UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2005/2006; 1st Semester
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Homework Set # 12 *Deadline:* December 13th, 2005 (Two weeks!)

Exercise 39 (total of eight points).

Let $\mathcal{L} := \{\dot{0}, \dot{1}, \dot{+}, \dot{\times}\}$ be the language of fields (*i.e.*, $\dot{0}$ and $\dot{1}$ are 0-ary function symbols, and $\dot{+}$ and $\dot{\times}$ are binary function symbols; if you don't know what a field is, please find out. On the other hand, the details are not really important for this exercise.) For a variable x, we define a term $x \cdot n$ by recursion: $x \cdot 1 := x$ and $x \cdot (n+1) := (x \cdot n) \dot{\times} x$. Let χ_n be the formula $\exists x(\neg(x = \dot{0}) \wedge x \cdot n = \dot{0})$. We say that a field $\mathbf{K}$ has **characteristic** $\leq n$ if $\mathbf{K} \models \chi_n$, that it has **characteristic** n if n is least such that it has characteristic $\leq n$, and that it has **characteristic zero** if for all $n > 0$, $\mathbf{K} \models \neg\chi_n$.

Remarks (not part of the exercise, but potentially useful). If you have never seen the notion of characteristic before, it might be useful to familiarize with it before starting the exercise. Notice that χ_1 is never true, so there can be no field with characteristic ≤ 1, even though there are fields of characteristic zero (it is a subtle notational hint that we write "characteristic zero" instead of "characteristic 0"). The operation $\cdot$ is not a binary (internal) operation on a field, but rather an operation linking fields to the natural numbers, so $\cdot : \mathbf{K} \times \mathbb{N} \to \mathbf{K}$. This operation satisfies the law of associativity as follows: $x \cdot (mn) = (x \cdot m) \cdot n$ ($\star$). It does not satisfy the other conceivable law of associativity: $(x \dot{\times} y) \cdot n = x \dot{\times} (y \cdot n)$. From ($\star$), it is easy to derive that the characteristic is always zero or a prime number: Suppose that $x \neq \dot{0}$, but $x \cdot mn = \dot{0}$. Then $(x \cdot m) \cdot n = \dot{0}$, which means that either $x \cdot m = \dot{0}$ (in which case, the characteristic is $\leq m$) or $y := x \cdot m \neq \dot{0}$ witnesses that the characteristic is $\leq n$. In both cases, the characteristic cannot be mn.

For every prime number p, there is a field of characteristic p: take the numbers $\{0, 1, 2, ..., p-1\}$ and define $\dot{+}$ and $\dot{\times}$ by computation modulo p. It's easy to check that this is a field of characteristic p.

(1) Prove: If φ is an $\mathcal{L}$-sentence that holds in all fields of characteristic zero, then there is some natural number n such that φ holds in all fields of characteristic n (4 points).

Hint. Use the compactness theorem.

(2) Let $\mathbb{P}$ be the set of all prime numbers. If $\mathbf{K}_p$ is a field of characteristic p and U is a non-principal ultrafilter on $\mathbb{P}$, what is the characteristic of the ultraproduct $\mathrm{Ult}(\langle \mathbf{K}_p\,;\, p \in \mathbb{P}\rangle, U)$ (4 points)?

Hint. Use Łoś's theorem.

Exercise 40 (total of six points).

What is an Erdős number? This can either be a large cardinal notion (give a one-sentence description; 1 point) or a property of researchers (give a full recursive definition of "X has Erdős number n"; 3 points).

Compute the Erdős number of Johan van Benthem and Bill Gates (in both cases, give the shortest path witnessing the upper bound; 1 point each).

Exercise 41 (total of eight points).

Let $\langle \mathbf{M}, V\rangle$ be a Kripke model. We define

$$\mathbf{not}\,\varphi := \Box\neg\varphi.$$

Let DN_0 (for "*duplex negatio*") be $(\mathbf{not}\,\mathbf{not}\,\varphi) \to \varphi$ and DN_1 be $\varphi \to \mathbf{not}\,\mathbf{not}\,\varphi$.

(1) Check (and give corresponding proofs) whether DN_0 and/or DN_1 hold in the class of all reflexive, transitive frames ("$\mathbf{S4}$-frames"; 2 points each)?

(2) Check (and give corresponding proofs) whether DN_0 and/or DN_1 hold in the class of all reflexive, symmetric, transitive frames ("$\mathbf{S5}$-frames"; 2 points each)?