String Theory

Take Home Set 3
Hand in on May 23, 2007

The Torus

In string theory in Euclidean signature, the world-sheet can have different topologies. It can be two-sphere, a torus, or a more complicated surface. When we studied the string world-sheet, we assumed that we could always choose a gauge such that $g_{\alpha\beta} = \eta_{\alpha\beta}$, or $g_{\alpha\beta} = \delta_{\alpha\beta}$ in Euclidean signature. However, one can not always make such a gauge choice globally. We will now study this for the case of the torus. We start with a two-torus, parametrized by two coordinates $(x^1, x^2) = (x, y)$ such that x is identified with $x+1$ and y is identified with $y+1$ as well. We will take an arbitrary metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ with $\alpha, \beta = 1, 2$, where $g_{\alpha\beta}(x, y) = g_{\alpha\beta}(x+1, y) = g_{\alpha\beta}(x, y+1)$.

1) Explain why this does describe a two-torus.

2) In any dimension the following identity holds:

$$\delta \int d^d x \sqrt{g} R = \int d^d x \delta g^{\mu\nu} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}).$$

This is how one derives the Einstein equations from an action principle. Use this to show that $\delta \int d^2 x \sqrt{g} R$ is invariant in two dimensions under small variations of the metric.

3) Use this to show that $\int d^2 x \sqrt{g} R = 0$ for all metrics on the two torus. (Hint: use the previous problem to deform the metric to a simple one and then explicitly evaluate $\int d^2 x \sqrt{g} R$.)

4) For a function $f(x^1, x^2)$, the Laplacian $\nabla^2 f$ is defined by the following equation (valid for all functions h)

$$\int d^2 x \sqrt{g} h \nabla^2 f \equiv - \int d^2 x \sqrt{g} g^{\alpha\beta} \partial_\alpha h \partial_\beta f.$$

This definition also shows that ∇^2 is an Hermitian operator and therefore it can be diagonalized. Show that the only eigenfunctions with eigenvalue zero are the constant functions.

5) Use this to show that for all Ricci scalars R that satisfy $\int d^2 x \sqrt{g} R = 0$, we can always find a function f such that $\nabla^2 f = R$.

It is straightforward to show that the Ricci scalar of the metric $e^f g_{\alpha\beta}$ equals

$$R[e^f g_{\alpha\beta}] = e^{-f}(R[g_{\alpha\beta}] - \nabla^2 f)$$

and therefore we have now shown that for any metric on the two-torus we can always find a Weyl transformation that makes the Ricci scalar identically zero. In two dimensions, vanishing Ricci scalar implies vanishing Riemann tensor, and if the Riemann tensor vanishes we can always find coordinates such that the metric is that of Euclidean $\mathbb{R}^2$. It is however not guaranteed that the periodic identifications of the coordinates are preserved. From now on we will work with complex coordinates $w, barw$ so that the standard metric of Euclidean $\mathbb{R}^2$ is $ds^2 = dwd\bar{w}$. Therefore, what we have demonstrated so far is that the metric of any two-torus can with the help of a Weyl transformation and a diffeomorphism be put in the form

$$ds^2 = dwd\bar{w} \tag{1}$$

with periodic identifications

$$w \sim w + w_0, \qquad w \sim w + w_1 \tag{2}$$

for some w_0, w_1. This can be simplified a bit further.

6) Show that we can always find a combination of a diffeomorphism of the form $w \to \lambda w$ and a Weyl rescaling that preserves the form of the metric (1) but makes $w_0 = 1$.

After we put $w_0 = 1$, there is only one free parameter left, namely w_1. This parameter is usually called τ. Two-tori that have different values of the parameter τ cannot in general be mapped into each other using a combination of diffeomorphisms and Weyl rescalings: there is a one-parameter family of conformally inequivalent tori. The integral over all metrics in the Polyakov path integral can be reduced to an integral over the single parameter τ by appropriate gauge fixing, but this final integral remains and has to be done by hand. The complex parameter τ determines the shape of the torus.

This parameter τ can be thought of as follows: a torus is created by taking a parallelogram, and gluing opposite edges together. This parallelogram is embedded in the complex plane such that one of the corners is the origin, and the other corners are at $1, \tau$ and $1 + \tau$ (see figure 1), where τ is in the upper right quarter of the plane. In other words, the parallellogram is defined by the relations

$$\begin{aligned} w &\equiv w + 1 \\ w &\equiv w + \tau \end{aligned} \tag{3}$$

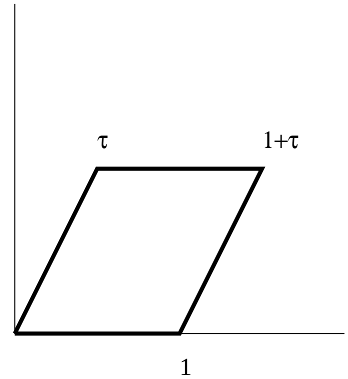

Figure 1: A torus is created by identifying opposite edges of a parallellogram.

If we conformally map w to $z = e^{2\pi i w}$, the first equivalence is automatically satisfied. The second equivalence translates into

$$z \equiv qz, \qquad q = e^{2\pi i \tau}. \tag{4}$$

In a picture, this looks like figure 2. We can take an annulus as the fundamental region, and identify the inner and outer circle. However, in this identification, there is a twist: the point 1 is identified with the point q.

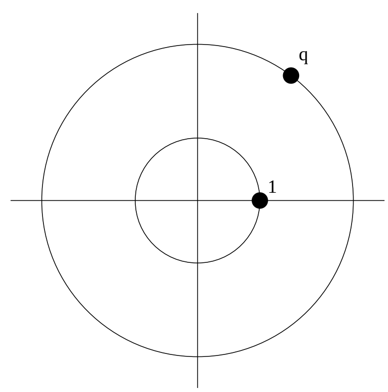

Figure 2: A torus as an annulus in the z-plane.

We will now calculate the partition function Z for a single free scalar field X on the torus. A priori, Z will depend on τ. In general, Z will not be a holomorphic function, so we write $Z(\tau, \bar{\tau})$. We also write $\tau = \tau_1 + i\tau_2$.

Once again, look at figure 2. If we view this picture as a string diagram, we see that what happens between the inner and the outer circle are two things. First of all, (remember time runs radially), there is a propagation in time. The duration of this propagation is $2\pi\tau_2$. Second of all, there is a translation along the spatial direction

(i. e. along the circle) of length $2\pi\tau_1$. The operator that achieves both these facts is

$$\begin{aligned} 2\pi i\tau_1 P - 2\pi\tau_2 H &= 2\pi L_0(i\tau_1 - \tau_2) + 2\pi\bar{L}_0(-i\tau_1 - \tau_2) + \frac{\pi\tau_2}{24} \\ &= 2\pi i L_0\tau - 2\pi i\bar{L}_0\bar{\tau} + \frac{\pi\tau_2}{24}, \end{aligned} \tag{5}$$

where we used $P = L_0 - \bar{L}_0$ and $H = L_0 + \bar{L}_0 - \frac{1}{12}$. The latter constant is the central charge for a single free bosonic field.

7) Verify this and explain the form of P and H.

If you think back about the standard (Feynman) derivation of the path integral for quantum mechanics, that derivation shows that

$$\langle x_f|\exp(i(t_1 - t_0)\, H/\hbar)|x_i\rangle = \int_{\phi(t_0)=x_i}^{\phi(t_1)=x_f} \mathcal{D}\phi(t)\exp(\frac{i}{\hbar}\int dt L(\phi,\dot{\phi})). \tag{6}$$

If we now put $x_f = x_i = x$ and integrate over x, and at the same time go to Euclidean time $t \to it_E$, then the right hand side of this equation becomes an Euclidean path integral with periodic boundary conditions. The left hand side becomes

$$\int dx\langle x|\exp(-\Delta t_E\, H/\hbar)|x\rangle \equiv \mathrm{Tr}(\exp(-\Delta t_E\, H/\hbar)) \tag{7}$$

which is the finite temperature partition function of the quantum mechanical system. This demonstrates the equivalence of the finite temperature partition function to the Euclidean path integral with periodic boundary conditions.

By generalizing these arguments to the case of a free scalar field on a Euclidean two-torus, one finds a similar result. It can be shown that the Euclidean partition function is equivalent to exponentiating (5) and taking the trace of this operator over the Hilbert space[1]:

$$Z(\tau,\bar{\tau}) = (q\bar{q})^{-1/24}\mathrm{Tr}(q^{L_0}\bar{q}^{\bar{L}_0}). \tag{8}$$

Taking the trace over the Hilbert space means sandwiching an operator between all basis states and summing (or if necessary integrating) over these states (as illustrated above in the case of quantum mechanics).

8) Show that for a finite dimensional Hilbert space, this prescription corresponds to the familiar definition of the trace of a matrix.

9) In evaluating this expression, we use the fact that L_0 on a certain state counts the right-moving oscillations and adds the square of the momentum; $\bar{L}_0$ does the same on the right-moving side. Show that this means that

$$Z(\tau,\bar{\tau}) = C(q\bar{q})^{-1/24}\int dk\exp(-\pi\tau_2\alpha' k^2)\prod_n \sum_{N_n,\bar{N}_n=0}^{\infty} q^{nN_n}\bar{q}^{n\bar{N}_n}, \tag{9}$$

[1]We will not prove this here; for an example, the reader is referred to *Evaluation of the one loop string path integral* by Polchinski; Commun. Math. Phys. 104 (1986), pp. 37-47

where C is a normalization constant, and N_n is the occupation number of the harmonic oscillator formed by α_n, α_{-n}, and $\bar{N}_n$ is the occupation number of the harmonic oscillator formed by $\tilde{\alpha}_n, \tilde{\alpha}_{-n}$.

The Gaussian integral is easily evaluated; for the sums we use the identity

$$\sum_{N=0}^{\infty} a^N = \frac{1}{1-a}. \tag{10}$$

Inserting this, we arrive at the final expression

$$Z(\tau,\bar{\tau}) = C(4\pi^2\alpha'\tau_2)^{-1/2} \left| \prod_{n=1}^{\infty} \frac{q^{-1/24}}{(1-q^n)} \right|^2 . \tag{11}$$

10) Verify this.

It turns out that this partition function enjoys a further discrete symmetry called modular invariance, in important property of string theory. The origin of this symmetry is the fact that not all different τ correspond to different tori. As an example, look at figure 3. If we want to construct a torus, we take some fundamental parallelogram in the plane, and impose periodicity – which is the same as gluing the opposite sides of the parallelogram together. However, note that there is no difference between taking the parallelogram in solid lines as the fundamental one and taking the parallelogram in dashed lines; both correspond to the same torus.

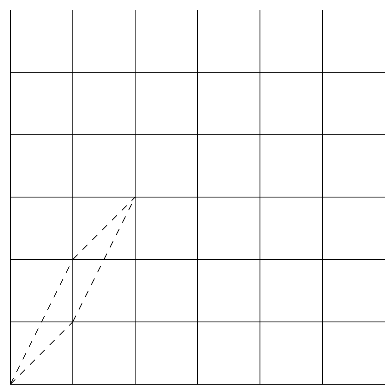

Figure 3: Two equivalent fundamental regions for a torus.

It can be shown that two fundamental regions with sides l_1, l_2 and l'_1, l'_2 correspond to the same torus if and only if

$$\begin{pmatrix} l'_1 \\ l'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \tag{12}$$

where the matrix is an element of $SL(2,\mathbb{Z})$, i. e. a, b, c, d are integers and the determinant $ad - bc = 1$. The proof of this fact is not very difficult; it consists of the following three observations:

- The endpoints of l_1' and l_2' should be identified with their starting points (i. e. the origin). This means these points have to be lattice points in the original lattice. In other words: a, b, c, d have to be integers.

- From elementary vector calculus, we know that the absolute value of the determinant of the transformation matrix (the Jacobian) gives the change in area of the parallelogram. Therefore, $ad - bc = \pm 1$. By changing the order of l_1' and l_2' (i. e. interchanging two rows), we can always choose the determinant to be $+1$.

- From the above two points, we see that any transformation that gives the same torus has to be an $SL(2, \mathbb{Z})$-transformation. To prove the converse, note that we can cover the entire plane with parallelograms with sides l_1', l_2'. Since the area of the new parallelogram is equal to the area of the old one, there cannot be any lattice points of the old lattice inside the fundamental area of the first one. The converse of course also holds, since the inverse of an $SL(2, \mathbb{Z})$-transformation is again an $SL(2, \mathbb{Z})$-transformation. Therefore, the two lattices consist of the same points, and hence the tori are the same.

11) Translate the equivalence between the sides of the parallelogram to an equivalence relation for the values of τ. These equivalence relations are the so-called modular transformations for τ.

12) (Optional, very difficult) Show that the partition function in equation (11) is invariant under this equivalence relation, in other words that it is modular invariant.

All of this means that we should not integrate τ over the entire complex plane, but only over a part of this space, which can be denoted by $\mathcal{F} = \mathbb{C}/SL(2, \mathbb{Z})$.