T; L_{2,0}(\Omega)),
$$

where $H^{-1}(\Omega) = H_0^1(\Omega)'$ with respect to the identification $L_2(\Omega)' \simeq L_2(\Omega)$. With this choice, the resulting space $\boldsymbol{\mathcal{H}}^1(\Omega)$ of divergence free spatial functions would read as $\{\mathbf{u} \in H_0^1(\Omega)^n : \mathrm{div}\,\mathbf{u} = 0\}$, with, as in Section 3, its closure $\boldsymbol{\mathcal{H}}^0(\Omega)$ in $L_2(\Omega)^n$ being $\{\mathbf{u} \in L_2(\Omega)^n : \mathrm{div}\,\mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$. Now when following the analysis from Section 3, the problem is that the $L_2(\Omega)^n$-orthogonal projector onto $\boldsymbol{\mathcal{H}}^0(\Omega)$, *i.e.*, the Leray projector, does not preserve no-slip boundary conditions, and therefore is not bounded on $H_0^1(\Omega)^n$.

In view of Remark 4.1, later, in Theorem 4.3, we will select trial- and test-spaces by making a shift in smoothness indices for the spatial variables.

Before that, first we study the stationary Stokes problem with homogeneous Dirichlet boundary conditions of finding $\mathbf{u} \in H_0^1(\Omega)^n$, $p \in L_{2,0}(\Omega)$ such that, for given $\mathbf{f} \in H^{-1}(\Omega)^n$, $g \in L_{2,0}(\Omega)$,

$$
\int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v}\, \mathrm{d}\mathbf{x} - \int_\Omega p\, \mathrm{div}\,\mathbf{v}\, \mathrm{d}\mathbf{x} + \int_\Omega q\, \mathrm{div}\,\mathbf{u}\, \mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{v}) + g(q) \quad ((\mathbf{v}, q) \in H_0^1(\Omega)^n \times L_{2,0}(\Omega)).
\tag{4.2}
$$

Since, using that Ω is a bounded Lipschitz domain, $\nabla \in \mathcal{L}(L_{2,0}(\Omega), H^{-1}(\Omega)^n)$ is homeomorphism onto its range, see (3.3), applications of Lemma 2.3, Proposition 2.2, and Theorem 2.1 confirm the well-known fact that this variational problem defines a boundedly invertible mapping between $H_0^1(\Omega)^n \times L_{2,0}(\Omega)$ and its dual.

We will need the following assumption on the domain Ω about $H^2(\Omega)^n \times H^1(\Omega)$-regularity of this stationary Stokes problem.

Assumption 4.2. The bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is such that for any $\mathbf{f} \in L_2(\Omega)^n$, $g \in H^1(\Omega)/\mathbb{R}$, the solution $(\mathbf{u}, p)$ of (4.2) belongs to $H^2(\Omega)^n \times H^1(\Omega)$ and $\|\mathbf{u}\|_{H^2(\Omega)^n} + \|p\|_{H^1(\Omega)} \lesssim \|\mathbf{f}\|_{L_2(\Omega)^n} + \|g\|_{H^1(\Omega)}$.

This assumption is known to be satisfied for domains Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ that either have a C^2-boundary, or that are convex with a piecewise smooth boundary. See [5, 11] for the two- or three-dimensional case, respectively.

Theorem 4.3. *With*

$$\mathbf{U} := L_2(0, T; L_2(\Omega)^n),$$

$$P := \left(L_2(0, T; H^1(\Omega)/\mathbb{R}) \cap H^1_{0,\{T\}}\big(0, T; (H^1(\Omega)/\mathbb{R})'\big) \right)',$$

$$\mathbf{V} := L_2(0, T; (H^1_0(\Omega) \cap H^2(\Omega))^n) \cap H^1_{0,\{T\}}(0, T; L_2(\Omega)^n),$$

$$Q := L_2(0, T; H^1(\Omega)/\mathbb{R}),$$

and under Assumption 4.2, the mapping $\mathbf{L} : (\mathbf{u}, p) \mapsto (\mathbf{f}, g)$ *as in* (2.1) *with bilinear forms from* (4.1) *defines a boundedly invertible linear mapping* $\mathbf{U} \times P \to \mathbf{V}' \times Q'$.

(Here, dual spaces should be interpreted with respect to the identifications $L_{2,0}(\Omega)' \simeq L_{2,0}(\Omega)$, or $L_2(0, T; L_{2,0}(\Omega))' \simeq L_2(0, T; L_{2,0}(\Omega))$, respectively.)

Before proving this theorem, we give some more auxiliary results dealing with the stationary problem.

Lemma 4.4. *It holds that*

$$\ker(\nabla' \in \mathcal{L}(L_2(\Omega)^n, (H^1(\Omega)/\mathbb{R})')) = \{\mathbf{u} \in L_2(\Omega)^n : \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \ on \ \partial\Omega\} := \check{\boldsymbol{\mathcal{H}}}^0(\Omega),$$

$$\ker(\nabla' \in \mathcal{L}(H^1_0(\Omega), L_{2,0}(\Omega))) = \{\mathbf{u} \in H^1_0(\Omega)^n : \operatorname{div} \mathbf{u} = 0\} := \check{\boldsymbol{\mathcal{H}}}^1(\Omega),$$

$$\ker(\nabla' \in \mathcal{L}((H^2(\Omega) \cap H^1_0(\Omega)^1)^n, H^1(\Omega)/\mathbb{R})) = \{\mathbf{u} \in (H^2(\Omega) \cap H^1_0(\Omega))^n : \operatorname{div} \mathbf{u} = 0\} := \check{\boldsymbol{\mathcal{H}}}^2(\Omega).$$

Proof. Since in the last two cases $\nabla = -\operatorname{div}'$ by definition, we only have to verify the first statement, *i.e.*, that

$$\boldsymbol{\mathcal{N}} := \ker(\nabla' \in \mathcal{L}(L_2(\Omega)^n, (H^1(\Omega)/\mathbb{R})')) = \check{\boldsymbol{\mathcal{H}}}^0(\Omega).$$

For $\mathbf{u} \in \check{\boldsymbol{\mathcal{H}}}^0(\Omega)$, $p \in H^1(\Omega)/\mathbb{R}$, one has $\int_\Omega \nabla p \cdot \mathbf{u} \, d\mathbf{x} = 0$, *i.e.*, $\check{\boldsymbol{\mathcal{H}}}^0(\Omega) \subset \boldsymbol{\mathcal{N}}$. To prove the reversed inclusion, we have to show that

$$\{\mathbf{u} \in L_2(\Omega)^n : \langle \mathbf{u}, \mathbf{w}\rangle_{L_2(\Omega)} = 0, \mathbf{w} \in \boldsymbol{\mathcal{N}}\} \supset \left\{\mathbf{u} \in L_2(\Omega)^n : \langle \mathbf{u}, \mathbf{w}\rangle_{L_2(\Omega)} = 0, \mathbf{w} \in \check{\boldsymbol{\mathcal{H}}}^0(\Omega)\right\}. \tag{4.3}$$

The set on the right is part of $\{\mathbf{u} \in L_2(\Omega)^n : \langle \mathbf{u}, \mathbf{w}\rangle_{L_2(\Omega)} = 0, \mathbf{w} \in \mathcal{D}(\Omega), \operatorname{div} \mathbf{w} = 0\}$. As shown by De Rham ([7], *cf.* [23], Chap. 1, Prop. 1.1), a distribution $\mathbf{u}$ that vanishes on all divergence-free test functions is a gradient of another distribution. If, additionally $\mathbf{u} \in L_2(\Omega)^n$, then necessarily $\mathbf{u} \in \nabla(H^1(\Omega)/\mathbb{R})$.

Since $\nabla : H^1(\Omega)/\mathbb{R} \to L_2(\Omega)^n$ is bounded, and so is closed, and moreover since this mapping is an homeomorphism with its image, which is therefore closed, the *closed range theorem* tells us that the space on the left in (4.3) is equal to $\nabla(H^1(\Omega)/\mathbb{R})$, which completes the proof. $\quad\square$

It holds that $\check{\boldsymbol{\mathcal{H}}}^2(\Omega) \hookrightarrow \check{\boldsymbol{\mathcal{H}}}^1(\Omega) \hookrightarrow \check{\boldsymbol{\mathcal{H}}}^0(\Omega)$ with dense embeddings. For $i \in \{1, 2\}$, we set $\check{\boldsymbol{\mathcal{H}}}^{-i}(\Omega) := (\check{\boldsymbol{\mathcal{H}}}^i(\Omega))'$, where this dual space should be interpreted with respect to the identification $(\check{\boldsymbol{\mathcal{H}}}^0(\Omega))' \simeq \check{\boldsymbol{\mathcal{H}}}^0(\Omega)$.

The stationary Stokes problem (4.2) with $g = 0$ can be reduced to a problem involving divergence-free velocities only. It reads as finding $\mathbf{u} \in \check{\boldsymbol{\mathcal{H}}}^1(\Omega)$ that solves

$$\int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} = \mathbf{f}(\mathbf{v}) \quad \left(\mathbf{v} \in \check{\boldsymbol{\mathcal{H}}}^1(\Omega)\right). \tag{4.4}$$

Under Assumption 4.2, for $\mathbf{f} \in L_2(\Omega)^n$ we have $\mathbf{u} \in \check{\boldsymbol{\mathcal{H}}}^2(\Omega)$ with

$$\|\mathbf{u}\|_{H^2(\Omega)^n} \lesssim \|\mathbf{f}\|_{L_2(\Omega)^n}. \tag{4.5}$$

After these preparations dealing with the stationary Stokes problem, we are ready to prove Theorem 4.3 by verifying the conditions of Theorem 2.1.

Recalling the definitions of the spaces $\mathbf{U}$, P, $\mathbf{V}$, and Q given in Theorem 4.3, similarly as in Section 3 one shows that the bilinear forms $a : \mathbf{U} \times \mathbf{V} \to \mathbb{R}$, $b : P \times \mathbf{V} \to \mathbb{R}$, and $c : \mathbf{U} \times Q \to \mathbb{R}$ are bounded.

With

$$\mathbf{B} := I \otimes \nabla' \in \mathcal{L}(\mathbf{V}, P'), \quad \mathbf{C} := I \otimes \nabla' \in \mathcal{L}(\mathbf{U}, Q'),$$

the operator $\mathbf{C}' \in \mathcal{L}(Q, \mathbf{U}')$ is a homeomorphism onto its range, which by Lemma 2.3 shows condition (ii) of Theorem 2.1.

As a second step, we verify Condition (i). As an easy consequence of Lemma 4.4, we have

$$\ker \mathbf{C} = L_2(0, T; \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)). \tag{4.6}$$

Similarly as in the proof of Lemma 3.3, we have

$$P \simeq L_2(0, T; H^1(\Omega)/\mathbb{R})' + H_{0,\{T\}}^1\big(0, T; (H^1(\Omega)/\mathbb{R})'\big)',$$

and consequently that

$$\begin{aligned}
\ker \mathbf{B} = {} & \ker(I \otimes \nabla' \in \mathcal{L}(L_2(0, T; (H_0^1(\Omega) \cap H^2(\Omega))^n), L_2(0, T; H^1(\Omega)/\mathbb{R})) \\
& \cap \ker(I \otimes \nabla' \in \mathcal{L}(H_{0,\{T\}}^1(0, T; L_2(\Omega)^n), H_{0,\{T\}}^1\big(0, T; (H^1(\Omega)/\mathbb{R})'\big) \\
= {} & L_2(0, T; \breve{\boldsymbol{\mathcal{H}}}^2(\Omega)) \cap H_{0,\{T\}}^1(0, T; \breve{\boldsymbol{\mathcal{H}}}^0(\Omega))
\end{aligned} \tag{4.7}$$

by an application of Lemma 4.4.

Theorem 4.5. *With*

$$\boldsymbol{\mathcal{X}}_1 := L_2(0, T; \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)), \quad \boldsymbol{\mathcal{Y}}_1 := L_2(0, T; \breve{\boldsymbol{\mathcal{H}}}^2(\Omega)) \cap H_{0,\{T\}}^1(0, T; \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)),$$

under Assumption 4.2, $\mathbf{A} : \mathbf{u} \mapsto (\mathbf{v} \mapsto a(\mathbf{u}, \mathbf{v}))$ defines a boundedly invertible linear mapping from $\boldsymbol{\mathcal{X}}_1$ to $\boldsymbol{\mathcal{Y}}_1'$.

Proof. We follow [21], proof of Theorem 4.2. The boundedness of $\mathbf{A}$ follows easily.

The boundedness of $\mathbf{A}^{-1}$ is equivalent to $(\mathbf{A}')^{-1} \in \mathcal{L}(\boldsymbol{\mathcal{X}}_1', \boldsymbol{\mathcal{Y}}_1)$. To demonstrate the latter, we have to show that for any $\mathbf{f} \in \boldsymbol{\mathcal{X}}_1 \simeq \boldsymbol{\mathcal{X}}_1'$, the variational problem of finding $\mathbf{z}$ such that

$$\int_0^T \int_\Omega -\mathbf{w} \cdot \frac{\partial \mathbf{z}}{\partial t} \, d\mathbf{x} dt + \int_0^T \int_\Omega \nu \nabla \mathbf{w} : \nabla \mathbf{z} \, d\mathbf{x} dt = \int_0^T \int_\Omega \mathbf{f} \cdot \mathbf{w} \, d\mathbf{x} dt \quad (\mathbf{w} \in \boldsymbol{\mathcal{X}}_1), \tag{4.8}$$

has a unique solution $\mathbf{z} \in \boldsymbol{\mathcal{Y}}_1$ with $\|\mathbf{z}\|_{\boldsymbol{\mathcal{Y}}_1} \lesssim \|\mathbf{f}\|_{\boldsymbol{\mathcal{X}}_1}$.

Although this result may follow from the theory of analytic semigroups, we give a more elementary derivation. With

$$\boldsymbol{\mathcal{X}}_0 := L_2(0, T; \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)), \quad \boldsymbol{\mathcal{Y}}_0 := L_2(0, T; \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)) \cap H_{0,\{T\}}^1\left(0, T; \breve{\boldsymbol{\mathcal{H}}}^{-1}\right),$$

similar to Theorem 3.6, we have that for $\mathbf{f} \in \boldsymbol{\mathcal{X}}_0' \supset \boldsymbol{\mathcal{X}}_1'$, (4.8), with test space $\boldsymbol{\mathcal{X}}_0$, has a unique solution $\mathbf{z} \in \boldsymbol{\mathcal{Y}}_0$. Below, we will show that for a subspace of sufficiently smooth $\mathbf{f}$, this solution is in $\boldsymbol{\mathcal{Y}}_1$, and thus that (4.8) holds for all $\mathbf{w} \in \boldsymbol{\mathcal{X}}_1$, and moreover that $\|\mathbf{z}\|_{\boldsymbol{\mathcal{Y}}_1} \lesssim \|\mathbf{f}\|_{\boldsymbol{\mathcal{X}}_1}$. Since the subspace of these smooth $\mathbf{f}$ will be dense in $\boldsymbol{\mathcal{X}}_1$, this will complete the proof.

Equation (4.8) is the variational formulation of the problem of finding, for $t \in (0, T)$, $\mathbf{z}(t, \cdot) \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)$ that satisfies

$$\begin{cases} \int_\Omega -\dfrac{\partial \mathbf{z}}{\partial t}(t, \cdot) \cdot \mathbf{w} \, d\mathbf{x} + \int_\Omega \nu \nabla \mathbf{w} : \nabla \mathbf{z}(t, \cdot) \, d\mathbf{x} = \int_\Omega \mathbf{f}(t, \cdot) \cdot \mathbf{w} \, d\mathbf{x} \quad \left(\mathbf{w} \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)\right), \\ \mathbf{z}(T, \cdot) = 0. \end{cases} \tag{4.9}$$

Note that as function of $\tilde{t} = T - t$, $\mathbf{z}$ satisfies a standard parabolic *initial* value problem. As shown in ([24], Chap. IV, Sect. 27), if $\mathbf{f} \in H^2(0, T; \breve{\mathcal{H}}^{-1}(\Omega))$ with $\mathbf{f}(T, \cdot) \in \breve{\mathcal{H}}^2(\Omega)$ and $\frac{\partial \mathbf{f}}{\partial t}(T, \cdot) \in \breve{\mathcal{H}}^0(\Omega)$, then its solution $\mathbf{z} \in H^2(0, T; \breve{\mathcal{H}}^1(\Omega))$.

By substituting $\mathbf{w} = -\frac{\partial \mathbf{z}}{\partial t}(t, \cdot) \in \breve{\mathcal{H}}^1(\Omega)$ in (4.9), we obtain

$$\left\| \frac{\partial \mathbf{z}}{\partial t}(t, \cdot) \right\|_{L_2(\Omega)^n}^2 - \frac{1}{2} \frac{\partial}{\partial t} \int_\Omega \nu \nabla \mathbf{z}(t, \cdot) : \nabla \mathbf{z}(t, \cdot) \, d\mathbf{x} = - \int_\Omega \mathbf{f}(t, \cdot) \cdot \frac{\partial \mathbf{z}}{\partial t}(t, \cdot) \, d\mathbf{x}.$$

By integrating this equality over time, applying $\mathbf{z}(T, \cdot) = 0$ and Cauchy−Schwarz' inequality, and by additionally assuming that $\mathbf{f} \in L_2(0, T; \breve{\mathcal{H}}^0(\Omega))$, we arrive at

$$\int_0^T \left\| \frac{\partial \mathbf{z}}{\partial t}(t, \cdot) \right\|_{L_2(\Omega)^n}^2 dt \leq \frac{1}{2} \int_0^T \|\mathbf{f}(t, \cdot)\|_{L_2(\Omega)^n}^2 dt + \frac{1}{2} \int_0^T \left\| \frac{\partial \mathbf{z}}{\partial t}(t, \cdot) \right\|_{L_2(\Omega)^n}^2 dt,$$

or

$$\int_0^T \left\| \frac{\partial \mathbf{z}}{\partial t}(t, \cdot) \right\|_{L_2(\Omega)^n}^2 dt \leq \int_0^T \left\| \mathbf{f}(t, \cdot) \right\|_{L_2(\Omega)^n}^2 dt. \tag{4.10}$$

By additionally assuming that $\mathbf{f}(t, \cdot) \in \breve{\mathcal{H}}^0(\Omega)$, from $\frac{\partial \mathbf{z}}{\partial t}(t, \cdot) \in \breve{\mathcal{H}}^1(\Omega) \subset \breve{\mathcal{H}}^0(\Omega)$ and (4.5), the first equation in (4.9) shows that $\mathbf{z}(t, \cdot) \in \breve{\mathcal{H}}^2(\Omega)$ and $\|\mathbf{z}(t, \cdot)\|_{H^2(\Omega)^n} \lesssim \|\mathbf{f}(t, \cdot)\|_{L_2(\Omega)^n} + \|\frac{\partial \mathbf{z}}{\partial t}(t, \cdot)\|_{L_2(\Omega)^n}$. By integrating this inequality over time and applying (4.10), we obtain that

$$\|\mathbf{z}\|_{L_2(0, T; \breve{\mathcal{H}}^2(\Omega))} \lesssim \|\mathbf{f}\|_{L_2(0, T; L_2(\Omega)^n)} \tag{4.11}$$

By combining (4.10) and (4.11), we have $\|\mathbf{z}\|_{\mathcal{Y}_1} \lesssim \|\mathbf{f}\|_{\mathcal{X}_1}$ and the proof is completed. $\qquad\square$

The characterizations of the kernels (4.6) and (4.7) together with Theorem 4.5 imply condition (i) of Theorem 2.1.

Condition (iii) of Theorem 2.1 is equivalent to (iii)′, which by Lemma 2.3, is equivalent to

$$\mathbf{B} = I \otimes \mathrm{div} \in \mathcal{L}\Big(L_2(0, T; (H_0^1(\Omega) \cap H^2(\Omega))^n) \cap H_{0,\{T\}}^1(0, T; L_2(\Omega)^n),$$
$$L_2(0, T; H^1(\Omega)/\mathbb{R}) \cap H_{0,\{T\}}^1\big(0, T; (H^1(\Omega)/\mathbb{R})'\big) \Big) \text{ is surjective.} \tag{4.12}$$

Below, we will construct a mapping div^+ with $\mathrm{div} \circ \mathrm{div}^+ = I$, such that

$$\mathrm{div}^+ \in \mathcal{L}(H^1(\Omega)/\mathbb{R}, (H_0^1(\Omega) \cap H^2(\Omega))^n), \ \mathrm{div}^+ \in \mathcal{L}((H^1(\Omega)/\mathbb{R})', L_2(\Omega)^n). \tag{4.13}$$

Since, consequently, $I \otimes \mathrm{div}_{\mathbf{x}}^+$ is a right-inverse for the mapping $\mathbf{B}$ defined in (4.12), this will imply the surjectivity of the latter mapping.

We define $\mathrm{div}^+ : g \mapsto \mathbf{u}$ by the solution map of the stationary Stokes problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = 0 & \text{on } \Omega \\ \mathrm{div}\, \mathbf{u} = g & \text{on } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases}$$

or, more precisely, by its variational formulation which reads: Find $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L_{2,0}(\Omega)$ such that

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \, \mathrm{div}\, \mathbf{v} + \int_\Omega q \, \mathrm{div}\, \mathbf{u} = g(q) \quad ((\mathbf{v}, q) \in H_0^1(\Omega)^n \times L_{2,0}(\Omega)). \tag{4.14}$$

Since the bilinear form on the left hand side of (4.2) is symmetric in $(\mathbf{u}, p)$ and $(\mathbf{v}, q)$, under Assumption 4.2 the mapping $(\mathbf{f}, g) \mapsto (\mathbf{u}, p)$ defined by (4.14) is not only in $\mathcal{L}(L_2(\Omega)^n \times H^1(\Omega)/\mathbb{R}, (H_0^1(\Omega) \cap H^2(\Omega))^n \times H^1(\Omega)/\mathbb{R})$, but, by taking the adjoint, it is also in $\mathcal{L}((H^{-1}(\Omega) \cap H^2(\Omega)')^n \times (H^1(\Omega)/\mathbb{R})', L_2(\Omega)^n \times (H^1(\Omega)/\mathbb{R})')$. We conclude (4.12) has been established, and thus that condition (iii) of Theorem 2.1 is valid.

Having verified all conditions of Theorem 2.1, the proof of Theorem 4.3 is now completed.

Similar to Theorem 3.7 for the free-slip boundary conditions case, for the instationary Stokes problem with no-slip boundary conditions and a *homogeneous initial condition*, a variational formulation of the form (2.1) can be derived without applying integration by parts. The bilinear forms and right-hand side read as

$$
\begin{cases}
a(\mathbf{u}, \mathbf{v}) = \int_0^T \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} \, d\mathbf{x}dt + \int_0^T \int_\Omega \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v} \, d\mathbf{x}dt, \\
b(p, \mathbf{v}) = \int_0^T \int_\Omega \mathbf{v} \cdot \nabla p \, d\mathbf{x}dt, \\
c(\mathbf{u}, q) = -\int_0^T \int_\Omega \mathbf{u} \cdot \nabla q \, d\mathbf{x}dt, \\
\mathbf{f}(\mathbf{v}) = \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{v} \, d\mathbf{x}dt, \\
g(q) = \int_0^T \int_\Omega g \, q \, d\mathbf{x}dt.
\end{cases}
\tag{4.15}
$$

and we have the following result:

Theorem 4.6. *With*

$$
\mathbf{U} := L_2(0, T; (H_0^1(\Omega) \cap H^2(\Omega))^n) \cap H_{0,\{0\}}^1(0, T; L_2(\Omega)^n),
$$
$$
P := L_2(0, T; H^1(\Omega)/\mathbb{R}),
$$
$$
\mathbf{V} := L_2(0, T; L_2(\Omega)^n),
$$
$$
Q := \left(L_2(0, T; H^1(\Omega)/\mathbb{R}) \cap H_{0,\{0\}}^1 \left(0, T; (H^1(\Omega)/\mathbb{R})'\right) \right)',
$$

the mapping $\mathbf{L} : (\mathbf{u}, p) \mapsto (\mathbf{f}, g)$ *as in* (2.1) *with bilinear forms from* (4.15) *defines a boundedly invertible linear mapping* $\mathbf{U} \times P \to \mathbf{V}' \times Q'$.

5. The instationary Navier−Stokes problem with homogeneous initial condition

With the spaces $\mathbf{U}$, P, $\mathbf{V}$, Q from either Theorem 4.6 (no slip boundary conditions) and $n \in \{2, 3\}$, or those from Theorem 3.7 (free-slip boundary conditions) and $n = 2$, we will show that any solution of the *Navier*-Stokes problem is locally unique in $\mathbf{U} \times P$, and that for sufficiently small data $(\mathbf{f}, g) \in \mathbf{V}' \times Q'$, such a solution exists.

Lemma 5.1. *For Banach spaces X and Y, let $B = L + N : X \to Y'$ where $L \in \mathcal{L}(X, Y')$ is boundedly invertible, and $N(0) = 0$.*

For some $R > 0$ and $\alpha < \|L^{-1}\|_{\mathcal{L}(Y', X)}^{-1}$, let

$$
\|N(x_1) - N(x_2)\|_{Y'} \leq \alpha \|x_1 - x_2\|_X \quad (x_1, x_2 \in B(0; R) := \{x \in X : \|x\|_X \leq R\}).
$$

Then for any $h \in Y'$ with $\|h\|_{Y'} \leq R(\|L^{-1}\|_{\mathcal{L}(Y', X)}^{-1} - \alpha)$, there exists a unique $x \in B(0; R)$ with $B(x) = h$.

Proof. $B(x) = h$ is equivalent to $x = T(x) := L^{-1}(h - N(x))$. For $\|h\|_{Y'} \leq R(\|L^{-1}\|_{\mathcal{L}(Y', X)}^{-1} - \alpha)$ and $x \in B(0; R)$, $\|T(x)\|_X \leq \|L^{-1}\|_{\mathcal{L}(Y', X)}(\|h\|_Y + \alpha \|x\|_X) \leq R$, and, for $x_1, x_2 \in B(0; R)$, $\|T(x_1) - T(x_2)\|_X \leq \|L^{-1}\|_{\mathcal{L}(Y', X)}\alpha \|x_1 - x_2\|$. The proof is completed by an application of *Banach's fixed point theorem*. □

5.1. No-slip boundary conditions

For a domain $\Omega \subset \mathbb{R}^n$, a vector field $\tilde{\mathbf{f}}$ on $(0,T) \times \Omega$, and a function g on $(0,T) \times \Omega$, we consider the instationary Navier–Stokes problem to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \tilde{\mathbf{f}} & \text{on } (0,T) \times \Omega, \\ \qquad\qquad\qquad\quad \operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } (0,T) \times \Omega, \\ \qquad\qquad\qquad\qquad\quad\ \mathbf{u} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \qquad\qquad\qquad\ \mathbf{u}(0,\cdot) = 0 & \text{on } \Omega. \end{cases} \tag{5.1}$$

It gives rise to a variational problem of the form (2.1) with an extra trilinear term $n(\cdot,\cdot,\cdot)$, that reads as finding $\mathbf{u} \in \mathbf{U}$, $p \in P$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + c(\mathbf{u}, q) = \mathbf{f}(\mathbf{v}) + g(q) - n(\mathbf{u}, \mathbf{u}, \mathbf{v}) \quad (\mathbf{v} \in \mathbf{V}, q \in Q), \tag{5.2}$$

where the Hilbert spaces $\mathbf{U}$, P, $\mathbf{V}$, Q, right-hand side functionals f and g, and bilinear forms a, b, c are as in Theorem 4.6 or (4.15), and

$$n(\mathbf{y}, \mathbf{z}, \mathbf{v}) := \int_0^T \int_\Omega \mathbf{y} \cdot \nabla_{\mathbf{x}} \mathbf{z} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \mathrm{d}t. \tag{5.3}$$

Theorem 5.2. *For $n = 2,3$, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain that satisfies Assumption 4.2. Then for sufficiently small $\mathbf{f} \in \mathbf{V}'$ and $g \in Q'$, (5.2) has a unique solution $(\mathbf{u}, p)$ in some ball in $\mathbf{U} \times P$ around the origin.*

Proof. By Theorem 4.6 and Lemma 5.1, it suffices to show that with $N(\mathbf{u})(\mathbf{v}) := n(\mathbf{u}, \mathbf{u}, \mathbf{v})$, it holds that $N : \mathbf{U} \to \mathbf{V}'$ with

$$\|N(\mathbf{u}) - N(\mathbf{w})\|_{\mathbf{V}'} \leq \zeta(\|\mathbf{u}\|_{\mathbf{U}}, \|\mathbf{w}\|_{\mathbf{U}}) \|\mathbf{u} - \mathbf{w}\|_{\mathbf{U}}$$

for some $\zeta : [0,\infty)^2 \to [0,\infty)$ with $\zeta(\alpha) \to 0$ if $\alpha \to 0$.

Recall that $\mathbf{U} = L_2(0,T;(H_0^1(\Omega) \cap H^2(\Omega))^n) \cap H_{0,\{0\}}^1(0,T;L_2(\Omega)^n)$ and $\mathbf{V} = L_2(0,T;L_2(\Omega)^n)$. Using twice a Hölder inequality, twice that $H^1(\Omega) \hookrightarrow L_6(\Omega)$ when $n \leq 3$, see *e.g.* ([23], Chap. II, Sect. 1.1) here the existence of an extension in $\mathcal{L}(H^1(\Omega), H^1(\mathbb{R}^n))$ is used, which holds true because Ω is a Lipschitz domain, and also twice that $\mathbf{U} \hookrightarrow C([0,T];H_0^1(\Omega)^n)$ ([6], Chap. XVIII, Sect. 1.3) being a consequence of $[L_2(\Omega), H_0^1(\Omega) \cap H^2(\Omega)]_{1/2} = H_0^1(\Omega)$ ([13], pp. 43, 64) for $\mathbf{y}, \mathbf{z} \in \mathbf{U}$ we find

$$\left(\sup_{0 \neq \mathbf{v} \in \mathbf{V}} \frac{|n(\mathbf{y}, \mathbf{z}, \mathbf{v})|}{\|\mathbf{v}\|_{\mathbf{V}}} \right)^2 = \int_0^T \|\mathbf{y}(t,\cdot) \cdot \nabla_{\mathbf{x}} \mathbf{z}(t,\cdot)\|_{L_2(\Omega)^n}^2 \, \mathrm{d}t$$

$$= \int_0^T \sum_{i=1}^n \int_\Omega |\mathbf{y} \cdot \nabla_{\mathbf{x}} z_i|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq \sum_{i=1}^n \int_0^T \int_\Omega |\mathbf{y}|^2 |\nabla_{\mathbf{x}} z_i|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

$$\leq \sum_{i=1}^n \int_0^T \|\mathbf{y}(t,\cdot)\|_{L_6(\Omega)^n}^2 \|\nabla z_i(t,\cdot)\|_{L_3(\Omega)^n}^2 \, \mathrm{d}t$$

$$\leq \sup_{t \in (0,T)} \|\mathbf{y}(t,\cdot)\|_{L_6(\Omega)^n}^2 \sum_{i=1}^n \int_0^T \left(\int_\Omega |\nabla_{\mathbf{x}} z_i|^{\frac{3}{2}} |\nabla_{\mathbf{x}} z_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} \, \mathrm{d}t$$

$$\leq \sup_{t \in (0,T)} \|\mathbf{y}(t,\cdot)\|_{L_6(\Omega)^n}^2 \sum_{i=1}^n \int_0^T \|\nabla_{\mathbf{x}} z_i(t,\cdot)\|_{L_2(\Omega)^n} \|\nabla_{\mathbf{x}} z_i(t,\cdot)\|_{L_6(\Omega)^n} \mathrm{d}t$$

$$\lesssim \sup_{t \in (0,T)} \|\mathbf{y}(t,\cdot)\|_{H^1(\Omega)^n}^2 \sup_{t \in (0,T)} \|\mathbf{z}(t,\cdot)\|_{H^1(\Omega)^n} \sqrt{T} \|\mathbf{z}\|_{L_2(0,T;H^2(\Omega)^n)} \lesssim \|\mathbf{y}\|_{\mathbf{U}}^2 \|\mathbf{z}\|_{\mathbf{U}}^2.$$

As a first consequence, we have $\|N(\mathbf{u})\|_{\mathbf{V}'}^2 \lesssim \|\mathbf{u}\|_{\mathbf{U}}^4$, and so in particular, $N : \mathbf{U} \to \mathbf{V}'$.

Secondly, from $n(\mathbf{u}, \mathbf{u}, \cdot) - n(\mathbf{w}, \mathbf{w}, \cdot) = n(\mathbf{u} - \mathbf{w}, \mathbf{u}, \cdot) + n(\mathbf{w}, \mathbf{u} - \mathbf{w}, \cdot)$, we find

$$\|N(\mathbf{u}) - N(\mathbf{w})\|^2_{\mathbf{V}'} \lesssim (\|\mathbf{u}\|^2_{\mathbf{U}} + \|\mathbf{w}\|^2_{\mathbf{U}}) \|\mathbf{u} - \mathbf{w}\|^2_{\mathbf{U}},$$

which completes the proof. $\qquad\square$

Besides existence and local uniqueness for sufficiently small data, we also have local uniqueness of any solution:

Theorem 5.3. *For $n = 2, 3$, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain that satisfies Assumption 4.2. Let $(\mathbf{u}, p)$ be a solution of (5.2), then for sufficiently small $\boldsymbol{\delta}\mathbf{f} \in \mathbf{V}'$, $\delta g \in Q'$, (5.2) with $(\mathbf{f}, g)$ reading as $(\mathbf{f} + \boldsymbol{\delta}\mathbf{f}, g + \delta g)$ has a unique solution in some ball in $\mathbf{U} \times P$ around $(\mathbf{u}, p)$.*

Proof. Writing the solution with perturbed data as $(\mathbf{u} + \boldsymbol{\delta}\mathbf{u}, p + \delta p)$, we find that $(\boldsymbol{\delta}\mathbf{u}, \delta p) \in \mathbf{U} \times V$ solves

$$a_{\mathbf{u}}(\boldsymbol{\delta}\mathbf{u}, \mathbf{v}) + b(\delta p, \mathbf{v}) + c(\boldsymbol{\delta}\mathbf{u}, q) = \boldsymbol{\delta}\mathbf{f}(\mathbf{v}) + \delta g(q) - n(\boldsymbol{\delta}\mathbf{u}, \boldsymbol{\delta}\mathbf{u}, \mathbf{v}) \quad (\mathbf{v} \in \mathbf{V}, \, q \in Q),$$

where

$$a_{\mathbf{u}}(\boldsymbol{\delta}\mathbf{u}, \mathbf{v}) := a(\boldsymbol{\delta}\mathbf{u}, \mathbf{v}) + n(\mathbf{u}, \boldsymbol{\delta}\mathbf{u}, \mathbf{v}) + n(\boldsymbol{\delta}\mathbf{u}, \mathbf{u}, \mathbf{v}).$$

The bilinear form $a_{\mathbf{u}}$ corresponds to the partial differential operator $\mathbf{w} \mapsto -\nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{w} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{w} + \mathbf{w} \cdot \nabla_{\mathbf{x}} \mathbf{u}$. Since the perturbations are of lower order, any result that we have proven for the Stokes equations is also valid for the modified Stokes equations with $-\nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{w}$ reading as $-\nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{w} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{w} + \mathbf{w} \cdot \nabla_{\mathbf{x}} \mathbf{u}$ (not uniformly in $\mathbf{u}$ though). We conclude that the statement is proven similarly to Theorem 5.2. $\qquad\square$

Remark 5.4. The point of the above proof is that with

$$\mathbf{B} : \mathbf{U} \times P \to \mathbf{V}' \times Q' : (\mathbf{u}, p) \mapsto \big((\mathbf{v}, q) \mapsto a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + c(\mathbf{u}, q) + n(\mathbf{u}, \mathbf{u}, \mathbf{v})\big),$$

the Fréchet derivative

$$D\mathbf{B}(\mathbf{u}, p) : (\boldsymbol{\delta}\mathbf{u}, \delta p) \mapsto \big((\mathbf{v}, q) \mapsto a_{\mathbf{u}}(\boldsymbol{\delta}\mathbf{u}, \mathbf{v}) + b(\delta p, \mathbf{v}) + c(\boldsymbol{\delta}\mathbf{u}, q)\big) \in \mathcal{L}(\mathbf{U} \times P, \mathbf{V}' \times Q')$$

is boundedly invertible, which is a crucial property for any method for *solving* the Navier−Stokes equations.

5.2. Free-slip boundary conditions

For a domain $\Omega \subset \mathbb{R}^n$, a vector field $\widetilde{\mathbf{f}}$ on $(0, T) \times \Omega$, we consider the instationary Navier–Stokes problem to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \widetilde{\mathbf{f}} & \text{on } (0, T) \times \Omega, \\[2mm] \operatorname{div}_{\mathbf{x}} \mathbf{u} = 0 & \text{on } (0, T) \times \Omega, \\[2mm] \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial\Omega, \\[2mm] \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\tau}_i = g_i & \text{on } (0, T) \times \partial\Omega, \, 1 \le i \le n - 1, \\[2mm] \mathbf{u}(0, \cdot) = \mathbf{u}_0 & \text{on } \Omega, \end{cases} \tag{5.4}$$

where $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ is an orthonormal set of tangent vectors.

It gives rise to a variational problem of the form (2.1) with an extra nonlinear term, that reads as finding $\mathbf{u} \in \mathbf{U}$, $p \in P$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + c(\mathbf{u}, q) = \mathbf{f}(\mathbf{v}) + n(\mathbf{u}, \mathbf{v}, \mathbf{u}) \quad (\mathbf{v} \in V, \, q \in Q), \tag{5.5}$$

where the spaces $\mathbf{U}$, P, $\mathbf{V}$, Q, right-hand side functional f, and bilinear forms a, b, c are as in Theorem 3.7 or (3.11), and the form n is as in (5.3).

We arrived at this variational formulation with $n(\mathbf{u}, \mathbf{v}, \mathbf{u})$, instead of the expected term $-n(\mathbf{u}, \mathbf{u}, \mathbf{v})$, by using that for smooth vector fields $\mathbf{u}$ on Ω that have vanishing normals at $\partial\Omega$ and that are divergence-free, and for smooth vector fields $\mathbf{v}$ on Ω,

$$n(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \sum_{i,j=1}^{2} \int_{\Omega} u_i(\partial_i v_j)v_j \, \mathrm{d}\mathbf{x} = \tfrac{1}{2} \sum_{i,j=1}^{2} \int_{\Omega} u_i \partial_i v_j^2 \, \mathrm{d}\mathbf{x} = \tfrac{1}{2} \left[\sum_j \int_{\Omega} -\mathrm{div}\, \mathbf{u}\, v_j^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} v_j^2\, \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s \right] = 0$$

Expanding $n(\mathbf{u}, \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w})$ for smooth vector fields $\mathbf{v}, \mathbf{w}$ on Ω, we arrive at $n(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -n(\mathbf{u}, \mathbf{w}, \mathbf{v})$. Note that it is essential that in (5.4) we have imposed $\mathbf{u} \cdot \mathbf{n} = 0$ on $(0, T) \times \partial\Omega$, instead of $\mathbf{u} \cdot \mathbf{n} = g$ on $(0, T) \times \partial\Omega$ for some general function.

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain that satisfies Assumption 3.1. Then for sufficiently small $\mathbf{f} \in \mathbf{V}'$, (5.5) has a unique solution $(\mathbf{u}, p)$ in some ball in $\mathbf{U} \times P$ around the origin.*

Proof. As shown in ([23], Chap. III, Sect. 3, Lem. 3.3), for $v \in H^1(\mathbb{R}^2)$ it holds that

$$\|v\|_{L_4(\mathbb{R}^2)} \leq 2^{1/4} \|v\|_{L_2(\mathbb{R}^2)}^{\frac{1}{2}} |v|_{H^1(\mathbb{R}^2)}^{\frac{1}{2}}.$$

Since $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, there exists an operator E that extends functions on Ω to functions on $\mathbb{R}^2$ with $E \in \mathcal{L}(L_2(\Omega), L_2(\mathbb{R}^2))$, $E \in \mathcal{L}(H^1(\Omega), H^1(\mathbb{R}^2))$ ([19], Chap. VI, Sect. 3, Thm. 5). We conclude that for $v \in H^1(\Omega)$,

$$\|v\|_{L_4(\Omega)} \leq \|Ev\|_{L_4(\mathbb{R}^2)} \leq 2^{1/4} \|Ev\|_{L_2(\mathbb{R}^2)}^{\frac{1}{2}} |Ev|_{H^1(\mathbb{R}^2)}^{\frac{1}{2}} \lesssim \|v\|_{L_2(\Omega)}^{\frac{1}{2}} \|v\|_{H^1(\Omega)}^{\frac{1}{2}}.$$

Using this result, and by a few applications of Cauchy−Schwarz inequality, for $\mathbf{y}, \mathbf{v}, \mathbf{z} \in H^1(\Omega)$ we have

$$\left| \int_{\Omega} \mathbf{y} \cdot \nabla_{\mathbf{x}} \mathbf{v} \cdot \mathbf{z} \, \mathrm{d}\mathbf{x} \right| = \left| \int_{\Omega} \sum_{i,j=1}^{2} y_i(\partial_i v_j)z_j \, \mathrm{d}\mathbf{x} \right| \leq \sum_{i,j} \|\partial_i v_j\|_{L_2(\Omega)} \|y_i\|_{L_4(\Omega)} \|z_j\|_{L_4(\Omega)}$$

$$\leq \sqrt{\sum_{i,j} \|\partial_i v_j\|_{L_2(\Omega)}^2} \sqrt{\sum_i \|y_i\|_{L_4(\Omega)}^2 \sum_j \|z_j\|_{L_4(\Omega)}^2}$$

$$\lesssim \|\mathbf{v}\|_{H^1(\Omega)^2} \|\mathbf{y}\|_{L_2(\Omega)^2}^{\frac{1}{2}} \|\mathbf{y}\|_{H^1(\Omega)^2}^{\frac{1}{2}} \|\mathbf{z}\|_{L_2(\Omega)^2}^{\frac{1}{2}} \|\mathbf{z}\|_{H^1(\Omega)^2}^{\frac{1}{2}}.$$

Recalling that $\mathbf{U} = L_2(0, T; \mathbf{H}^1(\Omega)) \cap H^1_{0,\{0\}}(0, T; \mathbf{H}^1(\Omega)')$ and $\mathbf{V} = L_2(0, T; \mathbf{H}^1(\Omega))$, for $\mathbf{y}, \mathbf{z} \in \mathbf{U}$, $\mathbf{v} \in \mathbf{V}$, from $\mathbf{U} \hookrightarrow C([0, T]; L_2(\Omega)^2)$ we obtain

$$|n(\mathbf{y}, \mathbf{v}, \mathbf{z})| \lesssim \int_0^T \|\mathbf{v}(t, \cdot)\|_{H^1(\Omega)^2} \|\mathbf{y}(t, \cdot)\|_{L_2(\Omega)^2}^{\frac{1}{2}} \|\mathbf{y}(t, \cdot)\|_{H^1(\Omega)^2}^{\frac{1}{2}} \|\mathbf{z}(t, \cdot)\|_{L_2(\Omega)^2}^{\frac{1}{2}} \|\mathbf{z}(t, \cdot)\|_{H^1(\Omega)^2}^{\frac{1}{2}} \, \mathrm{d}t$$

$$\leq \sup_{t \in [0,t]} \|\mathbf{y}(t, \cdot)\|_{L_2(\Omega)^2}^{\frac{1}{2}} \sup_{t \in [0,t]} \|\mathbf{z}(t, \cdot)\|_{L_2(\Omega)^2}^{\frac{1}{2}}$$

$$\times \int_0^T \|\mathbf{v}(t, \cdot)\|_{H^1(\Omega)^2} \|\mathbf{y}(t, \cdot)\|_{H^1(\Omega)^2}^{\frac{1}{2}} \|\mathbf{z}(t, \cdot)\|_{H^1(\Omega)^2}^{\frac{1}{2}} \, \mathrm{d}t$$

$$\lesssim \|\mathbf{y}\|_{\mathbf{U}}^{\frac{1}{2}} \|\mathbf{z}\|_{\mathbf{U}}^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{V}} \left(\int_0^T \|\mathbf{y}(t, \cdot)\|_{H^1(\Omega)^2}^2 \, \mathrm{d}t \right)^{\frac{1}{4}} \left(\int_0^T \|\mathbf{z}(t, \cdot)\|_{H^1(\Omega)^2}^2 \, \mathrm{d}t \right)^{\frac{1}{4}}$$

$$\lesssim \|\mathbf{y}\|_{\mathbf{U}} \|\mathbf{z}\|_{\mathbf{U}} \|\mathbf{v}\|_{\mathbf{V}},$$

or $\sup_{0\neq\mathbf{v}\in\mathbf{V}}\frac{|n(\mathbf{y},\mathbf{v},\mathbf{z})|}{\|\mathbf{v}\|_{\mathbf{V}}} \lesssim \|\mathbf{y}\|_{\mathbf{U}}\|\mathbf{z}\|_{\mathbf{U}}$. Similar to the proof of Theorem 5.2, the latter result together with Theorem 3.7 and Lemma 5.1 completes the proof. □

Remark 5.6. Compared to Section 5.1, the smoothness indices of the spatial Sobolev spaces incorporated in $\mathbf{U}$ or $\mathbf{V}$ has been lowered or raised by one, respectively. Because of the double or single occurrence of $\mathbf{u}\in\mathbf{U}$ or $\mathbf{v}\in\mathbf{V}$ as arguments in the trilinear form n, the task of bounding $n(\mathbf{u},\mathbf{u},\mathbf{v})$ is more difficult here, giving an explanation why the arguments in the present subsection only apply to space dimension $n=2$.

Similar to the no-slip case, besides existence and local uniqueness for sufficiently small data, we also have local uniqueness of any solution:

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain that satisfies Assumption 3.1. Let $(\mathbf{u}, p)$ be a solution of (5.5), then for sufficiently small $\boldsymbol{\delta f} \in \mathbf{V}'$, (5.2) with $\mathbf{f}$ reading as $\mathbf{f} + \boldsymbol{\delta f}$ has a unique solution in some ball in $\mathbf{U} \times P$ around $(\mathbf{u}, p)$.*

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