String Theory
Homework May 9, 2007

Yang Mills theories are theories that have a non-abelian gauge symmetry. This is a generalization of the abelian $U(1)$-symmetry we encounter in electrodynamics.

We begin with a complex scalar field $\phi(x)$. A generic Lagrangian for this kind of field is

$$L_1 = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{1}$$

Show that this Lagrangian is invariant under *global* $U(1)$-transformations

$$\phi \to e^{i\alpha} \phi, \tag{2}$$

but not under *local* $U(1)$-transformations. What is the change in the Lagrangian L_1 if we apply a local $U(1)$-transformation?

To make the action locally $U(1)$-invariant, we use a well-known trick. Introduce a new field $A^\mu(x)$ and a term

$$L_2 = -iA^\mu(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \tag{3}$$

in the Lagrangian. How should A^μ transform to cancel the change in the Lagrangian L_1? Notice that this does not solve our entire problem yet, since there are now two terms in the change of L_2 that are not canceled. Write down a term L_3 that will transform in the opposite way, so that the total Lagrangian $L_1 + L_2 + L_3$ is conserved under local $U(1)$-transformations.

The expression we found can be written down more elegantly if we define a *covariant derivative* $D_\mu \phi$ such that the action can be written as

$$L = D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi. \tag{4}$$

what is the expression for $D_\mu \phi$? Notice the name 'covariant derivative': $D_\mu \phi$ transforms in the same way as ϕ itself.

Notice that the field A^μ is non-dynamical: there is no kinetic term for A^μ in the Lagrangian. Show that the kinetic term

$$L_4 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{5}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge-invariant.

Now, we want to generalize this construction to a non-abelian gauge group. This means our scalar field will obtain an index i, and it will transform as

$$\phi^i \to M^{ij} \phi^j, \tag{6}$$

where the matrix M is a group element. All the matrices in a Lie group can be generated by a finite number of generators T^a:

$$M = e^{i\omega^a T^a}. \tag{7}$$

The ω^a are ordinary numbers; note that even though we supressed the matrix indices, the T^a are matrices! We can make local gauge transformations by making ω^a position-dependent.

The generators T^a form a closed algebra:

$$[T^a, T^b] = f^{abc} T^c, \tag{8}$$

where f^{abc} are called the *structure constants*. Derive a relation for these structure constants from the Jacobi identity

$$[[T^a, T^b], T^c] + \text{cycl. perm.} = 0 \tag{9}$$

Like in the abelian case, we want to construct a covariant derivative; i. e. a derivative that transforms like (6). With the abelian case in mind, we propose the derivative

$$D_\mu = \partial_\mu + i A^a_\mu T^a. \tag{10}$$

Show that this derivative is covariant for infinitesimal ω^a if we impose the transformation rule

$$A^a_\mu \to A^a_\mu - \partial_\mu \omega^a + i f^{abc} \omega^b A^c_\mu. \tag{11}$$

Using this covariant derivative, we can construct a locally invariant Lagrangian such as

$$L = \text{Tr}(D_\mu \phi^i D_\mu \phi^j - m^2 \phi^i \phi^j). \tag{12}$$

Next, we want to make the fields A^a_μ dynamical. Show that $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ is not covariant, but that

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f^{abc} A^b_\mu A^c_\nu \tag{13}$$

is. (N.B. By 'covariant', we mean that the group index a transforms in the same way as the gauge field group index in (11). However, there is no explicit gauge term $\partial_\mu \omega^a$.) We can therefore introduce a term like

$$L_4 = -\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \tag{14}$$

in our Lagrangian. In fact, often the coupling to the scalar field ϕ is omitted, and the Lagrangian L_4 itself is taken as a starting point for a theory. This is the Lagrangian of non-abelian Yang-Mills theory. It is precisely this theory with gauge group $U(N)$ that describes the space-time physics of open strings with N Chan-Paton charges.

Now study pages 198-201 in the book, and do exercise 6.2 and problem 6.2.