String Theory
Take Home Set 2
Hand in on March 11th, 2007

Problem 1

Upon gauge fixing the diffeomorphism and Weyl invariance of the Polyakov action using the Faddeev-Popov method, we encountered the bc system. In this problem, we will study a slight generalization of the system encountered in class which depends on a real parameter λ. The only singular OPE is between b and c,

$$b(z)c(w) \quad = \quad \frac{1}{z-w} + \textit{nonsing}\,.$$

The energy momentum tensor of this system is given by

$$T(z) \quad = \quad :(\partial b)c: - \lambda\, \partial :bc: \;.$$

a) What is the weight of the fields b, c as a function of λ? (Hint: you have to calculate two OPEs.)

b) What is the central charge of this system, as a function of λ? As a check on your results: for $\lambda = 2$, the central charge should cancel that of a system of 26 free bosons.

In your calculations, keep in mind that b and c are fermionic fields; they anticommute within normal ordering brackets.

Problem 2

In class, we discussed BRST quantization in the abstract. We used a condensed notation in which the symmetry transformations of the fields were denoted by $\epsilon^\alpha \delta_\alpha \phi$. A summation over α, or, in case α is a continuous parameter, an integration, is understood. We express the commutator of two symmetry transformations in terms of structure constants $f^\gamma_{\alpha\beta}$: $[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta}\delta_\gamma$.

a) Let's first consider diffeomorphism invariance in 1 dimension,

$$\tau \rightarrow \tau - \epsilon(\tau)\,.$$

A scalar field transforms as

$$\begin{aligned} \phi(\tau - \epsilon(\tau)) \quad &= \quad \phi(\tau) - \epsilon(\tau)\partial\phi(\tau) + \mathcal{O}(\epsilon^2) \\ &= \quad \phi(\tau) + (\epsilon^{\tilde{\tau}}\delta_{\tilde{\tau}}\phi)(\tau) + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

Here, ϵ^τ is just different notation for $\epsilon(\tau)$. The equality

$$\epsilon^{\tilde{\tau}}\delta_{\tilde{\tau}}\phi(\tau) = -\epsilon(\tau)\partial\phi(\tau)$$

implies

$$\delta_{\tilde{\tau}} = \partial_\tau \delta(\tau - \tilde{\tau})\,.$$

Note that the above two lines are simply shorthand for the following manipulations,

$$\begin{aligned}
\epsilon^{\tilde{\tau}} \delta_{\tilde{\tau}} \phi(\tau) &= \int d\tilde{\tau}\, \epsilon^{\tilde{\tau}} \partial_\tau \delta(\tau - \tilde{\tau}) \phi(\tau) \\
&= -\int d\tilde{\tau}\, \epsilon(\tilde{\tau}) \delta(\tau - \tilde{\tau}) \partial_\tau \phi(\tau) \\
&= -\epsilon(\tau) \partial \phi(\tau)\,.
\end{aligned}$$

Verify the structure constants

$$f^{\tau_3}_{\tau_1 \tau_2} = \delta(\tau_3 - \tau_1) \partial_{\tau_3} \delta(\tau_3 - \tau_2) - \delta(\tau_3 - \tau_2) \partial_{\tau_3} \delta(\tau_3 - \tau_1)$$

by calculating $[\delta_{\tau_1}, \delta_{\tau_2}]\phi(\tau)$.

b) Perform the same exercise for diffeomorphism invariance in 2 dimensions. Use complex coordinates $z, \bar{z}$, treating z and $\bar{z}$ as independent.

c) The BRST transformation for the fields X in the original (not gauge-fixed) action and for the c ghost are

$$\begin{aligned}
\delta_B X &= -i \epsilon c^\alpha \delta_\alpha X\,, \\
\delta_B c^\alpha &= \frac{i}{2} \epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\,.
\end{aligned}$$

Specialize these to the case of 2 dimensional diffeomorphism invariance.

d) The holomorphic part of the BRST current for the Polyakov action of the bosonic string is given by

$$j_B = c T^m + :b c \partial c: + \frac{3}{2} \partial^2 c\,,$$

where T^m is the energy momentum tensor of the matter sector (in the simplest case, a system of 26 free bosons). Verify that this current induces the transformation of the c field determined under c), under the identification $c^z = c(z)$.

e) The OPE of j_B with itself is given by

$$j_B(z) j_B(w) \sim \frac{c_{bc} + 18}{2(z-w)^3} c \partial c(w) + \frac{c_{bc} + 18}{4(z-w)^2} c \partial^2 c(w) + \frac{c_{bc} + 26}{12(z-w)} c \partial^3 c(w)\,,$$

where c_{bc} is the central charge of the bc system. Use this OPE to determine the anticommutator of the BRST charge with itself. For what value of c_{bc} does this vanish? What is the significance of this result?

f) *Extra credit:* Verify the OPE between two BRST currents as given in e).