ADAPTIVE PIECEWISE TENSOR PRODUCT WAVELETS SCHEME FOR LAPLACE-INTERFACE PROBLEMS

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ABSTRACT. A Laplace type boundary value problem is considered with a generally discontinuous diffusion coefficient. A domain decomposition technique is used to construct a piecewise tensor product wavelet basis that, when normalised w.r.t. the energy-norm, has Riesz constants that are bounded uniformly in the jumps. An adaptive wavelet Galerkin method is applied to solve the boundary value problem with the best nonlinear approximation rate from the basis, in linear computational complexity. Although the solutions are far from smooth, numerical experiments in two dimensions show rates as for a one-dimensional smooth solution, the latter being possible because of the tensor product construction.

1. INTRODUCTION

In this paper, we study second order linear elliptic problems, generally with a discontinuous diffusion coefficient, that are known as *Laplace-interface problems* or *transmission problems*. For some domain $\Omega \subset \mathbb{R}^n$, $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$, and given $f \in H^1_{0,\Gamma}(\Omega)'$, we consider the problem of finding $u \in H^1_{0,\Gamma}(\Omega)$ such that

$$a_\kappa(u,v) := \int_\Omega \kappa \nabla u \cdot \nabla v = f(v) \quad \forall v \in H^1_{0,\Gamma}(\Omega). \tag{1.1}$$

For some *fixed* N, and $0 \le i \le N$, let $\Omega_i \subset \Omega$ be mutually disjoint hypercubes such that $\bar{\Omega} = \cup_{i=0}^N \bar{\Omega}_i$. We assume that

$$\kappa|_{\Omega_i} = \kappa_i, \quad i = 0, \cdots, N, \tag{1.2}$$

where each κ_i is a positive constant, and that Γ is the closure of the union of facets of one or more Ω_i. The coefficient in problem (1.1) may have large jumps across interfaces between the hypercubes. Consequently, the solution can be expected to be non-smooth at these interfaces, in particular in directions normal to them.

Because of the non-smoothness of the solution, we solve the problem numerically with an adaptive method, where we take the *Adaptive Wavelet-Galerkin Method* (**AWGM**) ([CDD01, GHS07, Ste09]). To do so, we equip $H^1_0(\Omega)$ with a Riesz basis $\boldsymbol{\Psi} = \{\boldsymbol{\psi}_\lambda : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}$ that has Riesz constants that are bounded *uniformly* in $\kappa = (\kappa_i)_i > 0$ w.r.t. the energy-norm. This means that for some constants $0 < C_1 \le C_2 < \infty$, independent of κ, it holds that

$$C_1 \le \frac{a_\kappa(\boldsymbol{c}^\top \boldsymbol{\Psi}, \boldsymbol{c}^\top \boldsymbol{\Psi})}{\sum_{\lambda \in \boldsymbol{\nabla}} c_\lambda^2 a_k(\boldsymbol{\psi}_\lambda, \boldsymbol{\psi}_\lambda)} \le C_2, \ (\boldsymbol{c} \in \ell_2(\boldsymbol{\nabla}), \ \kappa = (\kappa_i)_i > 0).$$

Date: January 8, 2017.
2000 *Mathematics Subject Classification.* 15A69, 35R05 41A25, 41A63, 42C40, 65N12, 65T60.
Key words and phrases. Interface problem, adaptive wavelet scheme, tensor product approximation, domain decomposition, extension operators, best approximation rates.
The first author was partly supported by IGP, grant no. 95650063.

The best possible constants C_1, C_2, i.e., the largest C_1 and the smallest C_2, are called Riesz constants. We will be able to construct such a 'uniform' Riesz basis under additional assumptions on the signs of the jumps of the κ_i that will be specified later, but *not* on their moduli.

Writing $u = \mathbf{u}^\top \boldsymbol{\Psi} := \sum_{\lambda \in \boldsymbol{\nabla}} \mathbf{u}_\lambda \boldsymbol{\psi}_\lambda$, problem (1.1) can be equivalently formulated as the bi-infinite matrix-vector problem

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \tag{1.3}$$

where $\mathbf{f} := [f(\boldsymbol{\psi}_\lambda)]_{\lambda \in \boldsymbol{\nabla}}$ and $\mathbf{A} := [a_\kappa(\boldsymbol{\psi}_\mu, \boldsymbol{\psi}_\lambda)]_{\lambda,\mu \in \boldsymbol{\nabla}}$ is the stiffness matrix of $a_\kappa(\cdot,\cdot)$ w.r.t. $\boldsymbol{\Psi}$. We note that $\mathbf{A}$ is bounded, symmetric, and positive definite with, after preconditioning by its diagonal, a spectral condition number that is bounded by $\frac{C_2}{C_1}$. The **AWGM** applied to (1.3) will produce a sequence of approximations that converge with the best possible rate from the basis.

For making the aforementioned 'uniform' Riesz basis, as a first step, we equip $H^1_{0,\Gamma}(\Omega)$ with norm $|||\cdot||| := \sqrt{\int_\Omega \kappa |\nabla \cdot|^2 + \mu |\cdot|^2}$, with $\mu|_{\Omega_i}$ being positive constants depending on κ such that this norm is equivalent to the energy-norm, uniformly in the κ_i's that are allowed. Thanks to $\mu|_{\Omega_i} > 0$, a norm on $H^1(\Omega_i)$ is defined by $|||\cdot|||_i := \sqrt{\int_{\Omega_i} \kappa |\nabla \cdot|^2 + \mu |\cdot|^2}$.

Next, we construct a 'uniform' isomorphism between the Cartesian product of H^1-Sobolev spaces on the subdomains –subject to homogeneous boundary conditions on selected faces, and equipped with the norms $|||\cdot|||_i$–, and the space $H^1_{0,\Gamma}(\Omega)$ equipped with the energy-norm. This isomorphism will be built from H^1-bounded extensions of functions on a subdomain to functions on a neighbouring subdomain, extending the approach from [DS99a, KS06, CDFS13] to interface problems. No boundary conditions are prescribed on the outward face, and homogeneous boundary conditions on the inward face. Moreover, the extensions should be directed to go from larger to smaller diffusion coefficients. Because the subdomains are part of a Cartesian grid of hypercubes in $\mathbb{R}^n$, the extension operators can be chosen as univariate extensions in the direction normal to the interfaces. Simple reflections will be applied in our experiments.

What is left to construct are 'uniform' Riesz bases for the H^1-Sobolev spaces on the subdomains equipped with the norms $|||\cdot|||_i$. These bases will be constructed as n-fold *tensor products* of univariate wavelet bases. A crucial advantage of such bases is that dimension-independent convergence rates can be expected. The approach of applying tensor product approximation, known under the names *hyperbolic cross approximation* or *sparse grids*, is here thus extended in the sense that is combined with domain decomposition and adaptivity. The combination of tensor-product bases with univariate extensions allows for a convenient implementation. It will turn out that only wavelets that do not vanish at the interface have to be extended.

The univariate wavelet bases will be constructed to give rise to give truly sparse mass and stiffness matrices. We constructed such bases in [CS11] for homogeneous boundary conditions. Here they will be adapted to other boundary conditions. As a consequence, the stiffness matrix $\mathbf{A}$ corresponding to (1.1) and the piecewise tensor product wavelet Riesz basis will be close to being sparse. Only columns corresponding to wavelets that were extended over an interface contain more than a uniformly bounded number of non-zeros, and therefore have to be approximated in the evaluation of a matrix-vector product. Numerical results that will be obtained with the **AWGM** show the best possible 'univariate' converge rate uniformly in κ in linear computational complexity, largely improving upon non-adaptive methods.

This paper is organised as follows: In Sect. 2, we prepare for the construction of a Riesz basis from Riesz bases on the subdomains by extension by focussing on the two subdomain case.

In Sect. 3, dealing with the multi-subdomain case, we collect conditions on the ordering of the extensions to end up with a Riesz basis that has Riesz constants bounded uniformly in the sizes of the jumps. We show how to construct suitable extensions from univariate extensions, collect conditions on the univariate wavelets, and show that the extensions only have to be applied to wavelets that do not vanish at the interface.

Wavelets that satisfy all conditions, and that give rise to an almost sparse stiffness matrix are constructed in Sect. 4.

Finally, in Sect. 5, numerical results are presented that are obtained with the **AWGM** applied to various Laplace-interface problems.

2. CONSTRUCTION OF ISOMORPHISMS FOR A DISCONTINUOUS DIFFUSION COEFFICIENT

Let $\Omega \subset \mathbb{R}^n$ be a domain. For some *fixed* N, and all $0 \leq i \leq N$, let $\Omega_i \subset \Omega$ be a subdomain such that $\Omega_i \cap \Omega_j = \emptyset$ $(i \neq j)$ and $\bar{\Omega} = \cup_{i=0}^{N} \bar{\Omega}_i$. For a constant $\kappa_i = \kappa|_{\Omega_i} > 0$, we equip $H^1(\Omega_i)$ with squared semi-norm

$$|u|_{E,i}^2 := \kappa_i |u|_{H^1(\Omega_i)}^2,$$

and $H^1(\Omega)$ with squared semi-norm

$$|u|_E^2 := \sum_{i=0}^{N} |u|_{\Omega_i}|_{E,i}^2 \quad (= a_\kappa(u,u)).$$

Note that by Friedrich's inequality, $|\cdot|_E$ is actually a norm on $H^1_{0,\Gamma}(\Omega)$, which is the *energy-norm* corresponding to our boundary-value problem.

Although $|\cdot|_E$ is a norm on $H^1_{0,\Gamma}(\Omega)$, for $0 \neq u \in H^1_{0,\Gamma}(\Omega)$ and Ω_i such that $\partial\Omega_i \cap \partial\Omega = \emptyset$, it holds that possibly $|u|_{\Omega_i}|_{E,i} = 0$ even if $u|_{\Omega_i} \neq 0$. This causes problems in our analysis, and for that reason for some constant $\mu_i > 0$, we equip $H^1(\Omega_i)$ with squared norm

$$|||u|||_i^2 := \mu_i \|u\|_{L_2(\Omega_i)}^2 + |u|_{E,i}^2 \tag{2.1}$$

and $H^1(\Omega)$ with squared norm

$$|||u|||^2 := \sum_{i=0}^{N} |||u|_{\Omega_i}|||_i^2.$$

Norms on operators between spaces equipped with norms $|||\cdot|||$ or $|||\cdot|||_i$, or Cartesian products of such spaces equipped with $\sqrt{\sum_{i=0}^{N} |||\cdot|||_i^2}$, will simply be denoted as $|||\cdot|||$.

A sufficient condition for $|\!|\!|\cdot|\!|\!|$ being on norm on $H^1_{0,\Gamma}(\Omega)$ that is uniformly equivalent to the energy-norm is derived next:

Proposition 2.1. *Let $\vec{\kappa} = (\kappa_0, \cdots, \kappa_N)$, $\vec{\mu} = (\mu_0, \cdots, \mu_N)$ be such that for any $0 \leq i \leq N$, there exist $\{j_0, \cdots, j_M\} \subset \{0, \cdots, N\}$ with $j_0 = i$, $|\partial\Omega_{j_\ell} \cap \partial\Omega_{j_{\ell+1}}| > 0$ ($\ell \in \{0, \cdots, M-1\}$), $|\partial\Omega_{j_M} \cap \Gamma| > 0$ and $\mu_i \lesssim \min_{0\leq\ell\leq M} \kappa_{j_\ell}$. Then*

$$|\!|\!|\cdot|\!|\!| \eqsim |\cdot|_E \text{ on } H^1_{0,\Gamma}(\Omega).$$

Proof. Friedrich's inequality

$$\|\cdot\|_{L_2((\cup_{\ell=0}^M \overline{\Omega}_{j_\ell})^{\text{int}})} \lesssim |\cdot|_{H^1((\cup_{\ell=0}^M \overline{\Omega}_{j_\ell})^{\text{int}})},$$

on $H^1_{0,\partial\Omega_{j_M}\cap\Gamma}((\cup_{\ell=0}^M \overline{\Omega}_{j_\ell})^{\text{int}})$ shows that for $u \in H^1_{0,\Gamma}(\Omega)$, $\mu_i\|u\|^2_{L_2(\Omega_i)} \lesssim \sum_{\ell=0}^M |u|_{\Omega_{j_\ell}}|^2_{E,j_\ell}$. $\square$

Remark 2.2. Following [PS13, Theorem 2.7], likely the conditions of this proposition can be relaxed.

Remark 2.3. Note that the conditions of Proposition 2.1 require that $\mu_i \lesssim \kappa_i$.

Remark 2.4. The obvious choice $\mu_i = \kappa_i$ is not always appropriate. Indeed, as an example consider $u \in H^1_0(-1, 2)$ defined by

$$u(x) = \begin{cases} 1 + x & x \in \Omega_0 := (-1, 0), \\ 1 & x \in \Omega_1 := (0, 1), \\ 2 - x & x \in \Omega_2 := (1, 2), \end{cases} \tag{2.2}$$

and $\kappa_0 = 1, \kappa_1 \geq 1, \kappa_2 = 1$. Then

$$\lim_{\kappa_1\to\infty} \frac{\sqrt{\sum_{i=0}^2 |u|_{\Omega_i}|^2_{E,i} + \kappa_i\|u|_{\Omega_i}\|^2_{L_2(\Omega_i)}}}{|u|_E} = \infty,$$

whereas for $\mu_0 = \mu_1 = \mu_2 := 1$, it holds that $|\!|\!|\cdot|\!|\!| \eqsim |\cdot|_E$ on $H^1_0(-1, 2)$, uniformly in $\kappa_1 \geq 1$, $\kappa_0 = \kappa_2 = 1$.

Our aim will now be to construct a Riesz-basis for $\left(H^1_{0,\Gamma}(\Omega), |\!|\!|\cdot|\!|\!|\right)$ with Riesz constants that are bounded uniformly in $\vec{\kappa}$, $\vec{\mu}$ from "large" subsets of $(0, \infty)^{N+1}$. In the examples that will presented in the final Sect. 5, we will verify whether the conditions of Proposition 2.1 are satisfied so that $|\!|\!|\cdot|\!|\!| \eqsim |\cdot|_E$.

For our goal, we will construct an isomorphism, uniform in these $\vec{\kappa}$, $\vec{\mu}$, between the Cartesian product $\prod_{i=0}^N \left(H^1_{0,\Gamma_i}(\Omega_i), |\!|\!|\cdot|\!|\!|_i\right)$, for suitable $\Gamma_i \subset \partial\Omega_i$, and $\left(H^1_{0,\Gamma}(\Omega), |\!|\!|\cdot|\!|\!|\right)$. By applying such an isomorphism to the union of uniform Riesz bases for $\left(H^1_{0,\Gamma_i}(\Omega_i), |\!|\!|\cdot|\!|\!|_i\right)$, for $i = 0, \cdots, N$, the result is a Riesz-basis for $\left(H^1_{0,\Gamma}(\Omega), |\!|\!|\cdot|\!|\!|\right)$ with Riesz constants that are bounded uniformly in $\vec{\kappa}$, $\vec{\mu}$.

To construct this isomorphism, first we focus on the case of having *two subdomains* Ω_1 and Ω_2. Since joining of two subdomains will be applied recursively, while handling the two subdomains case we will drop the assumptions that, for $i = 1, 2$, μ_i, κ_i are constants. So $\mu_i, \kappa_i \in L_\infty(\Omega_i)$ with $\mu_i, \kappa_i > 0$ a.e., where in applications μ_i, κ_i will be *piecewise* constant. The norms $|\!|\!|u|\!|\!|_i$ should now be read as $|\!|\!|u|\!|\!|_i = \sqrt{\int_{\Omega_i} \mu_i(x)|u(x)|^2 + \kappa_i(x)|\nabla u(x)|^2 dx}$.

For $i \in \{1,2\}$, let R_i be the restriction of functions on Ω to Ω_i, and let η_2 be the extension by zero of functions on Ω_2 to functions on Ω. Suppose that E_1 is some extension of functions on Ω_1 to functions on Ω, mapping $H^1_{0,\Gamma_1}(\Omega_1)$ into $H^1_{0,\Gamma}(\Omega)$, where for some given, measurable $\Gamma \subset \partial\Omega$,

$$\begin{aligned}\Gamma_1 &:= \partial\Omega_1 \cap \Gamma,\\ \Gamma_2 &:= \partial\Omega_2 \cup \Gamma \cap (\partial\Omega_1 \cap \partial\Omega_2).\end{aligned}$$

See Figure 1, for an illustration. So the trivial zero extension η_2 will be applied to

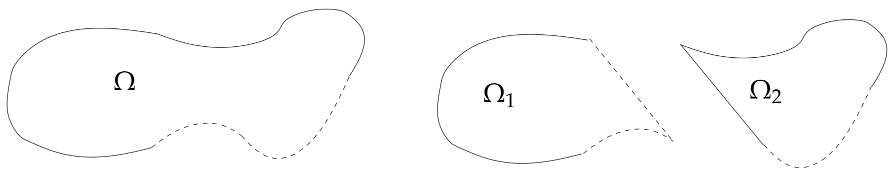

FIGURE 1. Left: $\partial\Omega \setminus \Gamma$ (dashed) and Γ (solid). Middle: $\partial\Omega_1 \setminus \Gamma_1$ (dashed) and Γ_1 (solid). Right: $\partial\Omega_2 \setminus \Gamma_2$ (dashed) and Γ_2 (solid).

functions on Ω_2 that vanish at the interface, whereas E_1 will be applied to functions on Ω_1 that generally do not vanish at this interface.

Proposition 2.5. *Setting*

$$E := [E_1 \quad \eta_2] : (H^1_{0,\Gamma_1}(\Omega_1), |||\cdot|||_1) \times (H^1_{0,\Gamma_2}(\Omega_2), |||\cdot|||_2) \to (H^1_{0,\Gamma}(\Omega), |||\cdot|||),$$

its inverse is given by

$$E^{-1} := \begin{bmatrix} R_1 \\ R_2(\mathrm{Id} - E_1R_1)\end{bmatrix} : (H^1_{0,\Gamma}(\Omega), |||\cdot|||) \to (H^1_{0,\Gamma_1}(\Omega_1), |||\cdot|||_1) \times (H^1_{0,\Gamma_2}(\Omega_2), |||\cdot|||_2).$$

It holds that

$$\begin{aligned}|||E||| &\leq \sqrt{1 + |||E_1|||^2},\\ |||E^{-1}||| &\leq \sqrt{\max(1 + 2|||E_1|||^2, 2)}.\end{aligned}$$

Proof. Let us first confirm the formula for E^{-1}. Using $R_1E_1 = \mathrm{Id}$, $R_1\eta_2 = 0$, $R_2\eta_2 = \mathrm{Id}$, we obtain

$$\begin{bmatrix} R_1 \\ R_2(\mathrm{Id} - E_1R_1)\end{bmatrix} [E_1 \quad \eta_2] = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id}\end{bmatrix},$$

and using that $\mathrm{ran}(\mathrm{Id} - E_1R_1) \subset \mathrm{ran}\,\eta_2$ and $R_2\eta_2 = \mathrm{Id}$, we have

$$[E_1 \quad \eta_2] \begin{bmatrix} R_1 \\ R_2(\mathrm{Id} - E_1R_1)\end{bmatrix} = E_1R_1 + \eta_2R_2(\mathrm{Id} - E_1R_1) = \mathrm{Id}.$$

Now we verify the boundedness of E and E^{-1}. For $u_i \in H_{0,\Gamma_i}(\Omega_i)$ $(i = 1,2)$, it holds that

$$\begin{aligned}|||E(u_1,u_2)||| = |||E_1u_1 + \eta_2u_2||| &\leq |||E_1|||\,|||u_1|||_1 + |||u_2|||_2\\ &\leq \sqrt{|||E_1|||^2 + 1} \cdot \sqrt{|||u_1|||_1^2 + |||u_2|||_2^2},\end{aligned}$$

and for $u \in H^1_{0,\Gamma}(\Omega)$, we have

$$\begin{aligned}
|||E^{-1}u|||^2 = |||R_1 u|||_1^2 + |||R_2(\mathrm{Id} - E_1R_1)u|||_2^2 &\leq |||R_1u|||_1^2 + 2|||R_2u|||_2^2 + 2|||R_2E_1R_1u|||_2^2 \\
&\leq |||R_1u|||_1^2 + 2|||R_2u|||_2^2 + 2|||E_1|||^2|||R_1u|||_2^2 \\
&= (1+2|||E_1|||^2)|||R_1u|||_1^2 + 2|||R_2u|||_2^2 \\
&\leq \max(1+2|||E_1|||^2, 2)|||u|||^2. \qquad \square
\end{aligned}$$

Besides the construction of a suitable E_1, the other question is how to construct a (uniform) Riesz basis for $(H^1_{0,\Gamma_i}(\Omega_i), |||\cdot|||_i)$. We will answer the latter question in the two possible cases $\mathrm{meas}(\Gamma_i) > 0$ and $\mathrm{meas}(\Gamma_i) = 0$. For $\mathrm{meas}(\Gamma_i) > 0$ and $0 < \mu_i \lesssim \kappa_i$, Friedrich's inequality shows that $|||\cdot|||_i \eqsim \sqrt{\kappa_i}\|\cdot\|_{H^1(\Omega_i)}$ on $H^1_{0,\Gamma_i}(\Omega_i)$. So a simple normalization of a Riesz basis for $H^1_{0,\Gamma_i}(\Omega_i)$ will do. In the following proposition we will discuss this issue for $\Gamma_i = \emptyset$, so that $H^1_{0,\Gamma_i}(\Omega_i) = H^1(\Omega_i)$.

Proposition 2.6. *Let $\Sigma \cup \{\mathbb{1}\}$ be a Riesz basis for $H^1(\Omega_i)$ (with standard norm) such that $\Sigma \subset H^1(\Omega_i)/\mathbb{R}$. Then, normalized, $\Sigma \cup \{\mathbb{1}\}$ is a Riesz basis for $(H^1(\Omega_i), |||\cdot|||_i)$ with Riesz constants that are bounded uniformly in $0 < \mu_i \lesssim \kappa_i$.*

Proof. Writing $\Sigma = \{\sigma_\lambda\}$, from $\langle \nabla\sigma_\lambda, \nabla\mathbb{1}\rangle_{L_2(\Omega_i)} = 0 = \langle \sigma_\lambda, \mathbb{1}\rangle_{L_2(\Omega_i)}$, for $\mathbf{c} \in \ell_2$ and $d \in \mathbb{R}$, we have

$$\begin{aligned}
|||\kappa_i^{-\frac{1}{2}}\mathbf{c}^\top\Sigma + \mu_i^{-\frac{1}{2}}d\mathbb{1}|||_i^2 &= |||\kappa_i^{-\frac{1}{2}}\mathbf{c}^\top\Sigma|||_i^2 + |||\mu_i^{-\frac{1}{2}}d\mathbb{1}|||_i^2 \\
&= |\mathbf{c}^\top\Sigma|^2_{H^1(\Omega_i)} + \frac{\mu_i}{\kappa_i}\|\mathbf{c}^\top\Sigma\|^2_{L_2(\Omega_i)} + \|d\mathbb{1}\|^2_{L_2(\Omega_i)} \\
&\eqsim \|\mathbf{c}^\top\Sigma\|^2_{H^1(\Omega_i)} + \|d\mathbb{1}\|^2_{H^1(\Omega_i)} \\
&\eqsim \|\mathbf{c}\|^2_{\ell_2} + |d|^2,
\end{aligned}$$

with the one but last $\eqsim$ being valid because of Poincaré's inequality, $\Sigma \subset H^1(\Omega_i)/\mathbb{R}$, and $\mu_i \lesssim \kappa_i$. In other words, $\kappa_i^{-\frac{1}{2}}\Sigma \cup \{\mu_i^{-\frac{1}{2}}\mathbb{1}\}$ is a Riesz basis for $(H^1(\Omega_i), |||\cdot|||_i)$ with Riesz constants that are bounded uniformly in $0 < \mu_i \lesssim \kappa_i$. $\square$

Finally in this section, we briefly discuss dual bases. In the situation of Proposition 2.5, let Ψ_i be a basis for $(H^1_{0,\Gamma_i}(\Omega_i), |||\cdot|||_i)$ so that $E_1\Psi_1 \cup \eta_2\Psi_2$ is a Riesz basis for $(H^1_{0,\Gamma}(\Omega), |||\cdot|||)$. Let $\tilde{\Psi}_1 \subset L_2(\Omega_1)$, $\tilde{\Psi}_2 \subset L_2(\Omega_2)$ be collections that are dual to Ψ_1 and Ψ_2, respectively. From $R_i = \eta_i^*$ and $R_i\eta_j = \delta_{ij}$ $(i, j \in \{1, 2\})$, we have

$$E^* = \begin{bmatrix} E_1^* \\ R_2 \end{bmatrix}, \quad E^{-*} = [\eta_1 \quad (\mathrm{Id} - \eta_1E_1^*)\eta_2].$$

We conclude that the collection in $L_2(\Omega)$ that is dual to $E_1\Psi_1 \cup \eta_2\Psi_2$ is given by

$$(\mathrm{Id} - \eta_1E_1^*)\eta_2\tilde{\Psi}_2 \cup \eta_1\tilde{\Psi}_1. \tag{2.3}$$

Since $(\mathrm{Id} - \eta_1E_1^*)\eta_2$ plays the role of the extension E_1 at the dual side, we will call it the *adjoint* extension.

3. Construction of a piecewise tensor product wavelet Riesz basis

3.1. **Construction of Riesz bases by extension.** In this subsection, we are going to apply the approach outlined in Propositions 2.5 iteratively, where the subdomains will be (unions of) hypercubes. Let $\mathcal{I} := (0,1)$, $\square := \mathcal{I}^n$. We assume that for a fixed, finite set of hypercubes $\{\square_0, \ldots, \square_N\}$ from $\{\tau + \square : \tau \in \mathbb{Z}^n\}$, it holds that $\cup_{i=0}^N \square_i \subset \Omega \subset (\cup_{i=0}^N \overline{\square}_i)^{\text{int}}$, and that $\partial\Omega$ is the union of (closed) facets of the $\square_i$'s. The case $\Omega \subsetneq (\cup_{i=0}^N \overline{\square}_i)^{\text{int}}$ corresponds to the situation that Ω has one or more cracks.

Recall the definition (2.1) of the norm $|\!|\!|\cdot|\!|\!|_i$ on $H^1(\Omega_i)$, i.e., on $H^1(\square_i)$. In view of Proposition 2.1, see also Remark 2.3, we will always assume that the μ_i's are chosen such that

$$0 < \mu_i \lesssim \kappa_i. \tag{3.1}$$

By applying extension operators, from Riesz bases for the corresponding Sobolev spaces on the subdomains $\square_i$, we will construct a Riesz basis for $\big(H^1_{0,\Gamma}(\Omega), |\!|\!|\cdot|\!|\!|\big)$, that, after normalization, has Riesz constants which, under some conditions, are bounded uniformly in $\{0 < \mu_i \lesssim \kappa_i : 0 \le i \le N\}$.

We set $\Omega_i^{(0)} := \square_i$, for $i = 0, \cdots, N$. Starting from the initial subdivision of Ω into hypercubes, we will create a sequence $(\{\Omega_i^{(q)} : q \le i \le N\})_{0 \le q \le N}$ of sets of polytopes, where each next entry in this sequence is created by joining two polytopes from the previous entry whose joint interface is part of a hyperplane. So we assume that for any $1 \le q \le N$, there exists $q-1 \le i_1 \ne i_2 \le N$ and $q \le \bar{i} \le N$ that satisfy

($\mathcal{D}_1$) $\Omega_{\bar{i}}^{(q)} := \Big(\overline{\Omega_{i_1}^{(q-1)} \cup \Omega_{i_2}^{(q-1)}} \setminus \partial\Omega\Big)^{\text{int}}$ is connected, and the interface $J := \Omega_{\bar{i}}^{(q)} \setminus (\Omega_{i_1}^{(q-1)} \cup \Omega_{i_2}^{(q-1)})$ is part of a hyperplane,

($\mathcal{D}_2$) $\{\Omega_i^{(q)} : q \le i \le N, i \ne \bar{i}\} = \{\Omega_i^{(q-1)} : q-1 \le i \le N, i \ne \{i_1, i_2\}\}$,

($\mathcal{D}_3$) $\Omega_N^{(N)} = \Omega$.

To each of the closed facets of the hypercubes $\square_j$, for $j = 0, 1, \cdots, N$, we associate a number 0 or 1 indicating the order of the homogeneous Dirichlet boundary condition on the facet (where order 0 means no boundary condition). Considering problem (1.1), we choose a first order homogeneous Dirichlet boundary condition (i.e. order 1) on all the closed facets that are on Γ, and order 0 on $\partial\Omega \setminus \Gamma$. The selection of the order 0 or 1 on the other, interior facets will follow from conditions ($\mathcal{D}_5$)–($\mathcal{D}_7$) given below.

By construction, the boundary of each $\Omega_i^{(q)}$ is a union of facets of hypercubes $\square_j$. We define $\mathring{H}^1(\Omega_i^{(q)})$ as the closure in $H^1(\Omega_i^{(q)})$ of the smooth functions on $\Omega_i^{(q)}$ that vanish on the union of the facets of the $\square_j$ on which homogeneous Dirichlet conditions are imposed, and that are part of $\partial\Omega_i^{(q)}$. Note that $\mathring{H}^1(\Omega_N^{(N)}) = H^1_{0,\Gamma}(\Omega)$.

We equip $\mathring{H}^1(\Omega_i^{(q)})$ with squared norm

$$|\!|\!|u|\!|\!|_{q,i}^2 := \sum_{\square_j \subset \Omega_i^{(q)}} |\!|\!|u|\!|\!|_j^2,$$

which for $i = N = q$ is the norm $|\!|\!|\cdot|\!|\!|$.

The boundary conditions on the hypercubes and the order in which polytopes are connected should be chosen such that

($\mathcal{D}_5$) on the $\Omega_{i_1}^{(q-1)}$ and $\Omega_{i_2}^{(q-1)}$ sides of J, the boundary conditions are of order 0 and 1, respectively,

($\mathcal{D}_6$) for any two hypercubes $\square_{j_1} \subset \Omega_{i_1}^{(q-1)}$ and $\square_{j_2} \subset \Omega_{i_2}^{(q-1)}$ from $\{\square_0, \ldots, \square_N\}$ that intersect each other at J, it holds that $\kappa_{j_2} \lesssim \mu_{j_1}$,

and, w.l.o.g. assuming that $J = \{0\} \times \breve{J}$ and $(0,1) \times \breve{J} \subset \Omega_{i_1}^{(q-1)}$,

($\mathcal{D}_7$) for any function in $\mathring{H}^1(\Omega_{i_1}^{(q-1)})$ that vanishes at $\{0,1\} \times \breve{J}$ its reflection in $\{0\} \times \mathbb{R}^{n-1}$ extended with zero, is in $\mathring{H}^1(\Omega_{i_2}^{(q-1)})$.

The condition ($\mathcal{D}_7$) can be formulated by saying that the order of the boundary condition at any subfacet of $\Omega_{i_1}^{(q-1)}$ adjacent to J should not be less than this order at its reflection in J, where in case this reflection is not part of $\partial\Omega_{i_2}^{(q-1)}$ the latter order should be read as 1; and vice versa, that the order of the boundary condition at any subfacet of $\Omega_{i_2}^{(q-1)}$ adjacent to J should not be larger than this order at its reflection in J, where in case this reflection is not part of $\partial\Omega_{i_1}^{(q-1)}$ the latter order should be read as 0.

Condition ($\mathcal{D}_7$) is meant to be able to construct, in the forthcoming Proposition 3.2, uniformly bounded extension operators $E_1^{(q)}$, generalizing the operator E_1 from Proposition 2.5 for the two subdomain case.

Example 3.1. Let Ω be subdivided into 5 squares $\Omega_0^{(0)}, \ldots, \Omega_4^{(0)}$, and let the Dirichlet boundary $\Gamma \subset \partial\Omega$ be as indicated as in Figure 2. Let $\kappa_1 \lesssim \kappa_2 \lesssim \kappa_3 \lesssim \kappa_4$, and $\kappa_1 \lesssim \kappa_0$. By taking $\mu_i = \kappa_i$ ($\forall i$), a valid ordering of the merging of the subdomains, with valid directions in which the extensions $E_1^{(q)}$ are applied, is illustrated in that figure.

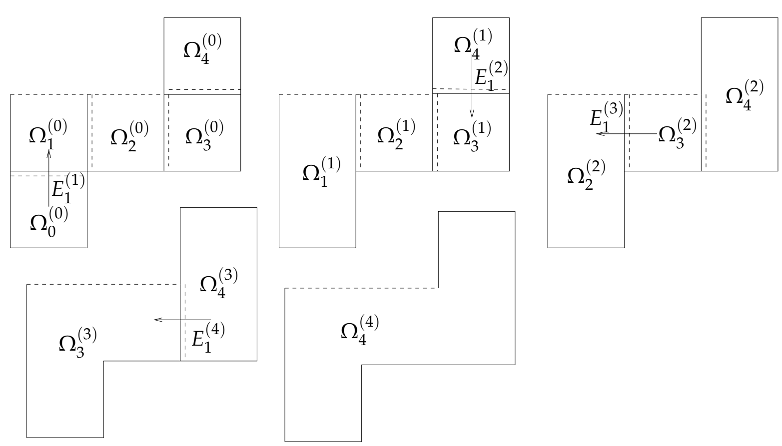

FIGURE 2. Solid facets indicate homogeneous Dirichlet boundary conditions, and no boundary conditions are indicated by dashed subfacets.

Given $1 \leq q \leq N$, for $j \in \{1,2\}$, let $R_j^{(q)}$ be the *restriction* of functions on $\Omega_{\bar{i}}^{(q)}$ to $\Omega_{i_j}^{(q-1)}$. Let $\eta_2^{(q)}$ be the trivial *extension* of functions on $\Omega_{i_2}^{(q-1)}$ to $\Omega_{\bar{i}}^{(q)}$ by zero, and let

$$E_1^{(q)} : \mathring{H}^1(\Omega_{i_1}^{(q-1)}) \to \mathring{H}^1(\Omega_{\bar{i}}^{(q)}),$$

be a suitable *extension* of functions on $\Omega_{i_1}^{(q-1)}$ to $\Omega_{\bar{i}}^{(q)}$. Following Proposition 2.5, we define

$$E^{(q)} := [E_1^{(q)} \quad \eta_2^{(q)}] : \mathring{H}^1(\Omega_{i_1}^{(q-1)}) \times \mathring{H}^1(\Omega_{i_2}^{(q-1)}) \to \mathring{H}^1(\Omega_{\bar{i}}^{(q)}). \tag{3.2}$$

Under the conditions $(\mathcal{D}_1)$-$(\mathcal{D}_7)$, the extensions $E_1^{(q)}$ can be constructed as tensor products of *univariate extensions* with identity operators in the other Cartesian directions:

Proposition 3.2. *In the setting of $(\mathcal{D}_1)$, w.l.o.g. let $J = \{0\} \times \check{J}$ and $(0,1) \times \check{J} \subset \Omega_{i_1}^{(q-1)}$. Let G_1 be an* extension *operator of functions on $(0,1)$ to functions on $(-1,1)$ such that*

$$G_1 \in B(L_2(0,1), L_2(-1,1)), \quad G_1 \in B(H^1(0,1), H^1_{0,\{-1\}}(-1,1)).$$

Then with $E_1^{(q)}$ defined as the composition of the restriction of functions on $\Omega_{i_1}^{(q-1)}$ to $(0,1) \times \check{J}$, followed by an application of

$$G_1 \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id},$$

followed by an extension by 0 to $\Omega_{i_2}^{(q-1)} \setminus (-1,0) \times \check{J}$, it holds that

$$E_1^{(q)} \in B((\mathring{H}^1(\Omega_{i_1}^{(q-1)}), \|\!|\cdot|\!\|_{q-1,i_1}), (\mathring{H}^1(\Omega_{\bar{i}}^{(q)}), \|\!|\cdot|\!\|_{q,\bar{i}})),$$

with a norm that is bounded uniformly in $\{0 < \mu_i \lesssim \kappa_i : 0 \leq i \leq N\}$ that satisfy $(\mathcal{D}_6)$.

Proof. Since the restriction and extension are bounded with norms equal to 1, it is sufficient to prove that $G_1 \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}$ is uniformly bounded from $H^1((0,1) \times \check{J})$ equipped with $\sqrt{\sum_{\square_j \subset (0,1)\times\check{J}} \|\!|\cdot|\!\|_j^2}$ to $H^1((-1,1) \times \check{J})$ equipped with $\sqrt{\sum_{\square_j \subset (-1,1)\times\check{J}} \|\!|\cdot|\!\|_j^2}$.

It is sufficient to prove this uniform boundedness for both coefficients μ_j and κ_j that define $\|\!|\cdot|\!\|_j$ being replaced by $\bar{\mu}_j$, where $\bar{\mu}_j = \begin{cases} \mu_j & \text{when } \square_j \subset (0,1) \times \check{J}, \\ \mu_{j'} & \text{when } \square_j \subset (-1,0) \times \check{J}, \end{cases}$ with j' being such that $\square_j$ and $\square_{j'}$ share a facet at $\check{J}$. Indeed, by $(\mathcal{D}_6)$ and (3.1), these replacements can make $\|\!|\cdot|\!\|_j$ for $\square_j$ at the right (left)-side of the interface only smaller (larger) (up to some constant factor).

With for an interval I, $A_I : H^1(I) \to H^1(I)'$ defined by $(A_I u)(v) = \langle u', v' \rangle_{L_2(I)} + \langle u, v \rangle_{L_2(I)}$, the assumption on G_1 means that $G_1, A_{(-1,1)}^{\frac{1}{2}} G_1 A_{(0,1)}^{-\frac{1}{2}} \in B(L_2(0,1), L_2(-1,1))$.

For notational simplicity only, let us consider the situation that $n = 2$, and let $\bar{\mu}$ be such that $\bar{\mu}(y) = \bar{\mu}_j$ when $(0,1) \times \{y\} \subset \square_j$. Denoting $u \mapsto (y \mapsto \sqrt{\bar{\mu}(y)} u(y))$ as ν, obviously $\nu A_{\check{J}}^{\frac{1}{2}} A_{\check{J}}^{-\frac{1}{2}} \nu^{-1}, \nu\nu^{-1} \in B(L_2(\check{J}), L_2(\check{J}))$.

We conclude that

$$(\mathrm{Id} \otimes \nu)(A_{(-1,1)}^{\frac{1}{2}} \otimes \mathrm{Id})(G_1 \otimes \mathrm{Id})(A_{(0,1)}^{-\frac{1}{2}} \otimes \mathrm{Id})(\mathrm{Id} \otimes \nu^{-1}),$$

$$(\mathrm{Id} \otimes \nu)(\mathrm{Id} \otimes A_{\check{J}}^{\frac{1}{2}})(G_1 \otimes \mathrm{Id})(\mathrm{Id} \otimes A_{\check{J}}^{-\frac{1}{2}})(\mathrm{Id} \otimes \nu^{-1}) \in B(L_2((0,1) \times \check{J}), L_2((-1,1) \times \check{J})),$$

independently of $\bar\mu$ with $\bar\mu, 1/\bar\mu \in L_\infty((-1,1)\times \check J)$. Boundedness of these operations mean that

$$\int_{-1}^{1}\int_{\check J}[((G_1\otimes \mathrm{Id})u)(x,y)^2+(\partial_x(G_1\otimes \mathrm{Id})u)(x,y)^2]\bar\mu(y)dydx \\ \lesssim \int_0^1\int_{\check J}[u(x,y)^2+(\partial_x u)(x,y)^2]\bar\mu(y)dydx,$$

or

$$\int_{-1}^{1}\int_{\check J}[((G_1\otimes \mathrm{Id})u)(x,y)^2+(\partial_y(G_1\otimes \mathrm{Id})u)(x,y)^2]\bar\mu(y)dydx \\ \lesssim \int_0^1\int_{\check J}[u(x,y)^2+(\partial_y u)(x,y)^2]\bar\mu(y)dydx,$$

and so

$$\int_{-1}^{1}\int_{\check J}[((G_1\otimes \mathrm{Id})u)(x,y)^2+(\partial_x(G_1\otimes \mathrm{Id})u)(x,y)^2+(\partial_y(G_1\otimes \mathrm{Id})u)(x,y)^2]\bar\mu(y)dydx \\ \lesssim \int_0^1\int_{\check J}[u(x,y)^2+(\partial_x u)(x,y)^2+(\partial_y u)(x,y)^2]\bar\mu(y)dydx,$$

for all $u \in H^1((0,1)\times\check J)$, independently of $\bar\mu$, which was to be proven. $\square$

By an application of Proposition 2.5, we have the following result.

Corollary 3.3. *Let $E_1^{(q)}$ be as constructed in Proposition 3.2. Then, for E being the composition for $q = 1,\dots,N$ of the mappings $E^{(q)}$ from* (3.2)*, trivially extended with identity operators in coordinates $i \in \{q-1,\dots,N\}\setminus\{i_1^{(q)}, i_2^{(q)}\}$, it holds that*

$$E \in B\Big(\prod_{i=0}^{n}(\overset{\circ}{H}{}^1(\square_i), |\!|\!|\cdot|\!|\!|_i), (\overset{\circ}{H}{}^1(\Omega), |\!|\!|\cdot|\!|\!|)\Big), \tag{3.3}$$

is boundedly invertible uniformly in $\{0<\mu_i\lesssim\kappa_i : 0\le i\le N\}$ that satisfy $(\mathcal{D}_6)$.

Corollary 3.4. *For $0 \le i \le N$, let $\mathbf{\Psi}_i$ be a Riesz basis for $L_2(\square_i)$, that renormalized in $(H^1(\square_i), |\!|\!|\cdot|\!|\!|_i)$, is a Riesz basis for $\overset{\circ}{H}{}^1(\square_i)$ with Riesz constants that are bounded uniformly in $0<\mu_i\lesssim\kappa_i$ (cf. Proposition 2.6 and the lines preceding it). Then with the isomorphism E from Corollary 3.3, the collection $E(\prod_{i=0}^N \mathbf{\Psi}_i)$, normalized in $|\!|\!|\cdot|\!|\!|$, is a Riesz basis for $(\overset{\circ}{H}{}^1(\Omega), |\!|\!|\cdot|\!|\!|)$ with Riesz constants that are bounded uniformly in $\{0<\mu_i\lesssim\kappa_i : 0\le i\le N\}$ that satisfy $(\mathcal{D}_6)$.*

3.2. **Tensor products of univariate wavelet functions.** As in [DS10], we are going to construct the bases $\mathbf{\Psi}_i$ as meant in Corollary 3.4 as tensor products of univariate wavelet bases. It is sufficient to consider the case that $\square_i = \mathcal{I}^n$.

For $\vec\sigma = (\sigma_\ell, \sigma_r) \in \{0,1\}^2$, where we will denote $(0,0)$ as $\vec 0$, let

$$H^1_{\vec\sigma}(\mathcal{I}) := \{v \in H^1(\mathcal{I}) : v(0) = 0 \text{ when } \sigma_\ell = 1, \text{ and } v(1) = 0 \text{ when } \sigma_r = 1\}.$$

We assume that *biorthogonal univariate primal and dual wavelet* collections

$$\Psi_{\vec\sigma} := \{\psi_\lambda^{\vec\sigma} : \lambda \in \nabla_{\vec\sigma}\} \subset H^1_{\vec\sigma}(\mathcal{I}),$$
$$\tilde\Psi_{\vec\sigma} := \{\tilde\psi_\lambda^{\vec\sigma} : \lambda \in \nabla_{\vec\sigma}\} \subset L_2(\mathcal{I}),$$

are available that satisfy the following properties:

($\mathcal{W}_1$) $\Psi_{\vec{\sigma}}$, and so $\tilde{\Psi}_{\vec{\sigma}}$, is a Riesz basis for $L_2(\mathcal{I})$,

($\mathcal{W}_2$) $\{2^{-|\lambda|}\psi_\lambda^{\vec{\sigma}} : \lambda \in \nabla_{\vec{\sigma}}\}$ is a Riesz basis for $H^1_{\vec{\sigma}}(\mathcal{I})$, where $|\lambda| \in \mathbb{N}_0$ denotes the *level* of λ,

($\mathcal{W}_3$) $|\langle\tilde{\psi}_\lambda^{\vec{\sigma}}, u\rangle_{L_2(\mathcal{I})}| \lesssim 2^{-|\lambda|d}\|u\|_{H^d(\operatorname{supp}\tilde{\psi}^{\vec{\sigma}})}$ ($u \in H^d(\mathcal{I}) \cap H^1_{\vec{\sigma}}(\mathcal{I})$), for some $\mathbb{N} \ni d > 1$,

($\mathcal{W}_4$)
$$\begin{aligned} 1 > \rho \quad &:= \quad \sup_{\lambda\in\nabla_{\vec{\sigma}}} 2^{|\lambda|}\max(\operatorname{diam}\operatorname{supp}\tilde{\psi}_\lambda^{\vec{\sigma}}, \operatorname{diam}\operatorname{supp}\psi_\lambda^{\vec{\sigma}}) \\ &\eqsim \quad \inf_{\lambda\in\nabla_{\vec{\sigma}}} 2^{|\lambda|}\max(\operatorname{diam}\operatorname{supp}\tilde{\psi}_\lambda^{\vec{\sigma}}, \operatorname{diam}\operatorname{supp}\psi_\lambda^{\vec{\sigma}}), \end{aligned}$$

($\mathcal{W}_5$) $\sup_{j,k\in\mathbb{N}_0} \#\{|\lambda| = j : [k2^{-j}, (k+1)2^{-j}] \cap (\operatorname{supp}\tilde{\psi}_\lambda^{\vec{\sigma}} \cup \operatorname{supp}\psi_\lambda^{\vec{\sigma}}) \neq \emptyset\} < \infty.$

($\mathcal{W}_6$) $\mathbb{1} \in \operatorname{span}\{\psi_\lambda^{\vec{0}} : \lambda \in \nabla_{\vec{0}}, |\lambda| = 0\}$,

($\mathcal{W}_7$) $\langle\psi_\lambda^{\vec{0}}, \mathbb{1}\rangle_{L_2(\mathcal{I})} = 0$ for $|\lambda| > 0$,

where the last two conditions thus only apply to the case $\vec{\sigma} = \vec{0}$.

Biorthogonal wavelets that satisfy the above conditions have been constructed in [DKU99, Dij09, Pri10]. The conditions ($\mathcal{W}_4$) and ($\mathcal{W}_5$) mean that *both primal and dual wavelets* are *local* and *locally finite*, respectively. The condition $\rho < 1$, which implies that a wavelet that is non-zero at one boundary vanishes at the other, can always be satisfied by increasing the coarsest scale.

For $\boldsymbol{\sigma} = (\vec{\sigma}_i = ((\sigma_i)_\ell, (\sigma_i)_r))_{1\leq i\leq n} \in (\{0,1\}^2)^n$, we define

$$H^1_{\boldsymbol{\sigma}}(\square) := H^1_{\vec{\sigma}_1}(\mathcal{I}) \otimes L_2(\mathcal{I}) \otimes \cdots \otimes L_2(\mathcal{I}) \cap \cdots \cap L_2(\mathcal{I}) \otimes \cdots \otimes L_2(\mathcal{I}) \otimes H^1_{\vec{\sigma}_n}(\mathcal{I}).$$

So $H^1_{\boldsymbol{\sigma}}(\square)$ is the space of H^1-functions on $\square$ that satisfy first order homogeneous Dirichlet boundary conditions on selected faces. The *tensor product wavelet* collection

$$\boldsymbol{\Psi}_{\boldsymbol{\sigma}} := \otimes_{i=1}^n \Psi_{\vec{\sigma}_i} = \big\{\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\boldsymbol{\sigma}} := \otimes_{i=1}^n \psi_{\lambda_i}^{\vec{\sigma}_i} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}} := \prod_{i=1}^n \nabla_{\vec{\sigma}_i}\big\}, \tag{3.4}$$

and its renormalized version $\big\{\big(\sum_{i=1}^n 4^{|\lambda_i|}\big)^{-1/2}\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\boldsymbol{\sigma}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}}\big\}$ are Riesz bases for $L_2(\square)$ and $H^1_{\boldsymbol{\sigma}}(\square)$, respectively.

The collection that is dual to $\boldsymbol{\Psi}_{\boldsymbol{\sigma}}$ reads as

$$\tilde{\boldsymbol{\Psi}}_{\boldsymbol{\sigma}} := \otimes_{i=1}^n \tilde{\Psi}_{\vec{\sigma}_i} = \big\{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{\boldsymbol{\sigma}} := \otimes_{i=1}^n \tilde{\psi}_{\lambda_i}^{\vec{\sigma}_i} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}}\big\},$$

and is a Riesz basis for $L_2(\square)$.

Possibly after a basis transformation that involves only basis functions on the coarsest scale, the conditions ($\mathcal{W}_6$) and ($\mathcal{W}_7$) guarantee that $\boldsymbol{\Psi}_{(\vec{0},\cdots,\vec{0})}$ satisfies the conditions of Proposition 2.6. We return to this issue at the end of Sect. 4.2.4.

3.3. **Construction of scale-dependent univariate extension operators.** We have observed that the extension operator $E_1^{(q)}$, for $1 \leq q \leq N$, can be built by applying an appropriate univariate extension operator G_1. Since, additionally, we apply tensor product wavelets on the hypercubes $\square_i$, the issue of constructing a suitable extension reduces to the issue in the univariate case. We are going to construct an extension operator that will act only on wavelets that are supported near the interface, so that the extension increases the diameter of the support of any wavelet by not more than a constant factor.

We consider the situation of the univariate domain $(-1,1)$ with two subdomains $(0,1)$ and $(-1,0)$, and we extend functions on $(0,1)$ to $(-1,1)$. We make the following additional assumptions on the univariate wavelets. For $\vec{\sigma} = (\sigma_\ell, \sigma_r) \in \{0,1\}^2$,

($\mathcal{W}_8$) $V_j^{\vec{\sigma}} := \mathrm{span}\{\psi_\lambda^{\vec{\sigma}} : \lambda \in \nabla_{\vec{\sigma}}, |\lambda| \leq j\} = V_j^{\vec{0}} \cap H^1_{\vec{\sigma}}((0,1))$,

($\mathcal{W}_9$) $\nabla_{\vec{\sigma}}$ is the disjoint union of $\nabla_{\sigma_\ell}^{(\ell)}$, $\nabla^{(I)}$, $\nabla_{\sigma_r}^{(r)}$ such that
   (i) $\sup_{\lambda \in \nabla_{\vec{\sigma}}^{(\ell)}, x \in \mathrm{supp}\, \psi_\lambda^{\vec{\sigma}}} 2^{|\lambda|}|x| \lesssim \rho$, $\sup_{\lambda \in \nabla_{\vec{\sigma}}^{(r)}, x \in \mathrm{supp}\, \psi_\lambda^{\vec{\sigma}}} 2^{|\lambda|}|1-x| \lesssim \rho$,
   (ii) for $\lambda \in \nabla^{(I)}$, $\psi_\lambda^{\vec{\sigma}} = \psi_\lambda^{\vec{0}}$, $\tilde{\psi}_\lambda^{\vec{\sigma}} = \tilde{\psi}_\lambda^{\vec{0}}$, and the extensions of $\psi_\lambda^{\vec{0}}$ and $\tilde{\psi}_\lambda^{\vec{0}}$ by zero are in $H^1(\mathbb{R})$ and $L_2(\mathbb{R})$, respectively.

($\mathcal{W}_{10}$) $\begin{cases} \mathrm{span}\{\psi_\lambda^{\vec{0}} : \lambda \in \nabla^{(I)},\, |\lambda| = j\} = \mathrm{span}\{\psi_\lambda^{\vec{0}}(1-\cdot) : \lambda \in \nabla^{(I)},\, |\lambda| = j\}, \\ \mathrm{span}\{\psi_\lambda^{(\sigma_r,\sigma_\ell)} : \lambda \in \nabla_{\sigma_r}^{(r)},\, |\lambda| = j\} = \mathrm{span}\{\psi_\lambda^{(\sigma_\ell,\sigma_r)}(1-\cdot) : \lambda \in \nabla_{\sigma_\ell}^{(\ell)},\, |\lambda| = j\} \end{cases}$

($\mathcal{W}_{11}$) $\begin{cases} \psi_\lambda^{\vec{\sigma}}(2^l\cdot) \in \mathrm{span}\{\psi_\nu^{\vec{\sigma}} : \nu \in \nabla_{\sigma_\ell}^{(\ell)}\} \quad (l \in \mathbb{N}_0,\, \lambda \in \nabla_{\sigma_\ell}^{(\ell)}), \\ \psi_\lambda^{\vec{0}}(2^l\cdot) \in \mathrm{span}\{\psi_\nu^{\vec{0}} : \nu \in \nabla^{(I)}\} \quad (l \in \mathbb{N}_0,\, \lambda \in \nabla^{(I)}). \end{cases}$

The biorthogonal wavelets constructed in [DKU99, Dij09, Pri10] satisfy these additional assumptions as well. Mainly in view of obtaining a bi-infinite stiffness matrix in which nearly all row and columns contain only finitely many non-zeros, in Sect. 4 we are going to present an alternative wavelet construction that also satisfies all conditions ($\mathcal{W}_1$)-($\mathcal{W}_{11}$).

To design a suitable extension, let us first consider the simple *reflection*

$$(\breve{G}_1 v)(x) := v(x) \qquad x \in (0,1)$$
$$(\breve{G}_1 v)(-x) := v(x) \qquad x \in (0,1) \tag{3.5}$$

for any $v \in L_2(0,1)$. It is clear that

$$\breve{G}_1 \in B(L_2(0,1), L_2(-1,1)),$$
$$\breve{G}_1 \in B(H^1(0,1), H^1(-1,1)). \tag{3.6}$$

Its dual reads as

$$(\breve{G}_1^* v)(x) := v(x) + v(-x) \quad (v \in L_2(-1,1),\, x \in (0,1)). \tag{3.7}$$

Let η_1 and η_2 denote the extensions by zero of functions on $(0,1)$ and on $(-1,0)$ to functions on $(-1,1)$, respectively, with R_1 and R_2 denoting their adjoints. The 'adjoint extension' of $\breve{G}_1$, denoted by $\breve{G}_2$, is defined by

$$\breve{G}_2 := (\mathrm{Id} - \eta_1 \breve{G}_1^*)\eta_2.$$

It satisfies

$$\breve{G}_2 v(x) = \begin{cases} v(x) & x \in (-1,0), \\ -v(-x) & x \in (0,1). \end{cases}$$

where $v \in L_2(-1,1)$.

The obvious choice of $G_1 = \breve{G}_1$, does not yield the desirable property of $\mathrm{diam}(\mathrm{supp}\, G_1 u) \lesssim \mathrm{diam}(\mathrm{supp}\, u)$. To solve this and the corresponding problem for the adjoint extension, in any case for u being any primal or dual wavelet, respectively, following

[CDFS13] we will apply our construction using the modified, *scale-dependent* univariate extension operator

$$G_1 : u \mapsto \sum_{\lambda \in \nabla_0^{(\ell)}} \langle u, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(0,1)} \breve{G}_1 \psi_\lambda^{\vec{0}} + \sum_{\lambda \in \nabla^{(I)} \cup \nabla_0^{(r)}} \langle u, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(0,1)} \eta_1 \psi_\lambda^{\vec{0}}. \tag{3.8}$$

So this operator reflects only wavelets that are supported near the interface.

Proposition 3.5. *For $\vec{\sigma} \in \{0,1\}^2$, the scale-dependent extension G_1 from (3.8) satisfies*

$$G_1 \psi_\iota^{\vec{\sigma}} = \begin{cases} \eta_1 \psi_\iota^{\vec{\sigma}} \text{ when } \iota \in \nabla^{(I)} \cup \nabla_{\sigma_r}^{(r)}, \\ \breve{G}_1 \psi_\iota^{\vec{\sigma}} \text{ when } \iota \in \nabla_0^{(\ell)}. \end{cases} \tag{3.9}$$

The resulting adjoint extension $G_2 := (\mathrm{Id} - \eta_1 G_1^*)\eta_2$ satisfies

$$G_2(\tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)) = \begin{cases} \eta_2(\tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)) \text{ when } \iota \in \nabla^{(I)} \cup \nabla_{\sigma_\ell}^{(\ell)}, \\ \breve{G}_2(\tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)) \text{ when } \iota \in \nabla_{\sigma_r}^{(r)}. \end{cases} \tag{3.10}$$

For $\iota \in \nabla_{\vec{\sigma}}$, it holds that

$$\begin{aligned} \operatorname{diam}(\operatorname{supp} G_1 \psi_\iota^{\vec{\sigma}}) &\lesssim \operatorname{diam}(\operatorname{supp} \psi_\iota^{\vec{\sigma}}), \\ \operatorname{diam}(\operatorname{supp} G_2 \tilde{\psi}_\iota^{\vec{\sigma}}) &\lesssim \operatorname{diam}(\operatorname{supp} \tilde{\psi}_\iota^{\vec{\sigma}}). \end{aligned} \tag{3.11}$$

Finally, the following results hold

$$\begin{aligned} G_1 &\in B(L_2(0,1), L_2(-1,1)), \\ G_1 &\in B(H^1(0,1), H^1_{0,\{-1\}}(-1,1)). \end{aligned}$$

Proof. By $(\mathcal{W}_9)$(ii), for $\iota \in \nabla^{(I)} \cup \nabla_{\sigma_r}^{(r)}$, $\lambda \in \nabla_0^{(\ell)}$, one has $\langle \psi_\iota^{\vec{\sigma}}, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(0,1)} = 0$, and so $G_1 \psi_\iota^{\vec{\sigma}} = \sum_{\lambda \in \nabla_{\vec{0}}} \langle \psi_\iota^{\vec{\sigma}}, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(0,1)} \eta_1 \psi_\lambda^{\vec{0}} = \eta_1 \psi_\iota^{\vec{\sigma}}$, the last equality from $\Psi^{\vec{0}}$ being a Riesz basis for $L_2(0,1)$, and η_1 being L_2-bounded.

Similarly, for $\iota \in \nabla_{\sigma_\ell}^{(\ell)}$, $\lambda \in \nabla^{(I)} \cup \nabla_0^{(r)}$, it holds that $\langle \psi_\iota^{\vec{\sigma}}, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(0,1)} = 0$, and so $G_1 \psi_\iota^{\vec{\sigma}} = \sum_{\lambda \in \nabla_{\vec{0}}} \langle \psi_\iota^{\vec{\sigma}}, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(0,1)} \breve{G}_1 \psi_\lambda^{\vec{0}} = \breve{G}_1 \psi_\iota^{\vec{\sigma}}$.

For $v \in L_2(0,1)$,

$$\begin{aligned} G_1^* \eta_2(v(1+\cdot)) &= \sum_{\lambda \in \nabla_{\vec{0}}} \langle G_1^* \eta_2(v(1+\cdot)), \psi_\lambda^{\vec{0}} \rangle_{L_2(0,1)} \tilde{\psi}_\lambda^{\vec{0}} \\ &= \sum_{\lambda \in \nabla_{\vec{0}}} \langle v(1-\cdot), (R_2 G_1 \psi_\lambda^{\vec{0}})(-\cdot) \rangle_{L_2(0,1)} \tilde{\psi}_\lambda^{\vec{0}} \\ &= \sum_{\lambda \in \nabla_0^{(\ell)}} \langle v(1-\cdot), (R_2 \breve{G}_1 \psi_\lambda^{\vec{0}})(-\cdot) \rangle_{L_2(0,1)} \tilde{\psi}_\lambda^{\vec{0}} \\ &= \sum_{\lambda \in \nabla_0^{(\ell)}} \left\langle v(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(0,1)} \tilde{\psi}_\lambda^{\vec{0}}. \end{aligned} \tag{3.12}$$

For $v = \tilde{\psi}_\iota^{\vec{\sigma}}$ and $\iota \in \nabla^{(I)} \cup \nabla_{\sigma_\ell}^{(\ell)}$, (3.12) is zero by $(\mathcal{W}_{10})$, $(\mathcal{W}_{11})$, and $(\mathcal{W}_9)$(ii). For $v = \tilde{\psi}_\iota^{\vec{\sigma}}$ and $\iota \in \nabla_{\sigma_r}^{(r)}$, one has $\left\langle v(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(0,1)} = 0$ for $\lambda \in \nabla^{(I)} \cup \nabla_0^{(r)}$ by $(\mathcal{W}_{10})$, $(\mathcal{W}_{11})$, and $(\mathcal{W}_9)$(ii). So for those ι, one has $G_1^* \eta_2(\tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)) = \breve{G}_1^* \eta_2(\tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot))$, which completes the proof of (3.10).

Since $\operatorname{span}\{\psi_\iota^{\vec{0}} : \iota \in \nabla^{(I)} \cup \nabla_0^{(r)}\} + \operatorname{span}\{\psi_\iota^{\vec{0}} : \iota \in \nabla_0^{(\ell)}\}$ defines a stable splitting of both $L_2(0,1)$ and $H^1(0,1)$ into two subspaces, the statements about the boundedness of G_1 follow from (3.9) with $\vec{\sigma} = \vec{0}$, (3.6), and $(\mathcal{W}_9)$(ii). $\square$

3.4. **A more efficient scale-dependent univariate extension operator.** Recall the definition of the scale-dependent extension operator in (3.8). In view of the definition of the splitting of the wavelet index set $\nabla_{\vec{\sigma}}$ into $\nabla_{\sigma_\ell}^{(\ell)}$, $\nabla^{(I)}$, and $\nabla_{\sigma_r}^{(r)}$, we infer that each $\psi_\lambda^{\vec{0}}$ has to be extended over the interface by the application of $\breve{G}_1$, for all λ for which either $\psi_\lambda^{\vec{0}}$ *or* $\tilde{\psi}_\lambda^{\vec{0}}$ is a boundary adapted wavelet.

The reason why, for λ with the dual wavelet $\tilde{\psi}_\lambda^{\vec{0}}$ being a boundary adapted wavelet, $\psi_\lambda^{\vec{0}}$ had to be extended is to ensure that the resulting dual wavelets are locally supported. In this subsection, under some additional condition, we will show that the application of $\breve{G}_1$ can be confined to *only those primal wavelets $\psi_\lambda^{\vec{0}}$ that do not vanish at the boundary, and nevertheless end up with locally supported dual wavelets*. Since dual wavelets have usually larger supports, this means that fewer wavelets have to be extended, making the construction more efficient.

Remark 3.6. Since dual wavelets do not enter the implementation of the adaptive wavelets scheme, one may wonder why it is important that they have local supports. The reason is that using these local supports, in [CDFS13, Thm. 5.6] we could prove optimal 'univariate' approximation rates in $H^1(\Omega)$ from the piecewise tensor product wavelet basis assuming only mild piecewise weighted anisotropic Sobolev smoothness, which result was appended with corresponding regularity results for the Poisson problem in [CDFS13, Sect. 6] using results from [CDN12]. Although in the current paper we do not make an attempt to generalize those results to the interface problem, in view of them it can be expected to be beneficial to have locally supported duals.

We add the following condition, that will be satisfied by our wavelet construction presented in Sect. 4:

$(\mathcal{W}_{12})$ for some constant $m \in \mathbb{N}$, for any $\vec{\sigma} \in \{0,1\}^2$, for $k \geq m$, $\operatorname{span}\{\psi_\lambda^{\vec{\sigma}} : |\lambda| = j+k\}$ is $L_2(0,1)$-orthogonal to $\operatorname{span}\{\tilde{\psi}_\lambda^{\vec{0}} : |\lambda| = j\}$ $(j \in \mathbb{N}_0)$.

Now, with

$$\bar{\nabla}_0^{(\ell)} := \{\lambda \in \nabla_0^{(\ell)} : \psi_\lambda^{\vec{0}}(0) \neq 0\}, \quad \hat{\nabla}_0^{(\ell)} := \nabla_0^{(\ell)} \backslash \bar{\nabla}_0^{(\ell)},$$

we redefine

$$G_1 : u \mapsto \sum_{\lambda \in \bar{\nabla}_0^{(\ell)}} \langle u, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} \breve{G}_1 \psi_\lambda^{\vec{0}} + \sum_{\lambda \in \hat{\nabla}_0^{(\ell)} \cup \nabla^{(I)} \cup \nabla_0^{(r)}} \langle u, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} \eta_1 \psi_\lambda^{\vec{0}}. \tag{3.13}$$

Note that, in view of $(\mathcal{D}_5)$, it is sufficient to consider the action of G_1 on primal wavelets $\psi_\iota^{\vec{\sigma}}$ for $\sigma_\ell = 0$, and that of the resulting dual extension G_2 on dual wavelets $\tilde{\psi}_\iota^{\vec{\sigma}}$ for $\sigma_r = 0$.

Proposition 3.7. *For $\sigma_\ell = 0$, the scale-dependent extension G_1 from* (3.13) *satisfies*

$$G_1 \psi_\iota^{\vec{\sigma}} = \begin{cases} \eta_1 \psi_\iota^{\vec{\sigma}} \ \textit{when}\ \iota \in \hat{\nabla}_0^{(\ell)} \cup \nabla^{(I)} \cup \nabla_{\sigma_r}^{(r)}, \\ \breve{G}_1 \psi_\iota^{\vec{\sigma}} \ \textit{when}\ \iota \in \bar{\nabla}_0^{(\ell)}, \end{cases} \tag{3.14}$$

and so in particular, for $\iota \in \nabla_{\vec{\sigma}}$*,*

$$\operatorname{diam}(\operatorname{supp} G_1 \psi_\iota^{\vec{\sigma}}) \lesssim \operatorname{diam}(\operatorname{supp} \psi_\iota^{\vec{\sigma}}).$$

For the resulting adjoint extension $G_2 := (\mathrm{Id} - \eta_1 G_1^*)\eta_2$*, and for* $\sigma_r = 0$*, it holds that*

$$\operatorname{diam}(\operatorname{supp} G_2 \tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)) \lesssim \operatorname{diam}(\operatorname{supp} \tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)). \tag{3.15}$$

Finally,

$$G_1 \in B(L_2(0,1), L_2(-1,1)),$$
$$G_1 \in B(H^1(0,1), H^1_{0,\{-1\}}(-1,1)).$$

Proof. By $(\mathcal{W}_9)$(ii), for $\iota \in \hat{\nabla}_0^{(\ell)} \cup \nabla^{(I)} \cup \nabla_{\sigma_r}^{(r)}$, $\lambda \in \bar{\nabla}_0^{(\ell)}$, it holds that $\langle \psi_\iota^{\vec{\sigma}}, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} = 0$, and so $G_1 \psi_\iota^{\vec{\sigma}} = \sum_{\lambda \in \nabla_{\vec{0}}} \langle \psi_\iota^{\vec{\sigma}}, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} \eta_1 \psi_\lambda^{\vec{0}} = \eta_1 \psi_\iota^{\vec{\sigma}}$. Also for $\iota \in \bar{\nabla}_0^{(\ell)}$, $\lambda \in \hat{\nabla}_0^{(\ell)} \cup \nabla^{(I)} \cup \nabla_{\sigma_r}^{(r)}$, one has $\langle \psi_\iota^{\vec{\sigma}}, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} = 0$, and so $G_1 \psi_\iota^{\vec{\sigma}} = \breve{G}_1 \psi_\iota^{\vec{\sigma}}$.

For $v \in L_2(\mathcal{I})$, we have

$$\begin{aligned} G_1^* \eta_2(v(1+\cdot)) &= \sum_{\lambda \in \nabla_{\vec{0}}} \langle G_1^* \eta_2(v(1+\cdot)), \psi_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}} \qquad (3.16) \\ &= \sum_{\lambda \in \nabla_{\vec{0}}} \langle v(1-\cdot), (R_2 G_1 \psi_\lambda^{\vec{0}})(-\cdot) \rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}} \\ &= \sum_{\lambda \in \bar{\nabla}_0^{(\ell)}} \langle v(1-\cdot), (R_2 \breve{G}_1 \psi_\lambda^{\vec{0}})(-\cdot) \rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}} \\ &= \sum_{\lambda \in \bar{\nabla}_0^{(\ell)}} \left\langle v(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}}. \qquad (3.17) \end{aligned}$$

Now we consider $v = \tilde{\psi}_\iota^{\vec{\sigma}}$ with $\sigma_r = 1$. For $\iota \in \nabla^{(I)} \cup \nabla_{\sigma_\ell}^{(\ell)}$, (3.17) is zero by $(\mathcal{W}_{10})$, $(\mathcal{W}_{11})$, and $(\mathcal{W}_9)$(ii), and so $G_2 \tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot) = \eta_2 \tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)$.

For $\iota \in \nabla_1^{(r)}$, one has $\left\langle \tilde{\psi}_\iota^{\vec{\sigma}}(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(\mathcal{I})} = 0$ for $\lambda \in \nabla^{(I)} \cup \nabla_0^{(r)}$ by $(\mathcal{W}_{10})$, $(\mathcal{W}_{11})$, and $(\mathcal{W}_9)$(ii). So for those ι, one has

$$\begin{aligned} G_1^* \eta_2(\tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)) &= \sum_{\lambda \in \bar{\nabla}_0^{(\ell)} \cup \nabla^{(I)} \cup \nabla_0^{(r)}} \left\langle \tilde{\psi}_\iota^{\vec{\sigma}}(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}} \\ &= \sum_{\lambda \in \nabla_{\vec{0}}} \left\langle \tilde{\psi}_\iota^{\vec{\sigma}}(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}} - \sum_{\lambda \in \hat{\nabla}_0^{(\ell)}} \left\langle \tilde{\psi}_\iota^{\vec{\sigma}}(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}} \\ &= \breve{G}_1^* \eta_2(\tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)) - \sum_{\lambda \in \hat{\nabla}_0^{(\ell)}} \left\langle \tilde{\psi}_\iota^{\vec{\sigma}}(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(\mathcal{I})} \tilde{\psi}_\lambda^{\vec{0}} \end{aligned}$$

From $(\mathcal{W}_{12})$, we have $\left\langle \tilde{\psi}_\iota^{\vec{\sigma}}(1-\cdot), \psi_\lambda^{\vec{0}} \right\rangle_{L_2(\mathcal{I})} = 0$, for $|\iota| + m \le |\lambda|$. For $\lambda \in \hat{\nabla}_0^{(\ell)}$, $\psi_\lambda^{\vec{0}}$ vanishes at 0, and thus is contained in the space $V_{|\lambda|}^{\vec{\sigma}}$. For $|\lambda| < |\iota|$, $\tilde{\psi}_\iota^{\vec{\sigma}}(1-\cdot)$ is orthogonal to this space, and so again the mentioned inner product is zero. Therefore, the last sum contains finitely many non-zero terms where $|\lambda| \eqsim |\iota|$, and thus $\operatorname{supp} G_2 \tilde{\psi}_\iota^{\vec{\sigma}}(1+\cdot)$ is local.

The proof of the final statement is as for the corresponding statement in Proposition 3.5. $\square$

4. Biorthogonal wavelet construction on the interval

In this section, for each $\vec{\sigma} \in \{0,1\}^2$ we construct univariate biorthogonal wavelet bases that satisfy all the conditions $(\mathcal{W}_1)$–$(\mathcal{W}_{12})$. In addition they will give rise to mass- and stiffness matrices for which nearly all their rows and columns contain finitely many non-zeros.

In order to construct a Riesz basis for a range of Sobolev spaces, in particular for $L_2(\mathcal{I})$ and $H^1_{\vec{\sigma}}(\mathcal{I})$, cf. $(\mathcal{W}_1)$-$(\mathcal{W}_2)$, we apply the following crucial result:

Theorem 4.1 (Biorthogonal space decompositions [DS99b, Ste03])**.** *Let*

$$V_0^{\vec{\sigma}} \subset V_1^{\vec{\sigma}} \subset \cdots \subset H^1_{\vec{\sigma}}(\mathcal{I}), \qquad \tilde{V}_0^{\vec{\sigma}} \subset \tilde{V}_1^{\vec{\sigma}} \subset \cdots \subset L_2(\mathcal{I}),$$

be sequences of primal and dual spaces such that $\dim V_j^{\vec{\sigma}} = \dim \tilde{V}_j^{\vec{\sigma}} < \infty$. *Let* $\Phi_j^{\vec{\sigma}}$ *and* $\tilde{\Phi}_j^{\vec{\sigma}}$ *be biorthogonal uniform* $L_2(\mathcal{I})$*-Riesz bases of* $V_j^{\vec{\sigma}}$ *and* $\tilde{V}_j^{\vec{\sigma}}$*, respectively. In addition, for some* $0 < \gamma < d$, $0 < \tilde{\gamma} < \tilde{d}$*, let*

$$\inf_{v_j \in V_j^{\vec{\sigma}}} \|v - v_j\|_{L_2(\mathcal{I})} \lesssim 2^{-jd} \|v\|_{H^d(\mathcal{I})} \quad (v \in H^d(\mathcal{I}) \cap H^1_{\vec{\sigma}}(\mathcal{I})),$$

$$\inf_{\tilde{v}_j \in \tilde{V}_j^{\vec{\sigma}}} \|v - \tilde{v}_j\|_{L_2(\mathcal{I})} \lesssim 2^{-j\tilde{d}} \|v\|_{H^{\tilde{d}}(\mathcal{I})} \quad (v \in H^{\tilde{d}}(\mathcal{I})), \tag{4.1}$$

(Jackson estimate)*, and*

$$\|v_j\|_{[L_2(\mathcal{I}), H^d(\mathcal{I}) \cap H^1_{\vec{\sigma}}(\mathcal{I})]_{s/d}} \lesssim 2^{js} \|v_j\|_{L_2(\mathcal{I})} \quad (v_j \in V_j^{\vec{\sigma}},\, s \in [0,\gamma)),$$

$$\|\tilde{v}_j\|_{[L_2(\mathcal{I}), H^{\tilde{d}}(\mathcal{I})]_{s/\tilde{d}}} \lesssim 2^{js} \|\tilde{v}_j\|_{L_2(\mathcal{I})} \quad (\tilde{v}_j \in \tilde{V}_j^{\vec{\sigma}},\, s \in [0,\tilde{\gamma})), \tag{4.2}$$

(Bernstein estimate)*. Then with* $\Phi_0^{\vec{\sigma}}$ *and* $\Psi_j^{\vec{\sigma}} = \{\psi_{j,k}^{\vec{\sigma}} \colon k \in J_j\}$ $(j \in \mathbb{N})$*, being uniform* $L_2(\mathcal{I})$*-Riesz bases for* $V_j^{\vec{\sigma}} \cap (\tilde{V}_{j-1}^{\vec{\sigma}})^{\perp_{L_2(\mathcal{I})}}$ (*wavelets*)*, for* $s \in (-\tilde{\gamma}, \gamma)$ *the collection*

$$\Psi_{\vec{\sigma}} := \Phi_0^{\vec{\sigma}} \cup \cup_{j \in \mathbb{N}} 2^{-sj} \Psi_j^{\vec{\sigma}},$$

is a Riesz basis for $\begin{cases} [L_2(\mathcal{I}), H^d(\mathcal{I}) \cap H^1_{\vec{\sigma}}(\mathcal{I})]_{s/d} & \textit{when } s \geq 0, \\ ([L_2(\mathcal{I}), H^{\tilde{d}}(\mathcal{I})]_{-s/\tilde{d}})' & \textit{when } s \leq 0. \end{cases}$

We will select $V_j^{(1,1)} \subset C^1(\mathcal{I}) \cap H^1_{(1,1)}(\mathcal{I})$ and $\tilde{V}_j^{(1,1)}$ such that

$$V_j^{(1,1)} + \dot{V}_j^{(1,1)} + \ddot{V}_j^{(1,1)} \subset \tilde{V}_{j+1}^{(1,1)}. \tag{4.3}$$

Here we use $\dot{V}_j^{(1,1)}$ and $\ddot{V}_j^{(1,1)}$ as notations for the linear spaces of the first or second derivatives of functions in $V_j^{(1,1)}$. Thanks to (4.3), for $\vec{\sigma} = (1,1)$, the bi-infinite matrices

$$\langle \Psi_{\vec{\sigma}}, \Psi_{\vec{\sigma}} \rangle_{L_2(\mathcal{I})}, \langle \dot{\Psi}_{\vec{\sigma}}, \dot{\Psi}_{\vec{\sigma}} \rangle_{L_2(\mathcal{I})}, \langle \dot{\Psi}_{\vec{\sigma}}, \Psi_{\vec{\sigma}} \rangle_{L_2(\mathcal{I})},$$

are truly sparse. With the exception of $\langle \Psi_{\vec{\sigma}}, \Psi_{\vec{\sigma}} \rangle_{L_2(\mathcal{I})}$, as we will see for $\vec{\sigma} = \{0,1\}^2 \backslash (1,1)$ this sparsity will generally be lost for column or row indices corresponding to "boundary adapted" wavelets.

4.1. **A realization.** In this subsection, we construct biorthogonal collections $\Phi_j^{\vec{\sigma}}$ and $\tilde{\Phi}_j^{\vec{\sigma}}$ as meant in Theorem 4.1, and so $V_j^{\vec{\sigma}}$ and $\tilde{V}_j^{\vec{\sigma}}$, for $\vec{\sigma} = (1,1), (0,1)$. The other cases, namely, $\vec{\sigma} = (0,0), (1,0)$, follow easily. The construction for the case $\vec{\sigma} = (1,1)$ was already given in [CS11]. For convenience we recall it here, before we describe the modifications needed for $\vec{\sigma} = (0,1)$. We take

$$V_j^{(1,1)} = \prod_{k=0}^{2^{j+1}-1} P_4(k2^{-(j+1)}, (k+1)2^{-(j+1)}) \cap C^1(\mathcal{I}) \cap H^1_{(1,1)}(\mathcal{I}), \tag{4.4}$$

of dimension $3 \cdot 2^{j+1}$. Then

$$V_j^{(1,1)} + \dot{V}_j^{(1,1)} + \ddot{V}_j^{(1,1)} \subset Z_j := \prod_{k=0}^{2^{j+1}-1} P_4(k2^{-(j+1)}, (k+1)2^{-(j+1)}), \tag{4.5}$$

of dimension $5 \cdot 2^{j+1}$. Following the idea of *intertwining multiresolution analyses* [DGH96], we select $\tilde{V}_{j+1}^{(1,1)}$ as the direct sum of Z_j and a subspace of Z_{j+1} of dimension 2^{j+1}, so that $\dim V_j^{(1,1)} = \dim \tilde{V}_j^{(1,1)}$. Since $(Z_j)_j$ is nested, so is $(\tilde{V}_j^{(1,1)})_j$, and Bernstein and Jackson estimates are satisfied at primal and dual side with parameters $d = \tilde{d} = 5$, $\gamma = \frac{5}{2}$, $\tilde{\gamma} = \frac{1}{2}$.

By setting $\hat{\mathcal{I}} = (-1,1)$ as a *reference macro element*, we consider the interpolatory basis functions $h_i, \ell_j \in P_4(-1,0) \times P_4(0,1) \cap C^1(\hat{\mathcal{I}})$ and $\tilde{\ell}_j \in P_4(\hat{\mathcal{I}})$ with the following properties

$$\begin{array}{ll} \dot{h}_i(\hat{\imath}) = \delta_{i\hat{\imath}}, & \dot{\ell}_j(\hat{\imath}) = 0, \\ h_i(\hat{\jmath}) = 0, & \ell_j(\hat{\jmath}) = \delta_{j\hat{\jmath}}, \end{array} \quad \tilde{\ell}_j(\hat{\jmath}) = \delta_{j\hat{\jmath}}, \qquad (\hat{\imath} \in H, \hat{\jmath} \in L),$$

where $H = \{-1, 0, 1\}$ and $L = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. The pictures of the interpolatory basis functions are given in Figure 3.

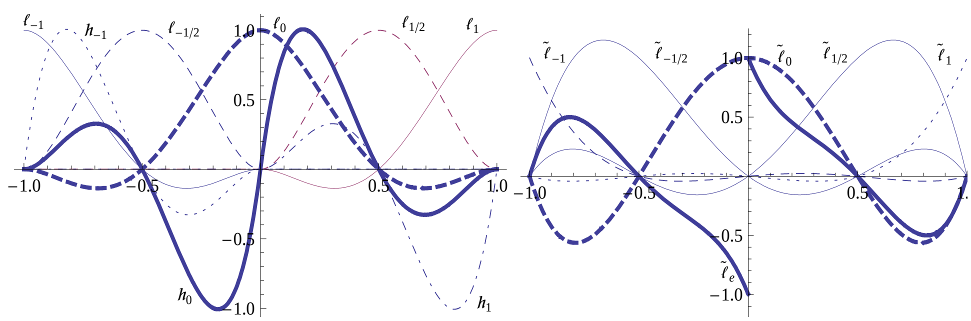

FIGURE 3. Interpolatory basis of $P_4(-1,0) \times P_4(0,1) \cap C^1(\hat{\mathcal{I}})$ (left picture) and that of $P_4(\hat{\mathcal{I}})$ (right picture), with discontinuous function $\tilde{\ell}_{e_1} \in P_4(-1,0) \times P_4(0,1)$ defined in step 2.

We apply a number of transformations to these bases, and enrich the collection at the dual side with one additional function. Then afterwards, the primal and dual collections will be duplicated on each individual "element" in the "finite element" mesh. At the primal side, functions will be "glued" over interfaces between elements in order to obtain C^1 functions.

The enrichment step will be performed by applying number of transformations and projections in the following steps.

Step 1. Select $\hat{h}_{-1}$ from span$\{h_{-1}, \ell_{-\frac{1}{2}}, h_0, \ell_0, \ell_{\frac{1}{2}}\}$ and $\hat{\ell}_{-1}$ from span$\{\ell_{-1}, \ell_{-\frac{1}{2}}, h_0, \ell_0, \ell_{\frac{1}{2}}\}$ such that

$$\langle \hat{h}_{-1}, \tilde{\ell}_{\hat{\jmath}} \rangle_{L_2(\hat{\mathcal{I}})} = \langle \hat{\ell}_{-1}, \tilde{\ell}_{\hat{\jmath}} \rangle_{L_2(\hat{\mathcal{I}})} = \tfrac{1}{2}\delta_{-1,\hat{\jmath}} \quad (\hat{\jmath} \in L).$$

Set $\hat{h}_1(x) = -\hat{h}_{-1}(-x)$ and $\hat{\ell}_1(x) = \hat{\ell}_{-1}(-x)$.

Step 2. Up to an irrelevant scaling, determine $\tilde{\ell}_{e_1} \in P_4(-1,0) \times P_4(0,1)$ uniquely such that it is $L_2(\hat{\mathcal{I}})$-orthogonal to $\{\hat{h}_{-1}, \hat{\ell}_{-1}, \hat{h}_1, \hat{\ell}_1\}$, and such that $\tilde{\ell}_{e_1}$ vanishes at $x = \pm\frac{1}{2}, \pm 1$, and furthermore $\tilde{\ell}_{e_1}(0^+) = -\tilde{\ell}_{e_1}(0^-)$.

Step 3. Define $\{\tilde{\tilde{\ell}}_{-\frac{1}{2}}, \tilde{\tilde{\ell}}_{e_1}, \tilde{\tilde{\ell}}_0, \tilde{\tilde{\ell}}_{\frac{1}{2}}\}$ from $\{\tilde{\ell}_{-\frac{1}{2}}, \tilde{\ell}_{e_1}, \tilde{\ell}_0, \tilde{\ell}_{\frac{1}{2}}\}$ by biorthogonalizing it with $\{\ell_{-\frac{1}{2}}, h_0, \ell_0, \ell_{\frac{1}{2}}\}$.

Step 4. For $j \in \{-1, 1\}$, define

$$\tilde{\tilde{\ell}}_j := \tilde{\ell}_j - \langle \tilde{\ell}_j, \ell_{-\frac{1}{2}} \rangle_{L_2(\hat{\mathcal{I}})} \tilde{\tilde{\ell}}_{-\frac{1}{2}} - \langle \tilde{\ell}_j, h_0 \rangle_{L_2(\hat{\mathcal{I}})} \tilde{\tilde{\ell}}_{e_1} - \langle \tilde{\ell}_j, \ell_0 \rangle_{L_2(\hat{\mathcal{I}})} \tilde{\tilde{\ell}}_0 - \langle \tilde{\ell}_j, \ell_{\frac{1}{2}} \rangle_{L_2(\hat{\mathcal{I}})} \tilde{\tilde{\ell}}_{\frac{1}{2}}.$$

Setting $\hat{\Phi} := \left[\hat{h}_{-1}\ \hat{\ell}_{-1}\ \ell_{-\frac{1}{2}}\ h_0\ \ell_0\ \ell_{\frac{1}{2}}\ \hat{h}_1\ \hat{\ell}_1\right]^\top$ and $\tilde{\hat{\Phi}} := \left[\tilde{\tilde{\ell}}_{-1}\ \tilde{\tilde{\ell}}_{-\frac{1}{2}}\ \tilde{\tilde{\ell}}_e\ \tilde{\tilde{\ell}}_0\ \tilde{\tilde{\ell}}_{\frac{1}{2}}\ \tilde{\tilde{\ell}}_1\right]^\top$, we have

$$\langle \hat{\Phi}, \tilde{\hat{\Phi}} \rangle_{L_2(\hat{\mathcal{I}})} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{4.6}$$

The primal and dual scaling functions on the reference macro element $\hat{\mathcal{I}}$, resulting from steps 1-4, have been illustrated in Figure 4 and their values in terms of the bases from Figure 3 can be found in [CS11, Tables 1 & 2].

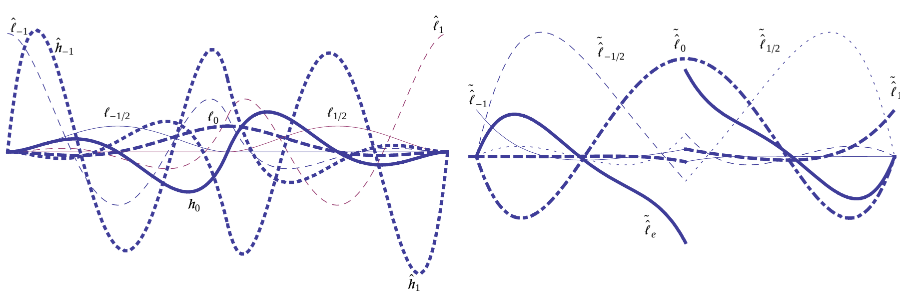

FIGURE 4. Biorthogonal collections $\hat{\Phi}$ and $\tilde{\hat{\Phi}}$ on the reference element.

Having defined primal and dual collections $\hat{\Phi}$ and $\tilde{\hat{\Phi}}$ on the reference macro-element and using (4.6), the collections $\Phi_j^{(1,1)}$ and $\tilde{\Phi}_j^{(1,1)}$ are assembled in the way known from finite elements. First, the collections $\hat{\Phi}$ and $\tilde{\hat{\Phi}}$ are lifted to any macro-element $[k2^{-(j+1)},(k+2)2^{-(j+1)}]$, and multiplied by $2^{(j+1)/2}$ to compensate for the change in length.

At the primal side, the function $\hat{h}_1$ from the left is connected to $\hat{h}_{-1}$ from the right, and $\hat{\ell}_1$ from the left is connected to $\hat{\ell}_{-1}$ from the right. In view of the boundary conditions, at the leftmost boundary the function $\hat{\ell}_{-1}$ is dropped, and at the rightmost boundary so is the function $\hat{\ell}_1$.

In view of the discontinuity between elements, at the dual side no degrees of freedom are identified. Instead a simple basis transformation is applied. At each internal interface, the pair of functions consisting of $\hat{\ell}_1$ from the left and $\hat{\ell}_{-1}$ from the right are replaced by the symmetric and anti-symmetric functions $\hat{\ell}_1+\hat{\ell}_{-1}$ and $-\hat{\ell}_1+\hat{\ell}_{-1}$, with double support lengths. Finally, at the left boundary $\hat{\ell}_{-1}$ is multiplied by 2, and at the right boundary $\hat{\ell}_1$ is multiplied by -2. In view of (4.6), one verifies that the resulting primal and dual collections, denoted as $\Phi_j^{(1,1)}$ and $\tilde{\Phi}_j^{(1,1)}$, are *biorthogonal*.

By construction, $\operatorname{span}\Phi_j^{(1,1)}=V_j^{(1,1)}$ and $Z_j\subset\tilde{V}_j^{(1,1)}:=\operatorname{span}\tilde{\Phi}_j^{(1,1)}\subset Z_{j+1}$. The collections $\Phi_j^{(1,1)}$ and $\tilde{\Phi}_j^{(1,1)}$ are uniformly local and uniformly $L_2(\mathcal{I})$-bounded.

When we try to adapt above construction for $\vec{\sigma}=(1,1)$ to the case that $\vec{\sigma}=(0,1)$, we realize that in this case we should not drop $\hat{\ell}_{-1}$ near the leftmost boundary, meaning that we should add a basis function at the dual side as well.

In view of this we revisit the construction of the primal and dual functions on the reference macro element giving modified collections to be used at the leftmost macro element in the actual mesh.

We start our modification after step 4.

Step 5. Define $\check{\ell}_{-1}:=\hat{\ell}_{-1}-\hat{h}_{-1}$ and $\tilde{\check{\ell}}_{-1}:=2\tilde{\hat{\ell}}_{-1}$. It is clear that $\check{\ell}_{-1}$ is orthogonal to $\{\tilde{\check{\ell}}_{-1},\tilde{\hat{\ell}}_{-\frac{1}{2}},\tilde{\hat{\ell}}_{e_1},\tilde{\hat{\ell}}_0,\tilde{\hat{\ell}}_{\frac{1}{2}},\tilde{\hat{\ell}}_1\}$.

At the dual side, we have to add a second additional function $\tilde{\ell}_{e_2}\in P_4(-1,0)\times P_4(0,1)$. Aiming at well-conditioned wavelet bases, the angle between primal and dual scaling function spaces should be as small as possible. In view of this aim, in the following steps the additional function $\tilde{\ell}_{e_2}$ will be selected such that the spaces $\operatorname{span}\{\hat{h}_{-1},\check{\ell}_{-1},\ell_{-\frac{1}{2}},h_0,\ell_0,\ell_{\frac{1}{2}},\hat{h}_1,\hat{\ell}_1\}$, $\operatorname{span}\{\tilde{\check{\ell}}_{-1},\tilde{\ell}_{e_2},\tilde{\hat{\ell}}_{-\frac{1}{2}},\tilde{\hat{\ell}}_{e_1},\tilde{\hat{\ell}}_0,\tilde{\hat{\ell}}_{\frac{1}{2}},\tilde{\hat{\ell}}_1\}$ are close.

Step 6. Set $S:=\operatorname{span}\{\hat{h}_{-1},\check{\ell}_{-1},\ell_{-\frac{1}{2}},h_0,\ell_0,\ell_{\frac{1}{2}}\}$, $\tilde{S}:=\operatorname{span}\{\tilde{\check{\ell}}_{-1},\tilde{\hat{\ell}}_{-\frac{1}{2}},\tilde{\hat{\ell}}_{e_1},\tilde{\hat{\ell}}_0,\tilde{\hat{\ell}}_{\frac{1}{2}}\}$. The space S can be decomposed as $S=T\oplus T^{\perp}$ with T being the orthogonal projection of $\operatorname{span}\tilde{S}$ onto $\operatorname{span}S$. Note that the subspace $T^{\perp}$ is an one dimensional space.

Step 7. Now consider the three dimensional subspace U of $P_4(-1,0)\times P_4(0,1)$ that is orthogonal to $\operatorname{span}\{\hat{h}_{-1},\ell_{-\frac{1}{2}},h_0,\ell_0,\ell_{\frac{1}{2}},\hat{h}_1,\hat{\ell}_1\}$. Define $\tilde{\ell}_{e_2}\in U$ such that $\operatorname{span}\{\tilde{\ell}_{e_2}\}$ is the image of $T^{\perp}$ under the orthogonal projection onto U, and furthermore such that $\langle\tilde{\ell}_{e_2},\check{\ell}_{-1}\rangle_{L_2(\hat{\mathcal{I}})}=1$.

The second additional dual function $\tilde{\ell}_{e_2}$ is illustrated in Figure 5. It happens to be continuous. Its values at $\frac{1}{4}\mathbb{Z}\cap[-1,1]$ are presented in Table 1. From steps 5-7, we conclude that

$$
\left\langle \begin{bmatrix} \hat{h}_{-1} \\ \check{\ell}_{-1} \\ \ell_{-\frac{1}{2}} \\ h_0 \\ \ell_0 \\ \ell_{\frac{1}{2}} \\ \hat{h}_1 \\ \hat{\ell}_1 \end{bmatrix}, \begin{bmatrix} \tilde{\check{\ell}}_{-1} \\ \tilde{\ell}_{e_2} \\ \tilde{\hat{\ell}}_{-\frac{1}{2}} \\ \tilde{\hat{\ell}}_{e_1} \\ \tilde{\hat{\ell}}_0 \\ \tilde{\hat{\ell}}_{\frac{1}{2}} \\ \tilde{\hat{\ell}}_1 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \tag{4.7}
$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L_2(\hat{\mathcal{I}})}$.

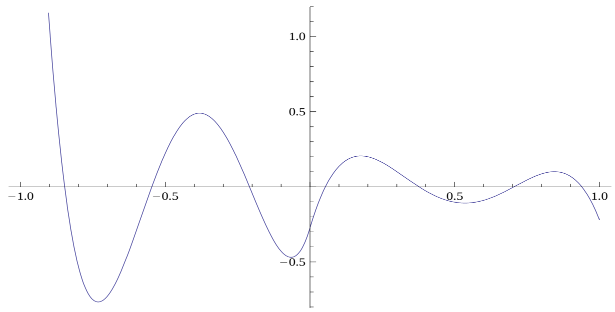

FIGURE 5. The second additional function $\tilde{\ell}_{e_2} \in P_4(-1, 0) \times P_4(0, 1)$.

	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\tilde{\ell}_{e_2}$	$\frac{276}{55}$	$-\frac{1059}{1408}$	$\frac{9}{40}$	$\frac{1281}{7040}$	$-\frac{3}{11}$	$\frac{1137}{7040}$	$-\frac{9}{88}$	$\frac{321}{7040}$	$-\frac{12}{55}$

TABLE 1. Values of $\tilde{\ell}_{e_2}$ at $\frac{1}{4}\mathbb{Z} \cap [-1, 1]$.

In view of (4.7), we define the biorthogonal collections

$$
\Phi_j^{(0,1)} := \Phi_j^{(1,1)} \cup \{\phi_{j,1}^{(0,1)}\},
$$
$$
\tilde{\Phi}_j^{(0,1)} := \tilde{\Phi}_j^{(1,1)} \cup \{\tilde{\phi}_{j,1}^{(0,1)}\},
$$

where $\phi_{j,1}^{(0,1)}$ and $\tilde{\phi}_{j,1}^{(0,1)}$ are defined as $\check{\ell}_{-1}$ and $\tilde{\ell}_{e_2}$ lifted to the leftmost macro element and multiplied by $2^{\frac{j+1}{2}}$, respectively.

We define $V_j^{(0,1)} := \operatorname{span} \Phi_j^{(0,1)}$ and set $\tilde{V}_j^{(0,1)} := \operatorname{span} \tilde{\Phi}_j^{(0,1)}$ so that $\tilde{V}_j^{(0,1)}$ is the direct sum of Z_j and a subspace of Z_{j+1}.

The collections $\Phi_j^{\vec{\sigma}}$ and $\Phi_j^{\vec{\sigma}}$ ($\vec{\sigma} = (1, 1), (1, 0)$) are uniformly local and uniformly $L_2(0, 1)$-bounded. Together with the biorthogonality, it implies that $\Phi_j^{\vec{\sigma}}$ and $\Phi_j^{\vec{\sigma}}$ ($\vec{\sigma} = (1, 1), (1, 0)$) are uniform $L_2(0, 1)$-Riesz bases for $V_j^{\vec{\sigma}}$ and $\tilde{V}_j^{\vec{\sigma}}$, respectively.

We note that the basis transformations between the interpolatory basis for $V_j^{\vec{\sigma}}$ and the basis $\Phi_j^{\vec{\sigma}}$ ($\vec{\sigma} = (1, 1), (0, 1)$) are uniformly local.

4.2. **Wavelets.** Since the spaces $V_j^{\vec{\sigma}}$ and $\tilde{V}_j^{\vec{\sigma}}$ have already been equipped with uniformly local, biorthogonal uniform $L_2(\mathcal{I})$-Riesz bases $\Phi_j^{\vec{\sigma}}$ and $\tilde{\Phi}_j^{\vec{\sigma}}$, for the construction of suitable biorthogonal wavelets the following result can be applied. For a proof we refer to [Ste03].

Proposition 4.2. *Let $\Xi_{j+1}^{\vec{\sigma}} \subset V_{j+1}^{\vec{\sigma}}$ be such that $\Phi_j^{\vec{\sigma}} \cup \Xi_{j+1}^{\vec{\sigma}}$ is a uniform $L_2(\mathcal{I})$-Riesz basis for $V_{j+1}^{\vec{\sigma}}$, and such that the basis transformations from $\Phi_j^{\vec{\sigma}} \cup \Xi_{j+1}^{\vec{\sigma}}$ to $\Phi_{j+1}^{\vec{\sigma}}$ and from $\Phi_{j+1}^{\vec{\sigma}}$ to $\Phi_j^{\vec{\sigma}} \cup \Xi_{j+1}^{\vec{\sigma}}$ are uniformly sparse. Then*

$$\Psi_{j+1}^{\vec{\sigma}} := \Xi_{j+1}^{\vec{\sigma}} - \langle \Xi_{j+1}^{\vec{\sigma}}, \tilde{\Phi}_j^{\vec{\sigma}} \rangle_{L_2(\mathcal{I})} \Phi_j^{\vec{\sigma}}, \tag{4.8}$$

and its unique dual collection $\tilde{\Psi}_{j+1}^{\vec{\sigma}}$ in $\tilde{V}_{j+1}^{\vec{\sigma}} \cap V_j^{\vec{\sigma}^{\perp_{L_2(\mathcal{I})}}}$ are biorthogonal, uniformly local, uniform $L_2(\mathcal{I})$-Riesz bases for $V_{j+1}^{\vec{\sigma}} \cap \tilde{V}_j^{\vec{\sigma}^{\perp_{L_2(\mathcal{I})}}}$ and $\tilde{V}_{j+1}^{\vec{\sigma}} \cap V_j^{\vec{\sigma}^{\perp_{L_2(\mathcal{I})}}}$, respectively.

In view of Proposition 4.2, in order to define the wavelets, it is sufficient to construct $\Xi_{j+1}^{\vec{\sigma}}$ such that $\Phi_j^{\vec{\sigma}} \cup \Xi_{j+1}^{\vec{\sigma}}$ is a uniform $L_2(\mathcal{I})$-Riesz basis for $V_{j+1}^{\vec{\sigma}}$. Moreover, in order to obtain uniformly local wavelets with uniformly local duals, we will select $\Xi_{j+1}^{\vec{\sigma}}$ such that the basis transformations between $\Phi_{j+1}^{\vec{\sigma}}$ and $\Phi_j^{\vec{\sigma}} \cup \Xi_{j+1}^{\vec{\sigma}}$ are uniformly local. Since the basis transformations between the interpolatory basis of $V_j^{\vec{\sigma}}$ and the basis $\Phi_j^{\vec{\sigma}}$ are uniformly local, the latter condition is equivalent to the locality of the basis transformations between the interpolatory basis for $V_{j+1}^{\vec{\sigma}}$ and the union of the interpolatory basis for $V_j^{\vec{\sigma}}$ and $\Xi_{j+1}^{\vec{\sigma}}$.

A natural choice for $\Xi_{j+1}^{\vec{\sigma}}$ is the subset of interpolatory basis functions for $V_{j+1}^{\vec{\sigma}}$ that correspond to the new degrees of freedom. With this choice, the basis transformations mentioned in Proposition 4.2 are uniformly local. Indeed, with I_ℓ being the canonical interpolation operator onto $V_\ell^{\vec{\sigma}}$, the argument is that for $u_{j+1} \in V_{j+1}^{\vec{\sigma}}$ the computation of the splitting $u_{j+1} = I_j u_{j+1} + I_{j+1}(u_{j+1} - I_j u_{j+1})$ requires local quantities only.

In order to reduce the support size of most of the resulting wavelets, we do not simply take $\Xi_{j+1}^{\vec{\sigma}}$ to be the above collection, but we construct it from that collection by applying a uniformly local transformation with uniformly local inverse. Our aim is to ensure that most functions in $\Xi_{j+1}^{\vec{\sigma}}$ are orthogonal to those duals scaling functions that correspond to primal scaling functions that have supports that extend to more than one macro-element. The resulting wavelets will then have no components in the directions of those scaling functions.

4.2.1. *The case $\vec{\sigma} = (1,1)$.* We recall the construction of $\Xi_{j+1}^{\vec{\sigma}}$ for the case $\vec{\sigma} = (1,1)$ already discussed in [CS11]. Similar to the previous subsection, it is sufficient to specify $\hat{\Xi}$ such that $\hat{\Phi} \cup \hat{\Xi}$ is a basis for $\sum_{p=-2}^{1} P_4(p/2, (p+1)/2) \cap C(\hat{\mathcal{I}})$. Then by lifting $\hat{\Xi}$ to any macro-element $[k2^{-(j+1)}, (k+2)2^{-(j+1)}]$ multiplying it by $2^{(j+1)/2}$, and taking the union over all macro-elements, the collection $\Xi_{j+1}^{(1,1)}$ is obtained. Since the function values and first order derivatives of any function in $\hat{\Xi}$ will vanish at $\partial\hat{\mathcal{I}}$, no degrees of freedom will have to be identified over the interfaces.

Let $\bar{H} = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ and $\bar{L} = \{-1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. We define the collection $\hat{\Sigma} := \{\bar{h}_i : \bar{H} \setminus H\} \cup \{\bar{\ell}_j : j \in \bar{L} \setminus L\} \subset \sum_{p=-2}^{1} P_4(p/2, (p+1)/2) \cap C(\hat{\mathcal{I}})$ as follows

$$\begin{aligned} \dot{\bar{h}}_i(\hat{\imath}) &= \delta_{i\hat{\imath}}, & \dot{\bar{\ell}}_j(\hat{\imath}) &= 0, \\ \bar{h}_i(\hat{\jmath}) &= 0, & \bar{\ell}_j(\hat{\jmath}) &= \delta_{j\hat{\jmath}}, \end{aligned} \qquad (\hat{\imath} \in \bar{H}, \hat{\jmath} \in \bar{L}),$$

So $\hat{\Sigma}$ consists of the interpolatory basis functions corresponding to the new degrees of freedom.

We will define $\hat{\Xi}$ by applying some basis transformations to $\hat{\Sigma}$. In view of our aforementioned aim, we look for $\hat{\Xi}$ such that only one of its elements is not orthogonal to $\tilde{\hat{\ell}}_1$ and only one other elements is not orthogonal to $\tilde{\hat{\ell}}_{-1}$.

Step 1. Determine $\hat{\xi}_{\pm\frac{1}{2}}$ as the best approximation to $\tilde{\hat{\ell}}_{\pm 1}$ from span $\hat{\Sigma}$, i.e.,

$$\hat{\xi}_{\pm\frac{1}{2}} = \langle \hat{\Sigma}, \tilde{\hat{\ell}}_{\pm 1} \rangle_{L_2(\hat{\mathcal{I}})} \langle \hat{\Sigma}, \hat{\Sigma} \rangle_{L_2(\hat{\mathcal{I}})}^{-1} \hat{\Sigma},$$

and then redefine the obtained $\{\hat{\xi}_{\pm\frac{1}{2}}\}$ by biorthogonalizing it with $\{\tilde{\hat{\ell}}_{\pm 1}\}$.

Step 2. Select $\{\hat{\xi}_{\pm\frac{3}{4}}, \hat{\xi}_{\pm\frac{1}{4}}\}$ from span $\hat{\Sigma} \cap \operatorname{span}\{\tilde{\hat{\ell}}_{\pm 1}\}^{\perp_{L_2(\hat{\mathcal{I}})}}$ by means of

$$\hat{\xi}_p := \bar{\ell}_p - \langle \bar{\ell}_p, \tilde{\hat{\ell}}_{-1} \rangle_{L_2(\bar{\mathcal{I}})} \hat{\xi}_{-\frac{1}{2}} - \langle \bar{\ell}_p, \tilde{\hat{\ell}}_1 \rangle_{L_2(\bar{\mathcal{I}})} \hat{\xi}_{\frac{1}{2}} \quad (p \in \{\pm\tfrac{3}{4}, \pm\tfrac{1}{4}\}).$$

Now it holds that span $\hat{\Xi} =$ span $\hat{\Sigma}$. The coefficients of $\hat{\Xi}$ in terms of $\hat{\Sigma}$ can be found in [CS11, Table 3].

The application of Proposition 4.2 to $\Phi_j^{(1,1)}$, $\tilde{\Phi}_j^{(1,1)}$, and $\Xi_{j+1}^{(1,1)}$ gives the wavelet collection $\Psi_{j+1}^{(1,1)}$. The wavelets from $\Psi_{j+1}^{(1,1)}$ can be subdivided into 4 categories:

W1 Groups of four "interior wavelets", which vanish outside $((k-1)2^{-j}, k2^{-j})$ for $k = 1, \cdots, 2^j$, and that are equal to each other modulo shifts,

W2 groups of two "interface wavelets", which vanish outside $((k-1)2^{-j}, (k+1)2^{-j})$ for $k = 1, \cdots, 2^j - 1$, and that are equal to each other modulo shifts,

–all these wavelets will be found by the construction for each $\vec{\sigma} \in \{0,1\}^2 \backslash (1,1)$–, and

W3 "right boundary wavelet" that vanishes outside $(1 - 2^{-j}, 1)$,

–this wavelet is also found by the construction for $\vec{\sigma} = (0,1)$–, and

W4 "left boundary wavelet" that vanishes outside $[0, 2^{-j})$,

–this wavelet is also found by the construction for $\vec{\sigma} = (1,0)$–.

To improve the conditioning of the collection $\Psi_{j+1}^{(1,1)}$, we use the following local transformations:

- Make **W1** mutually orthogonal.
- Make **W2** orthogonal to **W1**, and after that, make **W2** mutually orthogonal.
- Make **W3** and **W4** orthogonal to **W1**.

We note that since these local transformations and their inverses are local, the resulting collections $\Psi_j^{(1,1)}$ and $\tilde{\Psi}_j^{(1,1)}$ are uniformly local. The wavelets **W1**–**W4** are illustrated in [CS11, Figure 4].

Finally, in order to improve the conditioning of the basis in norms other than the $L_2(\mathcal{I})$-norm, in our definition of the wavelet collection $\Psi_{(1,1)}$, we replace the single-scale basis $\Phi_0^{(1,1)}$ for $V_0^{(1,1)}$ by a biorthogonal three-scale basis. We set $V_{-1}^{(1,1)} := P_4(\mathcal{I}) \cap H_0^1(\mathcal{I})$, $V_{-2}^{(1,1)} := \operatorname{span}\{x(1-x)\}$, $\tilde{V}_{-1}^{(1,1)} := P_2(\mathcal{I})$, and $\tilde{V}_{-2}^{(1,1)} := P_0(\mathcal{I})$, and construct a basis for $V_0^{(1,1)}$ as the union of the $L_2(\mathcal{I})$-orthogonal bases for $V_{-2}^{(1,1)}$, $V_{-1}^{(1,1)} \cap \tilde{V}_{-2}^{(1,1)^{\perp_{L_2(\mathcal{I})}}}$, and $V_0^{(1,1)} \cap \tilde{V}_{-1}^{(1,1)^{\perp_{L_2(\mathcal{I})}}}$. The resulting basis is illustrated in [CS11, Figure 5].

4.2.2. *The case $\vec{\sigma} = (0,1)$.* For this case, we revisit the construction in steps 1 and 2 for $\vec{\sigma} = (1,1)$ with the role of $\tilde{\tilde{\ell}}_{-1}$ now being played by $\tilde{\ell}_{e_2}$. As a consequence, only one wavelet will be non-zero at the left boundary, and so has to be reflected in case an extension is applied. The resulting set is denoted as $\check{\Xi}$.

The collection $\Xi_{j+1}^{(0,1)}$ is now obtained by lifting $\check{\Xi}$ to the leftmost macro-element, lifting $\hat{\Xi}$ to the remaining macro-elements, and by applying the some scaling as with $\Xi_{j+1}^{(1,1)}$.

Using $\Phi_j^{(0,1)}$, $\tilde{\Phi}_j^{(0,1)}$, and $\Xi_{j+1}^{(0,1)}$, the wavelets collections $\Psi_{j+1}^{(0,1)}$ are determined by Proposition 4.2. The wavelets from $\Psi_{j+1}^{(0,1)}$ can be subdivided into 6 categories that are **W1–W3**, together with

- **W5** four "leftmost interior wavelets" that vanish outside $(0, 2^{-j})$,
- **W6** two "leftmost interface wavelets" that vanish outside $(0, 2 \cdot 2^{-j})$,
- **W7** "left boundary wavelet" that vanishes outside $[0, 2^{-j})$.

In order to improve the conditioning of the resulting wavelets collection $\Psi_{j+1}^{(0,1)}$, we apply the following local transformations to **W5–W7**:

- Make **W5** mutually orthogonal. The wavelets resulting of this step, denoted as $\psi_i^L, i = 2, \ldots, 5$, are illustrated in Figure 6.
- Make **W6** orthogonal to **W1** and **W5**, and after that, make **W6** mutually orthogonal. The modified leftmost interface wavelets **W6**, denoted by ψ_6^L, ψ_7^L, are illustrated in Figure 6.
- Make **W7** orthogonal to **W5**. The left boundary wavelet, denoted by ψ_1^L, is illustrated in Figure 6.

Note that the resulting collections $\Psi_j^{(0,1)}$ and $\tilde{\Psi}_j^{(0,1)}$ are uniformly local.

Finally, in order to improve the conditioning of $\Psi_{(0,1)}$, we replace the single-scale basis $\Phi_0^{(0,1)}$ for $V_0^{(0,1)}$ by a biorthogonal three-scale basis. We define $V_{-2}^{(0,1)} := P_2(-1,1) \cap H^1_{0,\{1\}}(\hat{\mathcal{I}})$ and $V_{-1}^{(0,1)} := P_4(-1,1) \cap H^1_{0,\{1\}}(\hat{\mathcal{I}})$. On the dual side, we set $\tilde{V}_{-2}^{(0,1)} := P_1(-1,1)$ and $\tilde{V}_{-1}^{(0,1)} := P_3(-1,1)$. A basis for $V_0^{(0,1)}$ can be constructed by the union of $L^2(\hat{\mathcal{I}})$-orthogonal bases $\{\phi_1, \phi_2\}$ for $V_{-2}^{(0,1)}$, $\{\phi_3, \phi_4\}$ for $V_{-1}^{(0,1)} \cap \tilde{V}_{-2}^{(0,1)^{\perp_{L_2(\hat{\mathcal{I}})}}}$, and $\{\phi_5, \phi_6, \phi_7\}$ for $V_0^{(0,1)} \cap \tilde{V}_{-1}^{(0,1)^{\perp_{L_2(\hat{\mathcal{I}})}}}$. The resulting basis $\{\phi_i^{(0,1)} : 1 \leq i \leq 7\}$ is illustrated in Figure 7, and the values of the basis functions at $\frac{1}{8}\mathbb{N} \cap [0,1]$ are given in Table 2.

4.2.3. *The case $\vec{\sigma} = (1,0)$.* We define $\Psi_{(1,0)}(\cdot) := \Psi_{(0,1)}(1 - \cdot)$ as being the Riesz basis for the space $H^1_{(1,0)}(\mathcal{I})$.

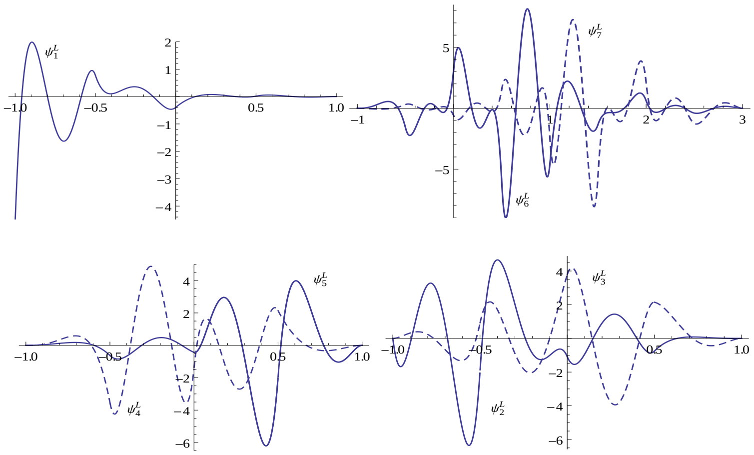

FIGURE 6. The 'leftmost mother' wavelets $\{\psi_i^L : i = 1, \ldots, 7\}$ on $[-1, 3]$ for the wavelet basis $\Psi^{(0,1)}$.

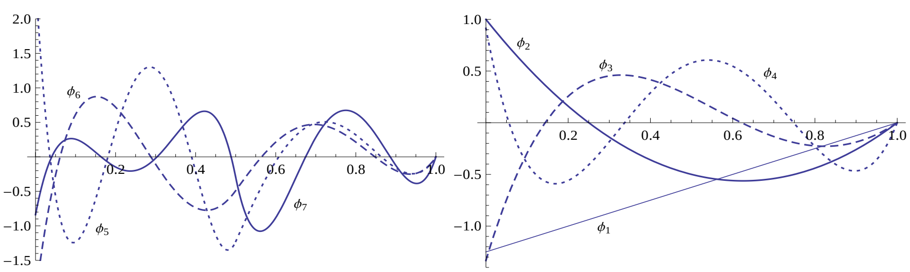

FIGURE 7. The three-scale basis $\{\phi_i = \phi^{(0,1)} : 1 \leq i \leq 7\}$ for $V_0^{(0,1)}$.

	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
ϕ_1	-2	$-\frac{7}{4}$	$-\frac{3}{2}$	$-\frac{5}{4}$	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$
ϕ_2	1	$\frac{7}{16}$	0	$-\frac{5}{16}$	$-\frac{1}{2}$	$-\frac{9}{16}$	$-\frac{1}{2}$	$-\frac{5}{16}$
ϕ_3	$-\frac{2}{3}$	$-\frac{119}{2048}$	$\frac{25}{128}$	$\frac{1355}{6144}$	$\frac{1}{8}$	$-\frac{7}{2048}$	$-\frac{37}{384}$	$-\frac{215}{2048}$
ϕ_4	$\frac{7}{115}$	$-\frac{16177}{471040}$	$-\frac{801}{29440}$	$\frac{1139}{94208}$	$\frac{71}{1840}$	$\frac{15327}{471040}$	$-\frac{17}{29440}$	$-\frac{14161}{471040}$
ϕ_5	$-\frac{862}{1715}$	$\frac{354667}{1756160}$	$\frac{1341}{15680}$	$-\frac{40339}{250880}$	$-\frac{839}{6860}$	$\frac{28863}{351232}$	$\frac{2159}{21952}$	$-\frac{1447}{50176}$
ϕ_6	$\frac{81912}{6347833}$	$-\frac{8307779}{1625045248}$	$\frac{572319}{101565328}$	$\frac{2466453}{1625045248}$	$-\frac{38981}{6347833}$	$\frac{1166781}{1625045248}$	$\frac{244307}{101565328}$	$-\frac{446411}{1625045248}$
ϕ_7	$-\frac{11}{1050}$	$\frac{4871}{2150400}$	$-\frac{331}{134400}$	$\frac{13103}{2150400}$	$-\frac{31}{8400}$	$-\frac{789}{102400}$	$\frac{151}{19200}$	$\frac{377}{307200}$

TABLE 2. Values of $\{\phi_i^{(0,1)} : 1 \leq i \leq 7\}$ at $\frac{1}{8}\mathbb{N} \cap [0, 1]$.

4.2.4. *The case* $\vec{\sigma} = (0, 0)$. The collection $\Psi_{j+1}^{(0,0)}$ is obtained by taking the 7 "leftmost" wavelets from $\Psi_{j+1}^{(0,1)}$, the 7 "rightmost" wavelets from $\Psi_{j+1}^{(1,0)}$, and by adding those that are in $\Psi_{j+1}^{(1,0)} \cap \Psi_{j+1}^{(0,1)}$.

Again, in order to improve the conditioning of $\Psi_{(0,0)}$, we replace the single-scale basis $\Phi_0^{(0,0)}$ for $V_0^{(0,0)}$ by a biorthogonal three-scale basis. We define $\tilde{V}_{-2}^{(0,0)} = V_{-2}^{(0,0)} := P_1(-1,1)$ and $\tilde{V}_{-1}^{(0,0)} = V_{-1}^{(0,0)} := P_4(-1,1)$. A basis for $V_0^{(0,0)}$ can be constructed by the union of $L^2(\hat{\mathcal{I}})$-orthogonal bases $\{\varphi_1, \varphi_2\}$ for $V_{-2}^{(0,0)}$, $\{\varphi_3, \varphi_4, \varphi_5\}$ for $V_{-1}^{(0,0)} \cap \tilde{V}_{-2}^{(0,0)\perp_{L_2(\hat{\mathcal{I}})}}$, and $\{\varphi_6, \varphi_7, \varphi_8\}$ for $V_0^{(0,0)} \cap \tilde{V}_{-1}^{(0,0)\perp_{L_2(\hat{\mathcal{I}})}}$. The resulting basis $\{\varphi_i : 1 \leq i \leq 8\}$ is illustrated in Figure 8, and the values of the basis functions at $\frac{1}{8}\mathbb{N} \cap [0,1]$ are given in Table 3.

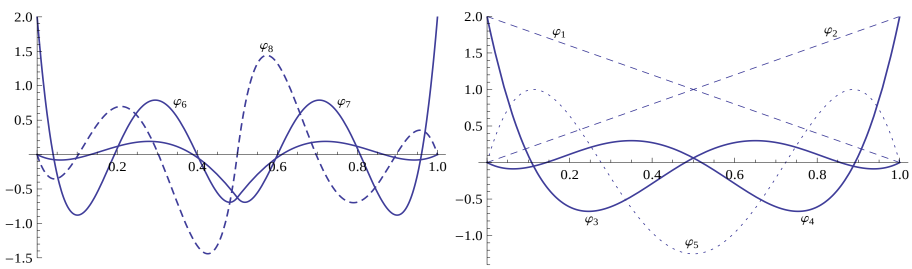

FIGURE 8. The three-scale basis $\{\varphi_i = \varphi_i^{(0,0)} : 1 \leq i \leq 8\}$ for $V_0^{(0,0)}$.

	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
ϕ_1	1	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0
ϕ_2	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
ϕ_3	2	$-\frac{707}{4096}$	$-\frac{171}{256}$	$-\frac{1555}{4096}$	$\frac{1}{16}$	$\frac{1197}{4096}$	$\frac{53}{256}$	$-\frac{131}{4096}$	0
ϕ_4	0	$-\frac{131}{4096}$	$\frac{53}{256}$	$\frac{1197}{4096}$	$\frac{1}{16}$	$-\frac{1555}{4096}$	$-\frac{171}{256}$	$-\frac{707}{4096}$	2
ϕ_5	0	$\frac{203}{4096}$	$\frac{3}{256}$	$-\frac{165}{4096}$	$-\frac{1}{16}$	$-\frac{165}{4096}$	$\frac{3}{256}$	$\frac{203}{4096}$	0
ϕ_6	2	$-\frac{1613}{2048}$	$\frac{79}{128}$	$\frac{595}{2048}$	$-\frac{5}{8}$	$\frac{99}{2048}$	$\frac{23}{128}$	$-\frac{29}{2048}$	0
ϕ_7	0	$-\frac{29}{2048}$	$\frac{23}{128}$	$\frac{99}{2048}$	$-\frac{5}{8}$	$\frac{595}{2048}$	$\frac{79}{128}$	$-\frac{1613}{2048}$	2
ϕ_8	0	$\frac{3}{256}$	$\frac{1}{32}$	$-\frac{15}{256}$	0	$\frac{15}{256}$	$-\frac{1}{32}$	$-\frac{3}{256}$	0

TABLE 3. Values of $\{\varphi_i^{(0,0)} : 1 \leq i \leq 8\}$ at $\frac{1}{8}\mathbb{N} \cap [0,1]$.

Finally, from Corollary 3.4 recall that our construction of a 'uniform' (wavelet) Riesz basis for $H^1_{0,\Gamma}(\Omega)$ using extension operators from (wavelet) Riesz bases Ψ_j on the subdomains $\square_j$ requires that, properly scaled, Ψ_j is a Riesz basis for $H_\sigma(\square_j)$, equipped with squared norm $\mu_i \|\cdot\|^2_{L_2(\square_j)} + \kappa_i |\cdot\|^2_{H^1(\square_j)}$, with Riesz constants that are bounded uniformly in $0 < \mu_i \lesssim \kappa_i$. Here $\boldsymbol{\sigma} = \boldsymbol{\sigma}(j) \in (\{0,1\}^n)^2$ encodes the boundary conditions on subdomain j.

As we discussed, for $\boldsymbol{\sigma} = (\vec{\sigma}_i)_{1 \leq i \leq n} \neq \vec{0}^n$ Friedrich's inequality implies that it is sufficient when, properly scaled, Ψ_j is a Riesz basis for $H_\sigma(\square_j)$ equipped with the standard $H^1(\square_j)$-norm. On the previous pages we constructed such a basis of the form $\otimes_{i=1}^n \Psi_{\vec{\sigma}_i}$ (cf. (3.4)) with $\Psi_{\vec{\sigma}_i}$ being, properly scaled, a Riesz basis for $L_2(\mathcal{I})$ and $H^1_{\vec{\sigma}_i}(\mathcal{I})$.

For $\vec{\sigma} = \vec{0}^n$, i.e. no boundary conditions on subdomain $\square_j$, it was shown in Proposition 2.6 that it is sufficient (actually it is also needed) that, properly scaled, Ψ_j is a Riesz basis for $H^1(\square_j)$ equipped with the standard $H^1(\square_j)$-norm, which is of type $\Psi_j = \{\mathbb{1}\} \cup \Psi_j \setminus \{\mathbb{1}\}$ with $\Psi_j \setminus \{\mathbb{1}\} \subset H^1(\square_j)/\mathbb{R}$. In view of our construction of Ψ_j of the form $\otimes_{i=1}^n \Psi_{(0,0)}$, it is therefore needed that the collection of univariate wavelets $\Psi_{(0,0)}$ is of type $\Psi_{(0,0)} = \{\mathbb{1}\} \cup \Psi_{(0,0)} \setminus \{\mathbb{1}\}$ with $\Psi_{(0,0)} \setminus \{\mathbb{1}\} \subset H^1(\mathcal{I})/\mathbb{R}$. The basis that we just have constructed is *not* of this type, but after replacing $\{\phi_1^{(0,0)}, \phi_2^{(0,0)}\}$ by $\{\phi_{1,\text{new}}^{(0,0)}, \phi_{2,\text{new}}^{(0,0)}\} := \{\phi_1^{(0,0)} + \phi_2^{(0,0)}, \phi_1^{(0,0)} - \phi_2^{(0,0)}\}$ it is.

Both these two new basis functions, however, are non-zero at both boundaries, and so the resulting collection $\Psi_{(0,0)}$ does not satisfy $(\mathcal{W}_4)$ for some $\rho < 1$. This condition was imposed in order that the biorthogonal pair $(\Psi_{(0,0)}, \tilde{\Psi}_{(0,0)})$ could be used to define the scale dependent univariate extension operator G_1 in (3.13).

To solve this problem, we will use the original biorthogonal pair $(\Psi_{(0,0)}, \tilde{\Psi}_{(0,0)})$ for defining G_1, and the new collection $\Psi_{(0,0)}$ for the definition of the tensor product wavelet basis for $H^1(\square_j)$. In effect it means that the univariate basis functions $\phi_{1,\text{new}}^{(0,0)}(= \mathbb{1})$, $\phi_{2,\text{new}}^{(0,0)}$ are not extended by reflection over the left or right boundary, but after expressing them as multiples of the original $\phi_1^{(0,0)}$ and $\phi_2^{(0,0)}$, the resulting multiple of $\phi_1^{(0,0)}$ or $\phi_2^{(0,0)}$ is reflected over the left- or right-boundary, respectively, see Figure 9.

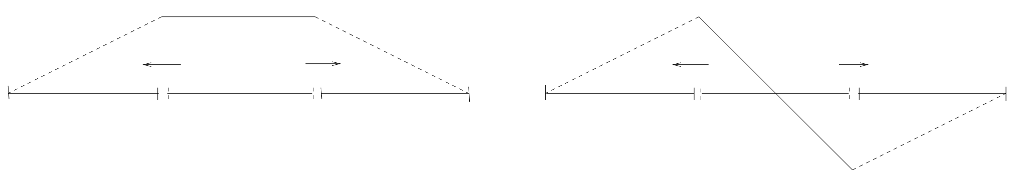

FIGURE 9. Extensions of the basis function $\phi_{1,\text{new}}^{(0,0)}$ (left), $\phi_{2,\text{new}}^{(0,0)}$ (right) over left and right boundaries.

4.3. **Dual wavelets.** Finally in this section, we will illustrate some dual wavelet functions that belong to the collections $\tilde{\Psi}_{\vec{\sigma}}$, $\vec{\sigma} = (1,1), (0,0)$. From Proposition 4.2, we conclude that there exist uniformly boundedly invertible matrices $M_j^{\vec{\sigma}}$ and $\tilde{M}_j^{\vec{\sigma}}$ ($j \in \mathbb{N}_0$) such that

$$\begin{aligned}
[(\Phi_j^{\vec{\sigma}})^\top \quad (\Psi_{j+1}^{\vec{\sigma}})^\top] &= (\Phi_{j+1}^{\vec{\sigma}})^\top M_j^{\vec{\sigma}}, \\
[(\tilde{\Phi}_j^{\vec{\sigma}})^\top \quad (\tilde{\Psi}_{j+1}^{\vec{\sigma}})^\top] &= (\tilde{\Phi}_{j+1}^{\vec{\sigma}})^\top \tilde{M}_j^{\vec{\sigma}}.
\end{aligned}$$

By setting $\left[\begin{array}{cc} \tilde{M}_{j,0}^{\vec{\sigma}} & \tilde{M}_{j,1}^{\vec{\sigma}} \end{array}\right] := \tilde{M}_j^{\vec{\sigma}}$, it holds that

$$\tilde{\Phi}_j^{\vec{\sigma}} = (\tilde{M}_{j,0}^{\vec{\sigma}})^\top \tilde{\Phi}_{j+1}^{\vec{\sigma}}, \; \tilde{\Psi}_{j+1}^{\vec{\sigma}} = (\tilde{M}_{j,1}^{\vec{\sigma}})^\top \tilde{\Phi}_{j+1}^{\vec{\sigma}}.$$

On account of biorthogonality between primal collections $\Phi_j^{\vec{\sigma}}$ and $\Psi_j^{\vec{\sigma}}$ and their duals $\tilde{\Phi}_j^{\vec{\sigma}}$ and $\tilde{\Psi}_j^{\vec{\sigma}}$, the following relations hold

$$M_j^{\vec{\sigma}} = \langle \tilde{\Phi}_{j+1}^{\vec{\sigma}}, [(\Phi_j^{\vec{\sigma}})^\top \quad (\Psi_{j+1}^{\vec{\sigma}})^\top]^\top \rangle_{L_2(\mathcal{I})}, \quad \tilde{M}_j^{\vec{\sigma}} = (M_j^{\vec{\sigma}})^{-\top}.$$

Since $\Phi_j^{\vec{\sigma}}$, $\Psi_j^{\vec{\sigma}}$ and their corresponding dual collections are uniformly local, $M_j^{\vec{\sigma}}$ and $\tilde{M}_j^{\vec{\sigma}}$ are uniformly sparse. For $\vec{\sigma} = (0,1)$ and $j = 4$, $M_j^{\vec{\sigma}}$ and $\tilde{M}_j^{\vec{\sigma}}$ are illustrated in Figure 11. Dual wavelets $\tilde{\psi}_\lambda^{(1,1)}$ for $\lambda \in \nabla_0^{(\ell)} \cup \nabla^{(I)}$, and $\tilde{\psi}_\lambda^{(0,1)}$ for $\lambda \in \nabla_0^{(\ell)}$ are illustrated in Figure 10.

FIGURE 10. Two rows at top: $\tilde{\psi}_\lambda^{(0,1)}$, $\lambda \in \nabla_0^{(\ell)}$. Two rows at middle: $\tilde{\psi}_\lambda^{(1,1)}$, $\lambda \in \nabla_0^{(\ell)}$. Two rows at bottom: $\tilde{\psi}_\lambda^{(1,1)}$, $\lambda \in \nabla^{(I)}$.

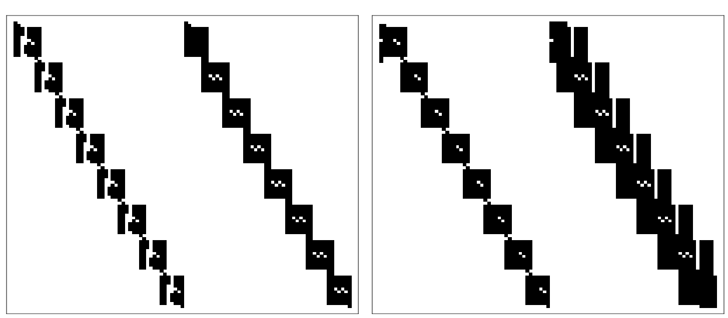

FIGURE 11. Non-zero block structures of $M_4^{(0,1)}$ at left and $\tilde{M}_4^{(0,1)}$ at right.

5. Numerical results

As an example in the one-dimensional case, we split the domain $\Omega = (-1, 2)$ into $\Omega_0 = (-1, 0)$, $\Omega_1 = (0, 1)$, and $\Omega_2 = (1, 2)$, with the diffusion coefficients κ_0, κ_1, and κ_2, respectively, such that $\kappa_0, \kappa_2 \lesssim \kappa_1$. We take $\mu_1 = \min(\kappa_1, \max(\kappa_0, \kappa_2))$ and $\mu_i = \kappa_i$, for $i = 0, 2$. An application of Proposition 2.1 shows that the norm $|||\cdot|||$ is equivalent to the energy-norm $|\cdot|_E : u \mapsto \sqrt{\sum_{i=0}^{2} \kappa_i |u|_{\Omega_i}|_{H^1(\Omega_i)}|^2}$ on $H_0^1(\Omega)$ uniformly in all $\kappa := (\kappa_0, \kappa_1, \kappa_2) \in (0, \infty)^3$.

The construction of a Riesz basis for $H_0^1(\Omega)$, that has Riesz constants w.r.t $|||\cdot|||$ (and thus w.r.t. $|\cdot|_E$) that are bounded uniformly in $\kappa_0, \kappa_2 \lesssim \kappa_1$, is determined once we have fixed the order in which the 3 subdomains are unified, and in which direction the non-trivial extensions are applied. Similar to Figure 2, we specify this by means of an illustration given in Figure 12. With this choice, $(\mathcal{D}_1)$–$(\mathcal{D}_7)$ are

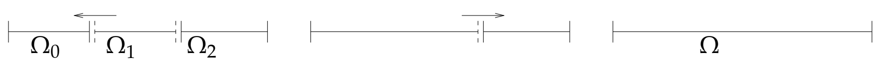

FIGURE 12. Solid edges indicate homogeneous Dirichlet boundary conditions, and no boundary conditions are indicated by dashed edges.

satisfied. Concerning $(\mathcal{D}_6)$, it means that necessarily $\kappa_0, \kappa_2 \lesssim \mu_1$, which follows from the definition of μ_1, and $\kappa_0, \kappa_2 \lesssim \kappa_1$.

So, initially, we equip $(-1, 0)$ with $\Psi_{(1,1)}(\cdot + 1)$, being a Riesz basis for $H_0^1(-1, 0)$, $(0, 1)$ with $\Psi_{(0,0)}$, being a Riesz basis for $H^1(0, 1)$, and $(1, 2)$ with $\Psi_{(1,1)}(\cdot - 1)$, being a Riesz basis for $H_0^1(1, 2)$. The resulting basis on $(-1, 2)$, denoted as $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$, is now obtained by reflecting over either 0 or 1, all wavelets from $\Psi_{(0,0)}$ that do not vanish at 0 or 1, respectively (and by applying the different extensions to the function $\phi_1^{(0,0)}$ and $\phi_2^{(0,0)}$ from Figure 9), and by taking the union with $\Psi_{(1,1)}(\cdot + 1)$ and $\Psi_{(1,1)}(\cdot - 1)$.

The numerically computed condition number of

$$A_J = \left[\frac{a_\kappa(\psi_\lambda, \psi_\mu)}{a_\kappa(\psi_\lambda, \psi_\lambda)^{\frac{1}{2}} a_\kappa(\psi_\mu, \psi_\mu)^{\frac{1}{2}}} \right]_{|\lambda|, |\mu| \leq J},$$

for $\kappa = (1, \kappa_1, 1)$, is given in Table 4.

TABLE 4. Condition number $\mathcal{K}(A_J)$, for $\kappa_0 = \kappa_2 = 1$ and $\kappa_1 = 1, 10, 100, 1000$.

$\kappa_1 \setminus J$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	22	32.0	36.5	40.0	42.8	44.9	47.0	48.6	50.8	51.9	52.3	53.0	53.7
10	9.79	14.9	17.3	19.0	20.4	21.6	22.6	23.4	24.2	25.0	25.4	25.9	26.2
100	8.53	13.1	15.2	16.8	18.1	19.1	20.0	20.8	21.4	22.0	22.5	23.0	23.3
1000	8.41	13.0	15.0	16.6	17.8	18.9	19.8	20.5	21.6	21.7	22.2	22.7	22.9

As a first example of a two-dimensional domain, we consider $\Omega = (0, 2)^2$ that is splitted into 4 squares $\Omega_0 = (0,1)^2$, $\Omega_1 = \{(1,0)\} + (0,1)^2$, $\Omega_2 = \{(0,1)\} + (0,1)^2$, and $\Omega_3 = (1,2)^2$. We consider diffusion coefficients κ_i, for $i = 0,1,2,3$, which satisfy

$$\kappa_1, \kappa_2 \lesssim \kappa_3,\ \kappa_0 \lesssim \kappa_1, \kappa_2. \tag{5.1}$$

We take $\mu_i = \kappa_i$, for $i = 0,1,2,3$. Then Proposition 2.1 shows that $|\!|\!|\cdot|\!|\!| \eqsim |\cdot|_E$ on $H_0^1(\Omega)$, uniformly in $(\kappa_i)_{0\leq i\leq 4} \in (0,\infty)^4$.

In Figure 13, we specify the imposed boundary conditions on the facets of the 4 subdomains, and the direction of the non-trivial extension operators. With this choice, $(\mathcal{D}_1)$–$(\mathcal{D}_7)$ are satisfied, where, in particular, $(\mathcal{D}_6)$ follows from $\mu_i = \kappa_i$ ($i = 0,1,2,3$) and (5.1).

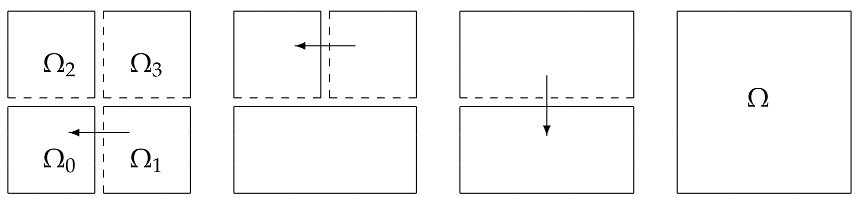

FIGURE 13. The solid subfacets indicate homogeneous Dirichlet boundary conditions, and no boundary conditions are indicated by dashed subfacets.

We equip Ω_0 with $\Psi_{(1,1)} \otimes \Psi_{(1,1)}$, Ω_1 with $\Psi_{(0,1)}(\cdot - 1) \otimes \Psi_{(1,1)}$, Ω_2 with $\Psi_{(1,1)} \otimes \Psi_{(0,1)}(\cdot - 1)$, and Ω_3 with $\Psi_{(0,1)}(\cdot - 1) \otimes \Psi_{(0,1)}(\cdot - 1)$. The piecewise tensor product basis for the space $H_0^1(\Omega)$ is obtained by taking the union of $\Psi_{(1,1)} \otimes \Psi_{(1,1)}$, $\Psi_{(1,1)} \otimes G_1(\Psi_{(0,1)}(\cdot - 1))$, $G_1(\Psi_{(0,1)}(\cdot - 1)) \otimes \Psi_{(1,1)}$, and that of $G_1(\Psi_{(0,1)}(\cdot - 1)) \otimes G_1(\Psi_{(0,1)}(\cdot - 1))$, where G_1 reflects in 1 all wavelets from $\Psi_{(0,1)}(\cdot - 1)$ that do not vanish at 1. The resulting basis has Riesz constants w.r.t $|\cdot|_E$ that are bounded uniformly in κ_i that satisfy (5.1).

Now as a second two-dimensional domain example, we consider $\Omega = (0,3)^2$ splitted into 9 subdomains Ω_i, for $i = 0,1,\cdots,8$, (see Figure 14) with the diffusion coefficients κ_i ($i = 0,1,\cdots,8$) such that

$$\begin{cases} \kappa_{i-1}, \kappa_{i+1} \lesssim \kappa_i, & i = 1,4,7, \\ \kappa_{i-3}, \kappa_{i+3} \lesssim \kappa_i, & i = 3,4,5. \end{cases} \tag{5.2}$$

We take $\mu_i = \kappa_i$, for $i = 0,\cdots,8$, and $\mu_4 = \min(\kappa_4, \max(\kappa_1,\kappa_3,\kappa_5,\kappa_7))$. In view of Proposition 2.1, $|\!|\!|\cdot|\!|\!| \eqsim |\cdot|$ on $H_0^1(\Omega)$ uniformly in $(\kappa_i)_{0\leq i\leq 8} \in (0,\infty)^8$. In Figure 14, we specify the imposed boundary conditions on the facets of the 9 subdomains, and the direction of the non-trivial extension operators. With this choice, $(\mathcal{D}_1)$–$(\mathcal{D}_7)$ are satisfied, where, in particular, $(\mathcal{D}_6)$ follows from (5.2) and our selection of the μ_i's.

The construction of the basis of $H_0^1(\Omega)$ from tensor product bases on the 9 subdomains follow similar steps as in the previous example. The bases on the subdomains have to satisfy the appropriate boundary conditions, and, in the order as

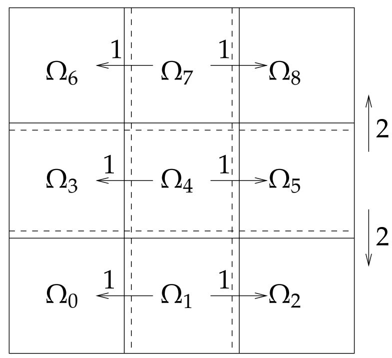

FIGURE 14. The solid subfacets indicate homogeneous Dirichlet boundary conditions, and no boundary conditions are indicated by dashed subfacets.

indicated in Figure 14, wavelets that do not vanish at an interface have to be reflected over this interface. No boundary conditions are prescribed on subdomain Ω_4. Consequently, as has been discussed in the last three paragraphs of Subsubsection 4.2.4, the construction of the tensor product basis on this subdomain requires a slight adaptation of the univariate scaling functions on the coarsest level, and the reflection operator requires a small adaption too. The resulting basis has Riesz constants w.r.t $|\cdot|$ that are bounded uniformly in the κ_i that satisfy (5.2).

We solved the Laplace-interface problem (1.1), with right-hand side $f = 1$, on both 2-dimensional domain examples by applying the *Adaptive Wavelet-Galerkin Method* (**AWGM**). This method, that is described and analyzed in detail in [CDD01, GHS07, Ste09], is an adaptive method applied to the representation (1.3) of (1.1) as the bi-infinite matrix-vector problem $\mathbf{Au} = \mathbf{f}$, where with $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ being the constructed 'uniform' Riesz basis, it holds that $\mathbf{f} = [f(\psi_\lambda)]_{\lambda\in\nabla}$, $\mathbf{A} = [a_\kappa(\psi_\mu, \psi_\lambda)]_{\lambda,\mu\in\nabla}$ and $u = \mathbf{u}^\top\Psi = \sum_{\lambda\in\nabla} \mathbf{u}_\lambda \psi_\lambda$. Pretending for simplicity that $\mathbf{A}$ is truly sparse, the **AWGM** reads as follows:

% let $\theta \in (0,1]$ be some constant
$\Lambda_0 := \emptyset$, $\mathbf{u}_0 := 0$, $i := 0$
`while` $\|\mathbf{f} - \mathbf{Au}_i\| >$ `TOL` `do`
  select a smallest $\Lambda_{i+1} \supset \Lambda_i$ such that
    $\|(\mathbf{f} - \mathbf{Au}_i)|_{\Lambda_{i+1}}\| \geq \theta\|\mathbf{f} - \mathbf{Au}_i\|$
  $i := i + 1$
  solve $\mathbf{A}|_{\Lambda_i\times\Lambda_i}\mathbf{u}_i = \mathbf{f}|_{\Lambda_i}$
`enddo`

For θ being sufficiently small, the sequence of approximate solutions $(\mathbf{u}_i)_{i\geq 0}$ converges to the exact solution with best possible nonlinear approximation rate from the basis. For a non truly sparse $\mathbf{A}$ the same holds true when the infinite residuals $\mathbf{f} - \mathbf{Au}_i$ are computed within a sufficiently small relative tolerance. Moreover, the exact Galerkin solutions $\mathbf{u}_i$ can be replaced by approximate solutions within a sufficiently small relative tolerance. In our current setting of having a matrix $\mathbf{A}$ that is very close to being sparse, the resulting algorithm can easily be implemented such that it runs in linear complexity, i.e., the cost of producing each approximation scales linearly with its support length.

In view of the tensor product wavelet construction, for sufficiently smooth functions, the aforementioned best nonlinear approximation rate is as large as $d-1=4$, compared to $\frac{d-1}{n}$ for a common isotropic wavelet basis of order d. In view of the results obtained in [CDFS13], there for $\kappa=1$, we expect moreover that this rate is achieved under very mild piecewise weighted Sobolev regularity conditions, with weights that vanish at the interfaces between the subdomains.

In the first example, we consider the following diffusion coefficients on $\Omega = (0,2)^2$ splitted into 4 squares,

$$\kappa_0 = 1, \quad \kappa_1 = 10^m, \quad \kappa_2 = 10^{m+2}, \quad \kappa_3 = 10^{m+4}, \tag{5.3}$$

for various $m \in \mathbb{N}_0$. In Figure 15, for $m = 0,2,4,8$, one finds the support length vs. the (relative) ℓ_2-norm of the residual of the approximations produced by the **AWGM**. Note that because of the Riesz basis property, the ℓ_2-norm of the residual is uniformly equivalent to the energy-norm of the error. For comparison, we included corresponding results obtained with the non-adaptive full and sparse-grids methods for $m = 2$. We observe that the **AWGM** realizes the optimal rate 4, whereas the sparse and full-grid methods do not converge at their best possible rates $d-1=4$ (up to some log-factors) and $\frac{d-1}{n}=2$, respectively, because of a lacking smoothness of the solution.

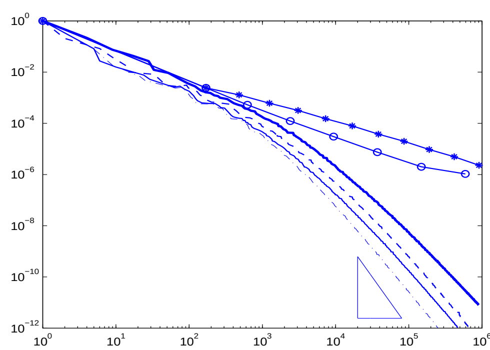

FIGURE 15. Support length vs. relative residual of the approximate solutions obtained by the **AWGM** for the Laplace-interface problem (1.1) with $f = 1$, and diffusion coefficients (5.3) for $m = 8,4,2,0$ (from left to right). The slope of the triangle is the best possible rate -4. The curves with $*$ and $\bullet$ show the convergence rates for $m = 2$ of the sparse and full-grid approximations, respectively.

At the end of these computations, the maximum levels of any univariate wavelet factor of the piecewise tensor product wavelets, that were selected by the **AWGM**, are $36, 26, 19, 19$ for $m = 0,2,4,8$, respectively. For the sparse and full-grids approximations and the case $m = 2$, these maximum levels were 10 and 7, respectively.

Centers of the supports of the piecewise tensor product wavelets, selected by the **AWGM**, are illustrated in Figure 16.

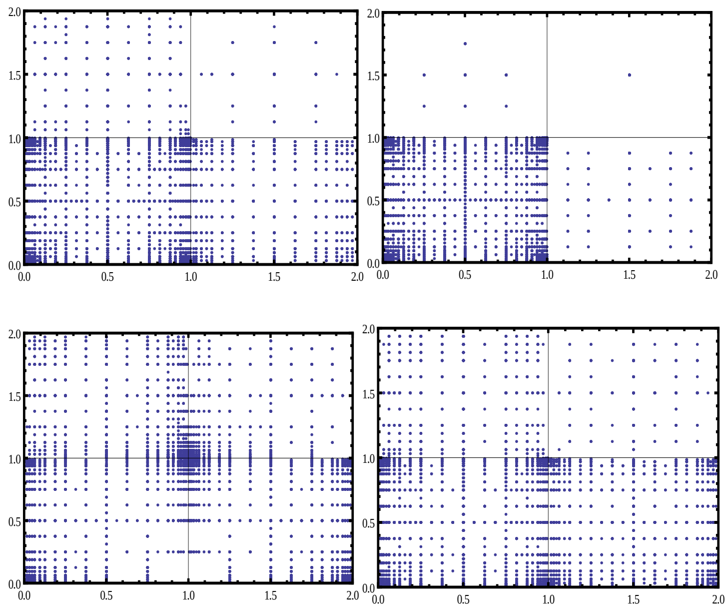

FIGURE 16. Centers of the supports of the piecewise tensor product wavelets, selected by the **AWGM** for the Laplace-interface problem (1.1) with $f = 1$, and diffusion coefficients (5.3), for $m = 0$ with 7100 wavelets at bottom-left, for $m = 2$ with 7170 wavelets at bottom-right, for $m = 4$ with 7223 wavelets at top-left, and for $m = 8$ with 7052 wavelets at top-right.

In Figure 17, we present the approximate solutions obtained by the **AWGM**, for the diffusion coefficients from (5.3) and $m = 0, 2, 4$, as well as for $\kappa_0 = 1$, $\kappa_1 = 4$, $\kappa_2 = 8$, $\kappa_4 = 12$.

In the second example, we consider the following diffusion coefficients on $\Omega = (0,3)^2$ splitted into 9 squares,

$$\kappa_4 = 10^m, \quad \kappa_i = 1 \ (0 \leq i \leq 8, i \neq 4), \tag{5.4}$$

for various $m \in \mathbb{N}_0$. In Figure 18, for $m = 1, 3, 6$, one finds the support length vs. the (relative) ℓ_2-norm of the residual of the approximations produced by the **AWGM**, as well as, for comparison, those that were obtained by the sparse and full grid methods for $m = 3$.

At the end of these computations, the maximum levels of any univariate wavelet factor of the piecewise tensor product wavelets, that were selected by the **AWGM**, are $28, 40, 38$, for $m = 0, 3, 6$, respectively. For the sparse and full-grids approximations, these maximum levels were 9 and 6, respectively.

Centers of the supports of the piecewise tensor product wavelets, selected by the **AWGM**, are illustrated in Figure 19.

The approximate solutions obtained by the **AWGM** are shown in Figure 20.

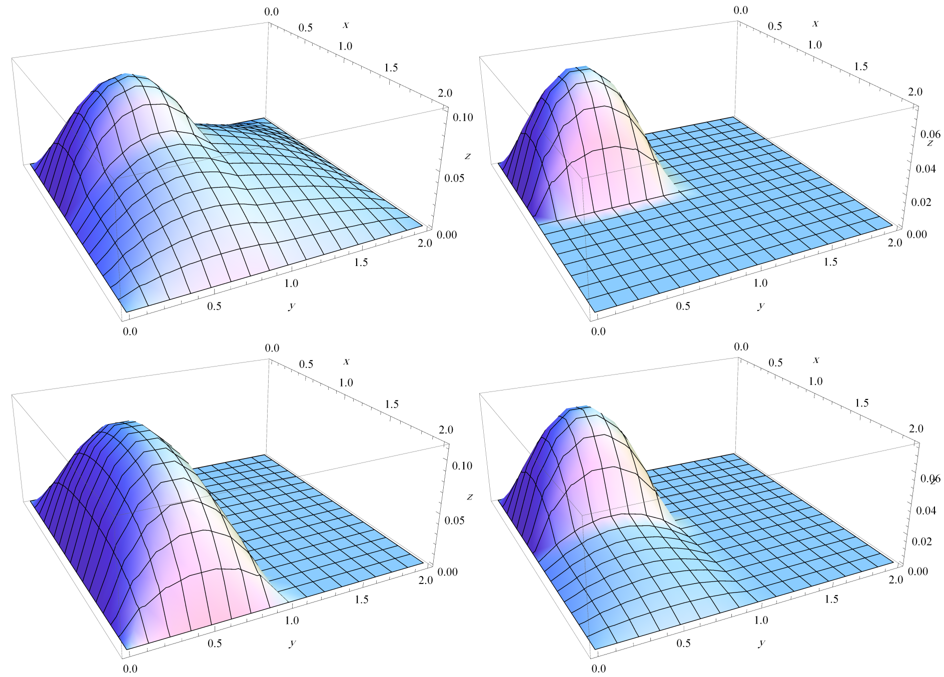

FIGURE 17. Approximate solutions computed by the **AWGM** of the Laplace-interface problem (1.1) with $f = 1$ for the diffusion coefficients (5.3) for $m = 0$ at bottom-left, $m = 2$ at bottom-right, and $m = 4$ at top-right, as well as for $\kappa_0 = 1$, $\kappa_1 = 4$, $\kappa_2 = 8$, $\kappa_4 = 12$ at top left.

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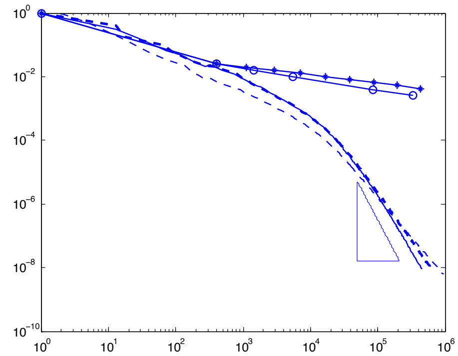

FIGURE 18. Support length vs. relative residual of the approximate solutions obtained by the **AWGM** for the Laplace-interface problem (1.1) with $f = 1$, and diffusion coefficients (5.4) for $m = 6, 3, 1$ (curves from left to right at the bottom of the figure). The slope of the triangle is the best possible rate -4. The curves with $*$ and $\bullet$ show the convergence rates for $m = 3$ of the sparse and full-grid approximations, respectively.

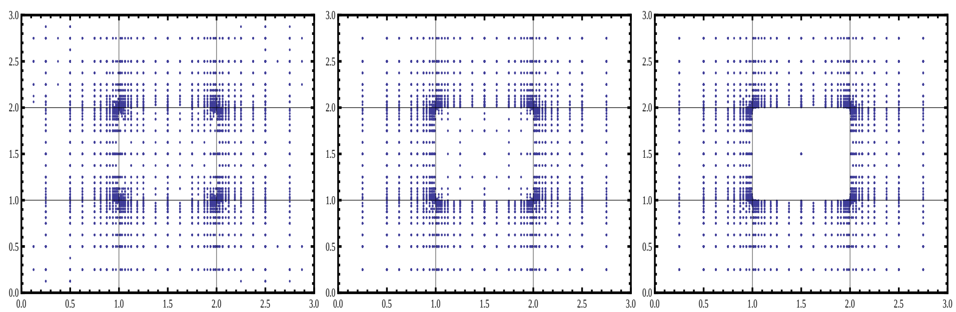

FIGURE 19. Centers of the supports of the piecewise tensor product wavelets that were selected by the **AWGM** for Laplace-interface problem (1.1) with $f = 1$, and diffusion coefficients (5.4), for $m = 1$ at left, $m = 3$ at middle, and $m = 6$ at right, in all cases with approximately 6000 wavelets.

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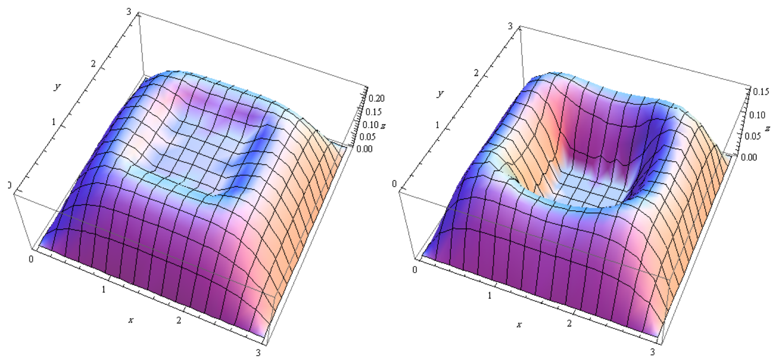

FIGURE 20. Approximate solutions for Laplace-interface problem with $f = 1$, and the diffusion coefficients (5.4), obtained by the **AWGM**, for $m = 1$ at left, and $m = 3$ at right.

Dedicated to Wolfgang Dahmen on the Occasion of his 60th Birthday, pages 543–598. Springer, Berlin, 2009.

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